Understanding Edge Excitations in Fractional Quantum Hall Systems
Key Notes

• Fractional quantum Hall states have gapless edge excitations.
• The structure of the edge excitations is determined by bulk topological orders.
• The topological orders of the fractional quantum Hall states can be measured by edge state transport experiments.

References:
Our Philosophy

• We ask the questions what kind of quantum Hall states can be formed at a specific filling fraction and what are the properties of such a state.

• The questions have been partially answered by constructing trial wave functions, the Chern-Simons topological field theory (in 2+1 dimensions), and the conformal field theory (on 1+1 dimensional edge).

• None of those answers however, out of various possible states, which one should form; this depends on energetics. In fact, they all answer the question of what are the possible solutions to an approximate (or ideal) problem.
Our Philosophy (continued)

• *Detail of interaction and competition between interaction and confinement is subtle* (e.g., different phases at half filling, edge instabilities).

• We want to solve *a more realistic problem* with both Coulomb interaction and background charge confining potential *without any presumptions* on which state should emerge as the global ground state.

• We must be *very careful about interpreting our numerical results* from small systems, and check them with, e.g., conformal field theory.
Outline

• Introduction
• (Abelian) Edge reconstruction – example of the complications arising from the competition between long-range interaction and short-distance edge confining potential
• (Nonabelian) Edge excitations of the Pfaffian state – understanding the results of a realistic model
• Summary and outlook

• Will not cover topological degeneracy, Chern number calculation, and universality of edge tunneling (in the presence of Coulomb interaction).
  – Sheng, XW, Rezayi, Yang, Bhatt & Haldane, PRL 90, 256802 (2003)
  – XW, Sheng, Rezayi, Yang, Bhatt & Haldane, PRB 72, 075325 (2005)
Fractional Quantum Hall Effect

- Discovered by Tsui, Stormer, Gossard (82)

- Laughlin sequence
  \[ \nu = \frac{1}{2p+1} = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \]

- Jain sequence
  \[ \nu = \frac{p}{2p+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots \]
Laughlin States

• In a magnetic field $B$, electron kinetic energy quantized to $(N+1/2)\hbar\omega_c$ (Landau levels)
  - $\nu \leq 1$, all states in the lowest Landau level
  - Single electron states in the LLL (symmetric gauge):
    $$\phi_m \sim z^m e^{-|x|^2/4l^2}, \quad z = x + iy; \quad m = 0,1,2,\ldots; \quad l = \sqrt{\hbar c / eB}$$

• Many-body states in the LLL:
  $$\Psi \sim f(z_1,z_2,\ldots,z_n)e^{-\sum |z_i|^2/4l^2}$$
  - $f(z_1, z_2, \ldots, z_n)$ analytic and antisymmetric
  - Laughlin’s choice
    $$f(z_1,z_2,\ldots,z_n) = \prod_{i<j} (z_i - z_j)^m, \quad m = 1,3,5,\ldots$$
  - Primary filling factor $\nu = \lim_{N \to \infty} N / m(N-1) = 1/m$
    - $m = 1$: Integer quantum Hall state at $\nu = 1$
    - $m = 3, 5, \ldots$: fractional quantum Hall states (Laughlin sequence)
Fractional Charge and Statistics

- Quasihole
  
  \[ \Psi_{\xi}^{qh}(z_j) = \prod_j (z_j - \xi) \prod_{i < j} (z_i - z_j)^3 \prod_i e^{-|z_i|^2/4} \]

  Hole of charge +1 = 3 x quasihole of charge 1/3

- Fractional statistics
  
  \[ \Psi_{\xi}^{2qh}(z_j) = \prod_j (z_j - \xi_a)(z_j - \xi_b) \prod_{i < j} (z_i - z_j)^3 \prod_i e^{-|z_i|^2/4} \]


Path equiv. in 3D; NOT equiv. in 2D
Why QHEs?

QHEs = Incompressibility + Disorder

- **Incompressibility**: \( \frac{d\mu}{dn} \rightarrow \infty \)
  - \( \Leftrightarrow \) energy gap for charged excitation (\( \Delta \mu \neq 0 \))

**Origin of incompressibility**
- IQHE: Single-body effect
  - Landau level quantization
- FQHE: Many-body effect
  - Coulomb interaction

- **Disorder**
  - Finite plateau widths
Edge States - Halperin (1982)

Chiral, gapless edge excitation

No backscattering.

Chiral Fermi liquid.

\[ B \pi r_m^2 = m\Phi_0 = \frac{mhc}{e} \]
Chiral Fermi Liquid

- Generated by $c_{k>0}^+$ and $c_{k\leq0}^-$, or equivalently

$$\rho_q = \sum_k c_{k+q}^+ c_k$$  (Haldane, 1981)

$$\left[\rho_q, \rho_{q'}\right] = \frac{q}{2\pi} \delta_{q+q'} $$

$$H = 2\pi V_F \sum_{q>0} \rho_q \rho_{-q} = qV_F \sum_{q>0} a_q^+ a_q$$

- Introduce $\phi(x)$:

$$\rho(x) = \frac{1}{2\pi} \partial_x \phi(x)$$

Electron operator: $\Psi(x) \sim e^{i\phi(x)}$  propagator: $G(x,t) \sim \frac{1}{x-V_F t}$
Chiral Luttinger Liquid

- For FQH states \( \nu = 1/m \), low-energy excited states are no longer generated by \( c_k \)'s.
- Nevertheless, \( \rho_q \)'s still generate low-energy excited states

\[
\left[ \rho_q, \rho_{q'} \right] = \frac{q}{2\pi m} \delta_{q+q'}
\]

\[
H = 2\pi m V \sum_{q>0} \rho_q \rho_{-q} = q V \sum_{q>0} a_q^+ a_q
\]

- Electron operator: \( \Psi(x) \sim e^{im\phi(x)} \)
- Electron propagator: \( G(x,t) \sim \frac{1}{(x-Vt)^m} \)

Luttinger liquid behavior

\[
\Psi_{gs} \sim \prod_{i<j} (z_i - z_j)^m e^{-\sum |z_i|^2/4l^2}
\]
Chern-Simons Topological Field Theory

• The system is (2+1)-dimensional system of electrons.
• The electromagnetic current is conserved:

$$\partial^\mu J^\text{em}_\mu = 0 \quad \Rightarrow \quad J^\text{em}_\mu = \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda$$

• Parity and time-reversal symmetry are broken by the magnetic field.
• We want an effective field theory for the long-distance, low-frequency behavior of the system:

$$\mathcal{L}_{\text{CS}} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda$$

K matrix, several gauge fields

Topological!
TFT and Conformal Field Theory

- Gauge transformation: 
  \[ a^\mu \mapsto a^\mu + \partial^\mu \Lambda \]
  action invariant up to a surface term

  \[ \delta \mathcal{L} = \varepsilon_{\mu \nu \lambda} \partial^\mu (a^\nu \partial^\lambda \Lambda) \]

- On a system with edge, cancelled by the gauge transformation of edge current

  \[ J_{\mu}^{\text{edge}} = \varepsilon_{\mu \nu} \partial^\nu \varphi \]

- The edge dynamics is described by a chiral Luttinger liquid

  \[ \partial_t \varphi = v \partial_u \varphi \]

  \[ \mathcal{L}_{\chi ll} = \frac{1}{4\pi} \int dt du ( (\partial_t \varphi)^2 - v^2 (\partial_u \varphi)^2 ) \]
TFT and CFT (continued)

• Using canonical quantization, the Fourier modes of the edge currents satisfy
  \[ [\hat{j}_m, \hat{j}_n] = \delta_{m+n,0} \sigma_H \]
  U(1) Kac-Moody algebra.

• Therefore, there is a connection between the topological field theory in 2+1 dimensions and the conformal field theory (a chiral Luttinger liquid) on the 1+1 dimensional edge of a quantum Hall sample.

Microscopic Theory of Edge Excitations

- Two-dimensional electron gas in the LLL
- Choose
  \[ H_{\text{Hardcore}} = \sum_{i<j} \partial^2 \delta(z_i - z_j) \]
- Laughlin state is the exact (zero-energy) ground state with the smallest angular momentum
  \[ \Psi_{\text{Laughlin}} = \sum_{i<j} (z_i - z_j)^3 e^{-\sum_i |z_i|^2/4}, \quad M = \frac{3N(N-1)}{2} \]
- Zero-energy states with larger angular momentum are edge states, which can be generated by multiplying symmetric polynomials to the Laughlin wave function.
Edge Excitations (Coulomb Interaction)

- Edge excitations generated by K-M algebra or symmetric polynomial
  \[ \Psi(\{z_i\}) = P(\{z_i\}) \prod_{i<j}(z_i - z_j)^m \prod_k e^{-\frac{1}{4}|k_i|^2} \]
  \[ \Psi(\{z_i\}) = \prod_i(z_i - \xi) \prod_{i<j}(z_i - z_j)^m \prod_k e^{-\frac{1}{4}|k_i|^2} \]
- Symmetric polynomial space spanned by
  \[ s_n = \sum_i z_i^n \]
  \[ \Delta M = 0 \quad s_0 \]
  \[ \Delta M = 1 \quad s_1 \]
  \[ \Delta M = 2 \quad s_2, s_1^2 \]
  \[ \ldots \quad \ldots \quad \ldots \]

Number of edge states: 1 1 2 3 5 7 11 …
Cleaved Edge Overgrowth

Pfeiffer, West, Stormer, Eisenstein, Baldwin, Gershoni, Spector (1990)
$m = 3$
$\alpha = 2.7$

$V_{\mu V}$

$10^{-2}$
$10^{0}$
$10^{2}$
$10^{4}$

$I$ (pA)

$\nu = 1/3$
$T = 25 mK$

$m = 1$
$\alpha = 1.2$

$V_{\mu V}$

$10^{-2}$
$10^{0}$
$10^{2}$
$10^{4}$

$I$ (pA)

$\nu = 1$
$T = 24 mK$
Grayson, Tsui, Pfeiffer, West, Chang (98)

να / 1≈55.0/ 2−≈να

Plateaulike?

Hilke, Tsui, Grayson, Pfeiffer, West (01)

α ≈ 2/ν − 0.55

Plateaulike?

No Universality!

Chang, Wu, Chi, Pfeiffer, West (01)
Reconstruction of IQH Edges

MacDonald, Yang, Johnson (93); Chamon, Wen (94)

• Strong confining potential/weak Coulomb interaction

\[
\langle n_k \rangle = \begin{cases} 
1 & k = k_F \\
0 & \text{otherwise}
\end{cases}
\]

\[
\rho(x) = \frac{\nu}{2d_B^2} \quad \sum_{k} K_{kn} K_{min} = K_{total}
\]

• Weak confining potential/strong Coulomb interaction

\[
\langle n_k \rangle = \begin{cases} 
1 & k = k_{s1}, k_{s2}, k_{s3} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\rho(x) = \frac{\nu}{2d_B^2} \quad \sum_{k} n_k k > K_{min} = K_{total}
\]

Edge reconstruction!
Chamon, Wen (94)

$K_{\text{tot}} = 45$

$K_{\text{tot}} = 50$

$K_{\text{tot}} = 45$

$K_{\text{tot}} = 55$
Model

- Disk geometry
- \( N = 4-12 \) electrons
- \( N_{\text{orb}} = M_{\text{max}} + 1 \) orbitals (sharp cleaved edge)
- Filling \( \nu = N / N_{\text{orb}} \)
- Uniform background charge on a disk, enclosing \( \Phi = N\Phi_0 / \nu \)
- Typically, \( d \sim 8-10 \) \( l_B \)
- Competition: U vs. V

\[
H = \frac{1}{2} \sum_{mnl} V_{mn} c_{m+l}^+ c_n^+ c_{n+l} c_m + \sum_m U_m c_m^+ c_m
\]
Reconstruction of FQH Edges

\[ d_c \approx 1.5 \pm 0.1 \]

\[ E_0 \text{ (in units of } e^2/4\pi) \]

\[ 2\pi |lB| \rho(r) \]

\[ M = 45, \text{ same as Laughlin state} \]

\[ d = 5.0 \]

\[ M = 65 \]

\[ d = 0.1 \]

\[ M = 45 \]
Before and After Reconstruction

N = 9  \quad d = 1.5  \quad m_{\text{max}} = 26

N = 9  \quad d = 1.7  \quad m_{\text{max}} = 26

E_0 \left( \frac{e^2}{\varepsilon B} \right)

2\pi\rho_0^2(r)

M = 108

M = 115
\[ \nu = 1/2 \]
Various Filling Factors

- \( M_{\text{max}} = 17 \) (sharp cleaved edge), \( d = 3.0 \, l_B, \nu = 1/3 \rightarrow 2/3 \)
- More electron density in edge piece (peak location roughly unchanged)
- Tendency of edge reconstruction \textit{generic} for properly confined electron at fractional fillings.

\begin{align*}
\text{Graph 1:} & \quad M = 69, \quad N = 7 \\
\text{Graph 2:} & \quad M = 94, \quad N = 9 \\
\text{Graph 3:} & \quad M = 114, \quad N = 11 \\
\text{Graph 4:} & \quad M = 84, \quad N = 8 \\
\text{Graph 5:} & \quad M = 104, \quad N = 10 \\
\text{Graph 6:} & \quad M = 123, \quad N = 12 
\end{align*}
Electrostatics for Edge Reconstruction

- Potential energy gain $\Delta E$ from moving an electron from $l_B$ inside the edge to the edge

$$\Delta E = \frac{2d}{l_B} \tan^{-1}\left(\frac{l_B}{d}\right) + \ln\left(1 + \frac{d^2}{l_B^2}\right)$$
Effective Theory for Reconstruction

• Chiral Luttinger theory

\[ H = 2\pi mV \sum_{q>0} \rho_q \rho_{-q} = \pi mV \int dx \rho^2(x) \]

• Take into account physics at shorter distance:

\[ H = \int dx \int dx' \rho(x)V(x-x') \rho(x') \]
\[ = \pi mV \int dx \left[ \rho^2(x) + a(\partial_x \rho(x))^2 + b(\partial_x^2 \rho(x))^2 + \cdots \right] \]

– Spectrum:

\[ E(k) = V\left(k + ak^3 + bk^5 + \cdots \right) \]

• If \( a < 0, b > 0: \)

\[ E(k) \approx \frac{(k-k_0)^2}{2m^*} - U \]
Energy Spectrum Evolution

$E (e^2/\theta_{ls})$

before Reconstruction after

d = 0.2

d = 1.4

$E (e^2/\theta_{ls})$

$E (e^2/\theta_{ls})$

$E (e^2/\theta_{ls})$
Single Electron Spectral Function

- CLL theory tested w/ Coulomb interaction (Palacios & MacDonald)
- With realistic edge confining potential
  \[ T(\{n_i\}) = \left| \left\langle \psi_{\{n_i\}}(N+1) c_{3N+\Delta M}^+ | \psi_0(N) \right\rangle \right|^2 \]
  - No reconstruction: Consistent with CLL theory
  - With reconstruction: One branch (total charge density mode) dominates, behaves like single branch before reconstruction

\[ \Rightarrow \text{Weak effects on structure of single electron spectral function} \]
Tunneling Spectral Weights \((N = 6)\)

\[
T(\{n_i\}) = \left| \langle \psi_{\{n_i\}}(N+1) | c_{3N+\Delta M}^\dagger | \psi_0(N) \rangle \right|^2
\]

<table>
<thead>
<tr>
<th>(\Delta M)</th>
<th>({n_i})</th>
<th>(d = 1.0) (no reconstruction)</th>
<th>(d = 1.6) (with reconstruction)</th>
<th>CLL theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0000}</td>
<td>0.0000</td>
<td>1.000</td>
<td>0.0000</td>
</tr>
<tr>
<td>1</td>
<td>{1000}</td>
<td>0.0317</td>
<td>2.791</td>
<td>0.0241</td>
</tr>
<tr>
<td>2</td>
<td>{2000}</td>
<td>0.0631</td>
<td>3.772</td>
<td>0.0477</td>
</tr>
<tr>
<td></td>
<td>{0100}</td>
<td>0.0434</td>
<td>1.383</td>
<td>0.0314</td>
</tr>
<tr>
<td>3</td>
<td>{3000}</td>
<td>0.0943</td>
<td>3.288</td>
<td>0.0714</td>
</tr>
<tr>
<td></td>
<td>{1100}</td>
<td>0.0738</td>
<td>3.863</td>
<td>0.0559</td>
</tr>
<tr>
<td></td>
<td>{0010}</td>
<td>0.0461</td>
<td>0.734</td>
<td>0.0306</td>
</tr>
<tr>
<td>4</td>
<td>{4000}</td>
<td>0.1252</td>
<td>2.083</td>
<td>0.0972</td>
</tr>
<tr>
<td></td>
<td>{2100}</td>
<td>0.1038</td>
<td>5.182</td>
<td>0.0797</td>
</tr>
<tr>
<td></td>
<td>{1010}</td>
<td>0.0756</td>
<td>2.529</td>
<td>0.0535</td>
</tr>
<tr>
<td></td>
<td>{0200}</td>
<td>0.0853</td>
<td>0.587</td>
<td>0.0621</td>
</tr>
<tr>
<td></td>
<td>{0001}</td>
<td>0.0465</td>
<td>0.402</td>
<td>0.0266</td>
</tr>
</tbody>
</table>

Effects of Layer Thickness

- Variational wave function (Fang & Howard)
  \[ \zeta_0(z) = 2(2b)^{-3/2}ze^{-z/2b} \]
  - Wave function peaked at 2b
  - Typically b = 30-50 Å
- Electron layer thickness not important; define an effective \( d \) between the layer and the background charged layer as
  \[ d^* = d + 2b \]
Landau Level Mixing

Hard Wall
\( (m_{\text{max}}) \)
\( \Phi = N\Phi_0 / \nu \)

Background charge (+Ne)

Electron layer (-Ne)

\( d = 0.5 \)

\( d = 1.5 \)
Another Application: 2DEG with Antidots

- Holes made by electron beam lithography and reactive-ion-beam etching.
- Depletion region diameter estimated to be 150-250 nm.

- “Geometric resonance”
- \[ \text{Re}(\sigma_{xx}) = \frac{W}{2Z_0d} \left| \ln\left(\frac{P}{P_0}\right) \right| \]

P: Transmitted power
Microwave Absorption in 2DEG with Antidots

- Increasing with $f$ while decreasing with $T$.
- Enhancement due to the additional low-energy edge modes.
- Characteristic $T \sim 0.05 \frac{e^2}{\varepsilon l_B}$, confirmed independently by finite-$T$ density profile and finite-$T$ Hartree-Fock calculation.

$N = 6, \text{Norb} = 18, d = 2.0$

$E_{\text{EMP}} > 100 \text{ GHz}$
Summary on Edge Reconstruction

• Edge reconstruction can and does occur in a fractional quantum Hall liquid, even for a sharp edge potential of a cleaved edge.
• Edge reconstruction leads to additional edge modes that are not maximally chiral, which can explain the non-universality of the tunneling exponent, and the enhanced microwave absorption at low temperatures in a sample with an antidot array.
• Edge reconstruction is suppressed above certain characteristic temperature, qualitatively consistent with observations in microwave absorption experiments.

References:
• XW, Yang & Rezayi, PRL 88, 056802 (2002)
• XW, Rezayi & Yang, PRB 68, 125307 (2003)
ν = 5/2: Experiments Then and Now

Willet et al., PRL (1987)

Pan et al., PRL (1999)
Pfaffian (Moore-Read) State

• \( \nu = 5/2 \): Quantum Hall state of \( p \)-wave paired fermions

\[
\Psi_{gs}(z_j) = \text{Pf} \left( \frac{1}{z_j - z_k} \right) \prod_{j<k} (z_j - z_k)^2 \prod_j e^{-|z_j|^2/4}
\]

\[
\text{Pf} \left( \frac{1}{z_j - z_k} \right) = \mathcal{A} \left( \prod_{i=\text{odd}} \frac{1}{z_i - z_{i+1}} \right) \quad \text{N even}
\]

• Exact ground state of a 3-body interaction, \( M_{\text{tot}} = N(2N-3)/2 \)

\[
H_{3\text{-body}} = V \sum_{i<j<k} \delta^2(z_i - z_j)\delta^2(z_i - z_k)
\]

• How good? Quasiparticles charge? Edge excitations?
Other Paired States

• Haldane-Rezayi state: singlet pairing state

\[
\Psi_{HR}\left(z_{1}^{\uparrow}, \ldots, z_{N/2}^{\uparrow}, z_{1}^{\downarrow}, \ldots, z_{N/2}^{\downarrow}\right) = \sum_{\sigma \in S_{N/2}} \text{sgn} \sigma \frac{1}{\left(z_{1}^{\uparrow} - z_{\sigma(1)}^{\downarrow}\right)^{2} \cdots \left(z_{N/2}^{\uparrow} - z_{\sigma(N/2)}^{\downarrow}\right)^{2}} \times \prod_{i<j} (z_{i} - z_{j})^{2} \exp\left[-\frac{1}{4} \sum_{i} |z_{i}|^{2}\right]
\]

• 331 state: two-component generalization of Laughlin state

\[
\Psi_{mm'n}\left(z_{1}^{\uparrow}, \ldots, z_{N/2}^{\uparrow}, z_{1}^{\downarrow}, \ldots, z_{N/2}^{\downarrow}\right) = \prod_{i<j} (z_{i}^{\uparrow} - z_{j}^{\uparrow})^{m} \prod_{k<l} (z_{k}^{\downarrow} - z_{l}^{\downarrow})^{m'} \prod_{r<s} (z_{r}^{\uparrow} - z_{s}^{\downarrow})^{n} \times \exp\left[-\frac{1}{4} \sum_{i} |z_{i}|^{2}\right]
\]

(Triplet pairing \(S_{z} = 0\))
Coulomb and 3-body Hamiltonian

\[ H_{\text{Coulomb}} = \frac{1}{2} \sum_{mn} V^l_{mn} c^+_m c^+_n c_n c_{n+l} + \sum_m U_m c^+_m c_m \]

Coulomb interaction

Confining potential

\[ H = (1 - \lambda) H_{\text{Coulomb}} + \lambda H_{3\text{-body}} \]

\[ H_{3\text{-body}} = V \sum_{i<j<k} \delta^2 (z_i - z_j) \delta^2 (z_i - z_k) \]
Overlaps with the Pfaffian State

N = 12 electrons, filling 1/2:

<table>
<thead>
<tr>
<th>Different systems</th>
<th>Overlap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without edge confining potential</td>
<td>0.455632432</td>
</tr>
<tr>
<td>With edge confining potential (d = 1.0)</td>
<td>0.601981011</td>
</tr>
<tr>
<td>$\delta V_1 = 0.02$, with edge confining potential (d = 1.0)</td>
<td>0.671972393</td>
</tr>
<tr>
<td>$\delta V_1 = 0.03$, with edge confining potential (d = 1.0)</td>
<td>0.700241671</td>
</tr>
<tr>
<td>$\delta V_1 = 0.04$, without edge confining potential</td>
<td>0.610470549</td>
</tr>
</tbody>
</table>

- For filling 1/3, overlaps of ground states with the Laughlin state are in general very close to 1.
- Note: overlap between many-body wave functions not very reliable.
Churn the FQH Liquid: Quasiholes

• Insert $\phi_0$ to create a charge $e/2$ quasihole

$$\prod_i(z_i - \xi) \mathcal{A}\left(\prod_{i=\text{odd}} \frac{1}{z_i - z_{i+1}}\right) \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i|^2 / 4}$$

$$\prod_i z_i \mathcal{A}\left(\prod_{i=\text{odd}} \frac{1}{z_i - z_{i+1}}\right) \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i|^2 / 4} \quad (\Delta M = N)$$

• Split the quasihole into two $e/4$ excitations

$$\mathcal{A}\left(\prod_{i=\text{odd}} \frac{(z_i - \xi_1)(z_{i+1} - \xi_2) + (\xi_1 \leftrightarrow \xi_2)}{z_i - z_{i+1}}\right) \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i|^2 / 4}$$

$$\mathcal{A}\left(\prod_{i=\text{odd}} \frac{z_i + z_{i+1}}{z_i - z_{i+1}}\right) \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i|^2 / 4} \quad (\Delta M = N / 2)$$
Observation of e/4 Quasihole (N = 10)
(F-fermions) Edge States

- \( F \) fermions created at the edge (\( \{n_1, \ldots, n_F\} \))

\[
\mathcal{A}\left( \prod_{k=1}^{F} z_{\sigma(k)}^{n_k} \left( z_{\sigma(F+1)} - z_{\sigma(F+2)} \right) \cdots \left( z_{\sigma(N-1)} - z_{\sigma(N)} \right) \right) \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i|^2/4}
\]

\[
M = \sum_{k=1}^{F} n_k + \frac{1}{2} [N(2N-3) + F]
\]

\[
\Delta M = \sum_{k=1}^{F} \left( n_k + \frac{1}{2} \right)
\]

Majorana-Weyl fermions on a circle with antiperiodic boundary conditions
Edge States Counting

• Edge states:
  – Chiral bosons (symmetrical polynomials) + Majorana fermions
    • $\Delta M$: 0 1 2 3 4 5 6 7 8
    • Bosonic: 1 1 2 3 5 7 11 15 22 ...
    • Fermionic: 0 0 1 1 2 2 3 3 5 ...
    • Total: 1 1 3 5 10 16 28 43 70 ...

Edge States: $\lambda = 0.5 \ (N = 12)$
Resolving Bosonic & Fermionic Modes

M

\[ dE(M) \]

N = 12

\[ \begin{array}{cccccc}
M & 0 & 1 & 2 & 3 & 4 & 5 \\
2 & \checkmark & \checkmark & & & & \\
3 & \checkmark & \checkmark & & & & \\
4 & \checkmark & \checkmark & \checkmark & & & \\
4 & & \checkmark & \checkmark & \checkmark & & \\
5 & \checkmark & \checkmark & \checkmark & \checkmark & & \\
5 & & \checkmark & \checkmark & \checkmark & \checkmark & \\
6 & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \\
6 & & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
6 & & & \checkmark & \checkmark & \checkmark & \checkmark \\
\end{array} \]
Bosonic and Fermionic Branches

Energy decreases with increasing $N_{\text{orb}}$

<table>
<thead>
<tr>
<th>$l$</th>
<th>bosonic $E_b(l)$</th>
<th>fermionic $E_f(l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000000</td>
<td>0.000324</td>
</tr>
<tr>
<td>1</td>
<td>0.022659</td>
<td>0.001676</td>
</tr>
<tr>
<td>2</td>
<td>0.030057</td>
<td>0.004117</td>
</tr>
<tr>
<td>3</td>
<td>0.029908</td>
<td>0.006011</td>
</tr>
<tr>
<td>4</td>
<td>0.024668</td>
<td></td>
</tr>
</tbody>
</table>
Edge States with 1QH at the Center

- \( F \) fermions created at the edge (\( \{n_1, \ldots, n_F\} \))

\[
\mathcal{A} \left( \prod_{k=1}^{F} Z_{\sigma(k)}^{n_k} \prod_{l=1}^{(N-F)/2} \left( Z_{\sigma(F+2l-1)} + Z_{\sigma(F+2l)} \right) \right) \prod_{i<j} \left( z_i - z_j \right)^2 e^{-\sum_i |z_i|^2 / 4}
\]

\[ M = \sum_{k=1}^{F} n_k + N(N-1) \]

\[ \Delta M = \sum_{k=1}^{F} n_k \]

Majorana-Weyl fermions on a circle with periodic boundary conditions
Numerical Data (N = 10)

$V_0 = 0.1, M_{\text{tot}} = 90$
$V_0 = 0.0, M_{\text{tot}} = 85$
Accu. diff.

$10, d = 0.5, 50\%$ 3-body

-7.07
-7.08
More Coulomb Interaction: $\lambda = 0.1$
Looking for Edge States (N = 12)

- We know the edge states for $\lambda = 0.5$
- Look for edge states for $\lambda = 0.1$
- Brute-force overlap calculations

<table>
<thead>
<tr>
<th></th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.017</td>
<td>0.827</td>
<td>0.034</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>#2</td>
<td>0.594</td>
<td>0.048</td>
<td>0.266</td>
<td>0.000</td>
<td>0.004</td>
</tr>
<tr>
<td>#3</td>
<td>0.230</td>
<td>0.003</td>
<td>0.470</td>
<td>0.112</td>
<td>0.068</td>
</tr>
<tr>
<td>#4</td>
<td>0.004</td>
<td>0.000</td>
<td>0.033</td>
<td>0.361</td>
<td>0.278</td>
</tr>
<tr>
<td>#5</td>
<td>0.000</td>
<td>0.056</td>
<td>0.002</td>
<td>0.003</td>
<td>0.026</td>
</tr>
</tbody>
</table>
Pure Coulomb Interaction: $\lambda = 0$
Summary and Outlook

• Studied the **stability** of the Pfaffian state as ground state in a model with Coulomb interaction and confining potential

• Observed the correct **edge excitations** of the Pfaffian state, developed a systematic method to identify edge modes

• Pointed out the significantly **different velocities of bosonic and fermionic modes**, raised possibility of various instabilities

• Observed **e/4 quasihole**, and the corresponding edge excitations in the presence of the quasihole at the center

• Quasiparticle?
• Edge reconstruction?
• Full Coulomb interaction?
• Roles of filled lowest Landau level?
Collaborators

On $\nu = 5/2$:

- Ed Rezayi, California State University
- Kun Yang, NHMFL, Florida State University

On projects related to topological orders and topological quantum numbers:

- Ravin Bhatt, Princeton University
- Qinghong Cui, Florida State University
- Ferdinand Evers, Forschungszentrum Karlsruhe
- Duncan Haldane, Princeton University
- Peter Schmitteckert, Universitaet Karlsruhe
- Donna Sheng, California State University

Acknowledgments:

- Dan Tsui, Matt Grayson, Wei Pan, Peide Ye, Mike Hilke
- Xiao-Gang Wen, Bert Halperin