SUPERPOSITION OF THE $\lambda$-PARAMETERIZED SQUEEZED STATES

XIAO-GUANG WANG
CCAST (World Lab.), P. O. Box 8730, Beijing 100080
and Laboratory of Optical Physics, Institute of Physics,
Chinese Academy of Sciences, Beijing 100080, P.R. China

HONGCHEN FU*
Quantum Processes Group, The Open University,
Milton Keynes, MK7 6AA, UK

Received 4 April 2000

The superposition states of the $\lambda$-parameterized squeezed states are introduced and investigated. These states are intermediate states interpolating between the number and Schrödinger cat states and admit algebraic characterization in terms of su(1, 1) algebra. It is shown that these states exhibit remarkable nonclassical properties.

1. Introduction

Various types of quantum states of the radiation field have been investigated in the literature. Among these states, the number states, coherent and squeezed states, as well as the Schrödinger cat states play important role in quantum optics and other fields such as gravitation wave detection. Recently, the so-called intermediate states, which interpolate between two fundamental states and degenerate to them in two different limits, attract much attention. One of the authors of this paper proposed the new type intermediate number-coherent states satisfying

$$\left(\sqrt{\eta}N + \sqrt{1-\eta}a\right)|\eta, \beta\rangle = \beta|\eta, \beta\rangle,$$

where $0 < \eta < 1$ is real and $a$ and $N$ are annihilation and number operators of the radiation field, respectively. When the eigenvalue $\beta = \sqrt{\eta}M$ ($M$ is a non-negative integer), we obtain the following solution

$$|\lambda, M\rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}}(a^\dagger + \lambda)^M|0\rangle,$$

*Corresponding author. On leave of absence from Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, P.R. China. E-mail: h.fu@open.ac.uk
where $\lambda \equiv \sqrt{(1 - \eta)/\eta}$ and $L_M(x)$ is the Laguerre polynomial,

$$L_M(x) = \sum_{n=0}^{M} \frac{1}{n!} \left( \frac{M}{M-n} \right)^n (-1)^n x^n.$$  \hspace{1cm} (3)

As an intermediate state, the state (2) tends to the number state $|M\rangle$ in the limit $\lambda \to 0$ and to the coherent states in a different limit $\lambda \to \infty$ and $M \to \infty$ keeping $\lambda^{-1}M = \alpha$. The state (2) can be experimentally generated and exhibits remarkable nonclassical properties.\hspace{1cm} For convenience, the state is referred to as the $\lambda$-parameterized squeezed state.

Equation (1) was also studied independently in a different context.\hspace{1cm} Beckers et al. proposed a new family of squeezed states by considering eigenstates of the following non-Hermitian Hamiltonian

$$H_\lambda = \frac{1}{2} \{a, a_\lambda^\dagger\} - \frac{1}{2} = N + \lambda a,$$  \hspace{1cm} (4)

where $a_\lambda^\dagger \equiv a^\dagger + \lambda$ is called the $\lambda$-parameterized bosonic creation operator. Eigenstates of this non-Hermitian Hamiltonian were obtained for any complex number $\beta$ by solving a differential equation.\hspace{1cm} A two-parameter generalization was given in Ref. 8.

In this paper, we will investigate the superposition of the $\lambda$-parameterized squeezed states and its various properties. The superposition state is proved to be an intermediate state interpolating between the number state and the Schrödinger cat state, and allows an algebraic characterization in terms of $su(1, 1)$ Lie algebra. The sub-Poissonian statistics can be enhanced by superposition and the squeezing effect exists for a large range of parameters involved.

2. Superposition State and Its Algebraic Characterization

Let us first introduce the superposition state of $\lambda$-parameterized squeezed states (2)

$$|\lambda, M, \phi\rangle = \mathcal{N}_{\lambda,M,\phi} (|\lambda, M\rangle + e^{i\phi}|-\lambda, M\rangle),$$  \hspace{1cm} (5)

where $0 < \phi < 2\pi$ and $\mathcal{N}_{\lambda,M,\phi}$ is the normalization constant. From the expansion of the state (2) in the number states\hspace{1cm}

$$|\lambda, M\rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} \sum_{n=0}^{M} \binom{M}{n} \lambda^{M-n} \sqrt{n!} |n\rangle,$$  \hspace{1cm} (6)

we can easily find the inner product

$$\langle -\lambda, M|\lambda, M\rangle = \langle \lambda, M|-\lambda, M\rangle = \frac{L_M(\lambda^2)}{L_M(-\lambda^2)}. $$  \hspace{1cm} (7)

Using Eq. (7), the normalization constant is obtained as

$$\mathcal{N}_{\lambda,M,\phi} = \left( \frac{L_M(-\lambda^2)}{2L_M(-\lambda^2) + \cos \phi L_M(\lambda^2)} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (8)
Inserting Eqs. (6) and (8) into Eq. (5), we obtain the expansion of the superposition state (5) in the number states

$$|\lambda, M, \phi\rangle = \frac{1}{\sqrt{2M!L_M(-\lambda^2) + \cos \phi L_M(\lambda^2)}} \times \sum_{n=0}^{M} \binom{M}{n} \sqrt{n!} \lambda^{M-n} \left[ 1 + e^{i\phi}(-1)^{M-n} \right] |n\rangle .$$

Note that the above state is a superposition of the number states in the \((M + 1)\)-dimensional subspace of Fock space. The photon number distribution of the superposition state is obtained as

$$P(n, \lambda, M, \phi) = \binom{M}{n}^2 \frac{n!\lambda^{2(M-n)} \left[ 1 + \cos \phi(-1)^{M-n} \right]}{M! \left[ L_M(-\lambda^2) + \cos \phi L_M(\lambda^2) \right]} ,$$

In a special case with \(\phi = \pi/2\), the photon number distribution (10) reduces to

$$P(n, \lambda, M, \pi/2) = \binom{M}{n}^2 \frac{n!\lambda^{2(M-n)}}{M!L_M(-\lambda^2)} ,$$

which is identical to that of the state \(|\lambda, M\rangle\).

When \(\phi = 0\) or \(\pi\), the superposition state is either the superposition of even number states (we refer it to as the \(\lambda\)-even squeezed state) or superposition of odd number states (we name it \(\lambda\)-odd squeezed state). It is easy to see that

\(\lambda\)-even squeezed state: \(|\lambda, 2K, 0\rangle\) or \(|\lambda, 2K + 1, \pi\rangle\);

\(\lambda\)-odd squeezed state: \(|\lambda, 2K, \pi\rangle\) or \(|\lambda, 2K + 1, 0\rangle\).

So the \(\lambda\)-even or \(\lambda\)-odd squeezed states depend not only on \(\phi\) but also on the parity of \(M\).

The superposition state (5) also admits an algebraic characterization. From Eq. (1), with \(\beta = \sqrt{\eta} M\), we can easily find that

$$a(|\lambda, M\rangle \pm e^{i\phi} |\lambda - M\rangle) = \lambda^{-1}(M - N)(|\lambda, M\rangle \mp e^{i\phi} |\lambda, M\rangle) .$$

Then, we obtain the eigenvalue equation satisfied by the superposition state (5)

$$[4\eta MK^0 - 4\eta (K^0)^2 + 2(1 - \eta)K^-] |\lambda, M, \phi\rangle = \eta \left( M^2 - \frac{1}{4} \right) |\lambda, M, \phi\rangle ,$$

in which

$$K^0 = \frac{1}{2} \left( N + \frac{1}{2} \right), \quad K^- = \frac{1}{2} a^2, \quad K^+ = \frac{1}{2} (a^\dagger)^2$$

generate the \(su(1,1)\) Lie algebra. The operator on the left-hand side of the eigenvalue equation (13) is an element in the universal enveloping algebra of \(su(1,1)\) algebra.
3. Limits to Number and Schrödinger Cat States

In this section, we prove that the superposition state (2) degenerates to the number and Schrödinger cat states in two different states. We first consider the limit $\eta \to 1$ (or $\lambda \to 0$). In this limit, the expansion coefficient in (6) degenerates to

$$\frac{1}{\sqrt{M!L_M(-\lambda^2)}}\left(\frac{M}{n}\right)^n \lambda^{-n} \sqrt{n!} \to \delta_{M,n},$$

which means $|\pm \lambda, M\rangle \to |M\rangle$. So, in the limit $\lambda \to 0$, the state $|\lambda, M, \phi\rangle$ degenerates to the number state $|M\rangle$ (up to a phase)

$$|\lambda, M, \phi\rangle \to \frac{1 + e^{i\phi}}{\sqrt{2(1 + \cos \phi)}} |M\rangle.$$

Then, let us consider a different limit $\eta \to 0$ ($\lambda \to \infty$), $M \to \infty$ keeping $\sqrt{\eta}M = \alpha$ a real constant. In this limit, we have

$$|\pm \lambda, M\rangle = \frac{(\pm \lambda)^M}{\sqrt{M!L_M(-\lambda^2)}} \left(1 \pm \frac{a^1}{\lambda}\right)^M |0\rangle \to |\pm \alpha\rangle = e^{-\alpha^2/2}e^{\pm \alpha^1} |0\rangle,$$

in which we have used the following limit formula

$$\lim_{\gamma \to \infty} \left(1 + \frac{A}{\gamma}\right)^\gamma = e^A.$$

Then, the state $|\lambda, M, \phi\rangle$ reduces to the Schrödinger cat states

$$\frac{1}{\sqrt{2(1 + \cos \phi e^{-2\alpha})}}(|\alpha + e^{i\phi}| - \alpha\rangle).$$

So, the superposition state of two $\lambda$-parameterized squeezed states interpolates between the number and Schrödinger cat states. In this sense, they can be regarded as the intermediate number–Schrödinger-cat states.

We note that in the limit $\lambda \to \infty$ with finite $M$, the state $|\lambda, M\rangle$ degenerates to the vacuum state $|0\rangle$ and so does the state $|\lambda, M, \phi\rangle$ (up to a constant).

In the Schrödinger cat state limit, Eq. (13) degenerates to the well-known eigenvalue equation satisfied by the Schrödinger cat state

$$a^2 |\infty, \infty, \phi\rangle = a^2 |\infty, \infty, \phi\rangle.$$

4. Nonclassical Properties

Now, we turn to the nonclassical properties of the superposition state (5). Let us begin with the photon statistics. From the following relation

$$a^k |\lambda, M\rangle = \sqrt{\frac{L_{M-k}(-\lambda^2)M!}{L_M(-\lambda)(M-k)!}} |\lambda, M-k\rangle,$$

it is easy to verify that

$$a^k |\lambda, M, \phi\rangle = \frac{\mathcal{N}_{\lambda,M,\phi}}{\mathcal{N}_{\lambda,M-k,\phi}} \sqrt{\frac{L_{M-k}(-\lambda^2)M!}{L_M(-\lambda)(M-k)!}} |\lambda, M-k, \phi\rangle.$$
Then, the expectation value of $a^k a^k$ is obtained as

$$
\langle a^k a^k \rangle_{\lambda, M, \phi} = \langle \lambda, M, \phi | a^k a^k | \lambda, M, \phi \rangle = \frac{M! [L_{M-k}(-\lambda^2) + \cos \phi L_{M-k}(\lambda^2)]}{(M-k)! [L_M(-\lambda^2) + \cos \phi L_M(\lambda^2)]}.
$$

(23)

The Mandel’s $Q$-parameter characterizing the photon statistics is obtained as

$$
Q = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} - 1 = \langle a^\dagger a \rangle_{\lambda, M-1, \phi} - \langle a^\dagger a \rangle_{\lambda, M, \phi}.
$$

(24)

When $Q < 0$, $Q = 0$ and $Q > 0$, the states are sub-Poissonian, Poissonian and super-Poissonian, respectively.

In Figs. 1 and 2, we give plots of the $Q$-parameter against $\lambda$ for different $M$ and $\phi$. When $\lambda = 0$, the superposition state is essentially the number state $|M\rangle$ (the phase does not change the statistical properties) and its $Q$-parameter is $-1$, as shown in Figs. 1 and 2. When $\phi = \pi/2$, the $Q$-parameter is the same as that of the state $|\lambda, M\rangle$ (Ref. 6) and the state is always sub-Poissonian. When $\lambda$ is small, the $Q$-parameter increases with $\lambda$ but does not depend on $\phi$ sensitively. However, when $\lambda$ increases to a large enough point, the $Q$-parameter of $\lambda$-even and $\lambda$-odd squeezed states have completely different behaviors: for $\lambda$-even states, the $Q$-parameter is an increasing function of $\lambda$ and there is a critical point $\lambda_0$ (at which $Q = 0$ and the state is Poissonian) such that the $\lambda$-even squeezed state is sub-Poissonian for $\lambda < \lambda_0$ and super-Poissonian for $\lambda > \lambda_0$; while the $Q$ parameter for $\lambda$-odd squeezed state becomes a decreasing function of $\lambda$ and the state is always sub-Poissonian and the sub-Poissonian statistics is enhanced by the superposition.

In Fig. 1, we give plots of $Q$-parameter for different $M$ ($\phi = 0$). We observe that, for $\lambda$-even squeezed state, the larger the $M$, the wider is the range of the sub-Poissonian.
Then, let us discuss the squeezing effect of the superposition state (5). To this end, we calculate the expectation value of $a^k$

$$
\langle \lambda, M, \phi | a^k | \lambda, M, \phi \rangle = \frac{N_{\lambda, M, \phi}}{N_{\lambda, M-k, \phi}} \sqrt{\frac{L_{M-k}(-\lambda^2)M!}{L_{M}(-\lambda^2)(M-k)!}} \langle \lambda, M, \phi | \lambda, M - k, \phi \rangle ,
$$

in which the inner product $\langle \lambda, M, \phi | \lambda, M - k, \phi \rangle$ is given by

$$
\langle \lambda, M, \phi | \lambda, M - k, \phi \rangle = \frac{N_{\lambda, M, \phi} N_{\lambda, M-k, \phi} \lambda^k \sqrt{(M-k)!}}{\sqrt{M!L_{M}(-\lambda^2)L_{M-k}(-\lambda^2)}}
\times \left( [1 + (-1)^k] L_{M-k}^{(k)}(-\lambda^2) + [e^{i\phi} + (-1)^k e^{-i\phi}] L_{M-k}^{(k)}(\lambda^2) \right) ,
$$

where $L_m^{(k)}(x)$ is the associated Laguerre polynomial

$$
L_m^{(k)}(x) = \sum_{n=0}^{m} \frac{(m+k)!}{(m-n)!n!(n+k)!}(-x)^n, \quad (k > -1).
$$

For $k = 1, 2$, we have

$$
\langle a \rangle = 2i \lambda \sin \phi N_{\lambda, M, \phi}^2 \frac{L_{M-1}^{(1)}(\lambda^2)}{L_{M}(-\lambda^2)} ,
$$

$$
\langle a^2 \rangle = 2\lambda^2 N_{\lambda, M, \phi}^2 \frac{L_{M-2}^{(2)}(-\lambda^2) + \cos \phi L_{M-2}^{(2)}(\lambda^2)}{L_{M}(-\lambda^2)} .
$$

We define the quadrature operators $X = (a + a^\dagger)/2$ and $Y = (a - a^\dagger)/(2i)$. Then, their variances are

$$
(\Delta X)^2 = \frac{1}{4} + \frac{1}{2} \left[ \langle a^2 \rangle - \langle a \rangle \right] ,
$$

$$
(\Delta Y)^2 = \frac{1}{4} + \frac{1}{2} \left[ \langle a \rangle - \langle a^2 \rangle - 2(\text{Im}\langle a \rangle)^2 \right] ,
$$
where we have used the fact that $\langle n^2 \rangle$ is real and $\langle a \rangle$ is pure imaginary. The quadrature $X$ (or $Y$) is considered to be squeezed if the variance $(\Delta X)^2$ [or $(\Delta Y)^2$] is below the vacuum level, that is, $(\Delta X)^2 < 1/4$ [or $(\Delta Y)^2 < 1/4$].

![Graph showing variance as a function of $\phi$ and $\lambda$ for $M = 9$ and 10.](image)

**Fig. 3.** Variance $(\Delta Y)^2$ as a function of $\phi$ and $\lambda$ for $M = 9$ and 10.

Figure 3 shows the dependence of $(\Delta Y)^2$ on $\lambda$ and $\phi$. From Eq. (28), we see that $(\Delta Y)^2 = (\Delta Y)^2_{2\pi - \phi}$, so we choose $0 \leq \phi \leq \pi$ in the figures. It can be seen that there exist squeezing for large ranges of parameters $\lambda$ and $\phi$. However, the $\lambda$-even and $\lambda$-odd squeezed states have different squeezing properties: the $\lambda$-even squeezed state is squeezed (see $|\lambda, 9, \pi\rangle$ and $|\lambda, 10, 0\rangle$), while the $\lambda$-odd squeezed state is not (see $|\lambda, 9, 0\rangle$ and $|\lambda, 10, \pi\rangle$). Figure 4 is a plot of $(\Delta Y)^2$ against $\lambda$ for different $M$.

The squeezing occurs when $\lambda$ is larger than the critical value $\lambda_c$, and $\lambda_c$ increases as $M$ increases. We also observe that the squeezing degree becomes deeper when $M$ increases.

![Graph showing variance as a function of $\lambda$ for $\phi = 0$ and different $M$.](image)

**Fig. 4.** Variance $(\Delta Y)^2$ as a function of $\lambda$ for $\phi = 0$ and different $M$. 
5. Conclusion

In this paper, we have introduced the superposition state of two $\lambda$-dependent squeezed states and investigated their various properties. We conclude that (1) the superposition state is an intermediate state between the number state and the Schrödinger cat state; (2) it can be characterized algebraically by su(1, 1) algebra; and (3) it exhibits strong nonclassical properties. However, the $\lambda$-even and $\lambda$-odd squeezed states have very different properties and depend on the parity of parameter $M$. The sub-Poissonian statistics can be enhanced by superposition, and the squeezing of the superposition states occurs for a large range of parameters.

Acknowledgments

This work is supported in part by the National Natural Science Foundation of China through the Northeast Normal University (19875008) and the State Key Laboratory of Theoretical and Computational Chemistry, Jilin University.

References