Quantum computing with four-particle decoherence-free states in an ion trap

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(Received 2 October 2001; published 2 April 2002)

Quantum computing gates are proposed to apply on trapped ions in decoherence-free states. As the phase changes due to time evolution of components with different eigenenergies of quantum superposition are completely frozen, quantum computing based on this model would be perfect. Possible applications of our scheme in future ion-trap quantum computers is discussed.

DOI: 10.1103/PhysRevA.65.044304 PACS number(s): 03.67.Lx, 32.80.Lg

Much effort has been put into quantum computing over the past few years due to great advantages of quantum computing over the computation made in existing computers for solving classically intractable problems \cite{1,2} and finding tractable solutions more rapidly \cite{3}. Some systems \cite{4} such as nuclear magnetic resonance, trapped ions, cavity quantum electrodynamics, and optical photons have been proven to meet the requirement of quantum computing.

There are some special advantages for each of the systems referred to above, while we will focus in the present work on the discussion of quantum gates on trapped ions. Since the proposal of Cirac and Zoller \cite{5}, many elaborate ideas have been put forward on quantum computing with trapped ions \cite{6,7} and some experiments showing the possibility of quantum computing with a few trapped ions have been achieved \cite{8–10}. However, to our knowledge, no specific robust scheme of quantum computing with trapped ions has been proposed so far, which can keep decoherence away from qubits in quantum-information processing, although various proposals \cite{11–13} have been put forward to resist the attack of decoherence by using decoherence-free states (DFS). We noticed an experiment \cite{14} made recently with trapped ions that encodes qubits in DFS. Although what it demonstrated is the simplest case of DFS, i.e., two-qubit DFS resisting collective dephasing, the effectiveness of DFS in information storage was clearly shown. However, it is natural for us to ask how to use these DFS for quantum computing and if it is possible to build an actual ion-trap quantum computer based on DFS. These questions are the motivation of the present work. In a recent work \cite{7}, the effective quantum computing gates have been performed on two identical two-level ions both illuminated with two lasers of different frequencies $\omega_{1,2} = \omega_0 \pm \delta$, where $\omega_0$ is the resonant transition frequency of the ion and $\delta$ is the detuning, not far from the trap frequency $\nu$. With the choice of laser detunings, the only energy-conserving transition is from $|ggg,n\rangle$ to $|ee,n\rangle$ or from $|eg,n\rangle$ to $|eg,n\rangle$, where the first (second) letter denotes the internal state $e$ or $g$ of the $i$th ($j$th) ion and $n$ is the quantum number for the vibrational state of the ion. As we consider $\nu \gg \eta \Omega$ where $\eta$ is the Lamb-Dicke parameter and $\Omega$ is the Rabi frequency denoting the interaction of ions with lasers, there is only negligible population being transferred to the intermediate states with vibrational quantum number $n \pm 1$. Therefore this two-photon process has nothing to do with the vibrational state $|n\rangle$ and quantum computing with such a configuration is valid even in the case when ions are in thermal states. Based on that model, the entangled state of four ions with the form of $1/\sqrt{2}(|eeee\rangle + i|gggg\rangle)$ has been experimentally realized \cite{9}. In fact, by setting different initial conditions, we would obtain other entangled states with decoherence-free forms of $1/\sqrt{2}(|egeg\rangle + i|gege\rangle)$ and $1/\sqrt{2}(|egge\rangle + i|gege\rangle)$, and so on. In what follows, we will show how to achieve basic gates of quantum computing on those states and discuss how to use our scheme in actual ion-trap quantum computing.

The effective Rabi frequency for transitions between product states of four-ion strings can be calculated by means of the second-order perturbation formula

$$\Omega = 2 \frac{(egeg, n|H_{int}|gggg, n+2)(gggg, n+2|H_{int}|gege, n) - (egeg, n|H_{int}|eeee, n-2)(eeee, n-2|H_{int}|gege, n)}{(2 \nu - \delta)}$$

$$= (2n+1) \frac{(\Omega \, \eta)^2}{2 \nu - \delta},$$

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where

\[
H_{int} = \frac{\Omega}{2} \sum_{j,k} \sum_{m=1}^{4} \sum_{n=1}^{2} (\sigma_j \sigma_k \exp{-i[\eta(a^2 + a^4) - \omega_{mn}^{(m)}]}) + \text{H.c.}
\]

(2)

with \(\Omega\) being the usual Rabi frequency of laser-ion interaction, \(\eta\) is the Lamb-Dicke parameter, and \(\omega_{mn}^{(m)}\) is the frequency of the \(m\)th laser. For the transition between \(|egge\rangle\) and \(|geeg\rangle\), we can obtain the same value of \(\tilde{\Omega}\) as in Eq. (1). It is obvious from Eq. (1) that transitions between different four-particle states are dependent on the vibrational state of ions. So to some extent, it is not a true hot-ion quantum computing. However, as \(n\) stays unchanged at the beginning and at the end of the process, as long as \(\Omega \eta \leq (2 \nu - \delta)\), the hot-ion quantum computing can still be carried out in the case when \(n\) is not very large. If \(n = 0\), the effective Rabi frequency of the four-particle case is almost half that of the two-particle case. Moreover, straightforward deduction from Eq. (2) can yield the time evolution of the four-particle case, similar to those shown in Refs. [7,9],

\[
|egge\rangle \Rightarrow \cos(\frac{\tilde{\Omega}t}{2})|egge\rangle - i \sin(\frac{\tilde{\Omega}t}{2})|gege\rangle,
\]

\[
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\]

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\]

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|geeg\rangle \Rightarrow \cos(\frac{\tilde{\Omega}t}{2})|geeg\rangle - i \sin(\frac{\tilde{\Omega}t}{2})|egge\rangle.
\]

(3)

To show that our scheme is available to implement quantum computing, we have to demonstrate how to achieve a controlled-NOT gate and a Hadamard one [15]. As it is relatively easy to implement a Hadamard gate, the main work in this respect is how to achieve a controlled-NOT gate. Similar to Ref. [7], in which the controlled-NOT operation is performed on two identical trapped ions experiencing a single-mode laser beam, we can also carry out the controlled-NOT operation on the four identical ions based on Eq. (3). If the computational space is formed by four states \(|egge\rangle_{1234}, |gege\rangle_{1234}, |egge\rangle_{1234}, \text{ and } |geeg\rangle_{1234}\), where subscripts mean the labeling of ions, with following sequences of operation: \(H_{34}, P_{34}, R, P_{34}, H_{12}, H_{34}\), and \(P_{34}^{-1}\), we will obtain the controlled-NOT operation as follows:

\[
|eg\rangle_{12}|ge\rangle_{34} \rightarrow |eg\rangle_{12}|eg\rangle_{34},
\]

\[
|ge\rangle_{12}|ge\rangle_{34} \rightarrow |ge\rangle_{12}|ge\rangle_{34},
\]

\[
|eg\rangle_{12}|ge\rangle_{34} \rightarrow |ge\rangle_{12}|eg\rangle_{34},
\]

and

\[
|ge\rangle_{12}|eg\rangle_{34} \rightarrow |ge\rangle_{12}|eg\rangle_{34},
\]

where \(H_{ij}\) denotes the transformations \(|eg\rangle_{ij} \rightarrow (|eg\rangle_{ij} - i|ge\rangle_{ij})/\sqrt{2}\) and \(|ge\rangle_{ij} \rightarrow (|ge\rangle_{ij} - i|eg\rangle_{ij})/\sqrt{2}\) made in Ref. [9] by radiating the two ions \(i\) and \(j\) simultaneously. \(P_{ij}\) is a \(\pi/2\) phase change of \(|eg\rangle_{ij}\) and \(R\) is the evolution shown in Eq. (3) with \(\Omega t = 3\pi/4\).

The Hadamard gate on an ion pair can be easily realized by \(H_{ij}\) in addition to \(P_{ij}\), that is, \(|eg\rangle \rightarrow i(|eg\rangle - |ge\rangle)/\sqrt{2}\) and \(|ge\rangle \rightarrow (|ge\rangle + |eg\rangle)/\sqrt{2}\). Therefore any quantum computing operation can be constructed with our scheme. However, the main problem of our scheme is that the controlled-NOT gate is restricted to be performed on two neighboring ion pairs, which is also the common drawback of various hot-ion quantum-computing schemes. It is recalled that in the Cirac-Zoller scheme and other similar models, the center-of-mass state of trapped ions plays the role of a data bus in performing the controlled-NOT operation on two separate qu-bits. In contrast the vibrational states of ions are excluded from the computational space in our scheme. Although we can implement a Grover search by using such a controlled-NOT gate, like in Ref. [16], it is impossible for us to accomplish Shor’s algorithm because Shor’s algorithm includes the quantum Fourier transform, and to perform the quantum Fourier transform we have to carry out a series of Hadamard and controlled-NOT operation, where most controlled-NOT operations are not performed on neighboring qu-bits. To solve this problem, we can transfer the states of qu-bits by operations of swapping. But we can do that more directly, as explained later.

It is still a great challenge both theoretically and experimentally to scale up the technique suitable for a few ions to a large-scale quantum computer with thousands of ions [17]. The computing operation on ions would become more and more complicated with the increase of the number of ions [18,19]. Moreover, with the increase of the number of ions, the spatial separation of trapped ions decreases, which makes their spatial resolution more and more difficult [20]. So to solve the problem, we have to use multiple traps. One of the promising schemes in this respect was proposed by Kielpinski et al. [21], in which each ion-trap quantum computer consists of two regions. One of them is called the storage region, which stores thousands of ions and acts as a quantum register. The other is called the accumulator region, in which quantum logic operations take place. As we only work with a few qubits at a time in the accumulator region, we can avoid the slow gate speed and detection inefficiency. Suppose that we implement our ion-pair scheme in such a quantum computer. After an amount of time for quantum computing, we hope to transfer the message (i.e., the state representing a certain result of quantum computing that we have made) of an ion pair (for example, ions 1 and 2) in the accumulator region to another ion pair (ions 3 and 4) in the storage region, so that ions 1 and 2 can be reset to new initial states and used for the coming computation. We can use teleportation [22] for doing that job. To this end, we prepare many four-ion strings with entangled states \(1/\sqrt{2}(|egge\rangle - i|geeg\rangle)\) in the storage region. Suppose the state of ions 1
and 2 to be $\Psi = |eg\rangle + e^{i\theta}|ge\rangle$ [23] and the four ions 3, 4, 5, and 6 to be entangled as $1/\sqrt{2}(|egge\rangle - i|geeg\rangle)$ in the storage region. By setting $|I\rangle = |eg\rangle$, $|0\rangle = |ge\rangle$ and using Bell states of $|I\overline{I}\rangle \pm i|\overline{I}I\rangle$ and $|\overline{I}0\rangle \pm i|0\overline{I}\rangle$, moving adiabatically ions 5 and 6 to the accumulator region will transfer the message from ions 1 and 2 to ions 3 and 4 by means of the idea of teleportation. Of course with the transfer of states between different ion pairs, the effect of controlled-NOT gate can be on two separate ion pairs. As both quantum computing and teleportation can be made without any affect from dephasing, our scheme would be toward the building of an actual ion-trap quantum computer with more qubits.

Another problem is that the number of ions used for quantum computing in our scheme is twice that in former schemes, which is also a challenge to the present ion-trap technique unavailable to confine large numbers of ions. Besides, we also note that, although DFS keep decoherence away from our desired states theoretically, some unclear source of decoherence existing in the present experiments would affect our scheme. As referred to in Ref. [9], even if both the center-of-mass and stretch states are cooled to ground states, this kind of decoherence affects the four-ion experiment more strongly than the two-ion one. It is speculated that this kind of decoherence results mainly from laser fluctuations. But in the present scheme we speculate that the four-ion state transitions related to the vibrational state would be probably another source of decoherence. Although mathematically we can neglect the vibrational state as it remains unchanged at the beginning and at the end of our scheme, the influence from the vibrational state does exist due to the incomplete destructive interference between the paths of $\pm (2 \nu - \delta)$. Therefore, for experimentally achieving our scheme, a more advanced technique is highly in demand to remove the laser fluctuation and keep the vibrational state coherent for a longer time.

Nevertheless, compared with various former schemes, the distinct character of our scheme is that quantum computing would be made more perfectly, without any possible phase change and to the largest extent, excluding the detrimental effect of decoherence. More importantly, unlike both the incomplete measurement of Bell states due to the lack of effective photon-photon interaction [24] and the partial distinguishability of Bell states in a cavity-atom system [25], we can make a complete measurement of Bell states in teleportation with our scheme. From Eq. (3) we know that a laser pulse on ions 1, 2, 5, and 6 with the period of $t = \pi/2\Omega$ would yield $(|geeg\rangle + i|gege\rangle) \longrightarrow |geeg\rangle$, $(|geeg\rangle - i|gege\rangle) \longrightarrow -i|gege\rangle$, $(|geeg\rangle + i|gege\rangle) \longrightarrow |geeg\rangle$, and $(|geeg\rangle - i|gege\rangle) \longrightarrow -i|gege\rangle$. So by detecting internal levels of ions 1, 2, 5, and 6, respectively, we know that, if the result is $|geeg\rangle$, ions 3 and 4 have been in the desired state of $\Psi$. But if results are $|gege\rangle$, $|geeg\rangle$, or $|geeg\rangle$, we have to perform operations $\sigma^z_3, \sigma^z_4, \sigma^x_3$, and $\sigma^z_3 \sigma^z_4 \sigma^x_3$, respectively, on ions 3 and 4 to get the state of $\Psi$. As this teleportation is made with 100% success probability, the message can be transferred safely and completely from the accumulator region to the storage one.

Let us have a brief discussion for the possibility of experimental realization of our scheme. To achieve our scheme, laser manipulation of individual ions is necessary because we need to prepare initial states of each ion at the beginning of quantum computing and detect the final states of each ion at the end of teleportation. Fortunately, the separate addressing of individual ions is experimentally available for the case of the confinement of few ions [10]. But if we want to avoid the individual addressing of ion in the preparation of initial states of each ion, we can adopt the method used in Ref. [14], in which simultaneous manipulation of two ions can give the net effect of rotation of one of the ions. To perform a controlled-NOT operation, we need time at least

$$t = t_1 + t_2 = \frac{3\pi}{4(2n+1)} \frac{2\nu - \delta}{(\Omega \eta)^2} + \frac{3\pi \delta}{4(\Omega \eta)^2};$$

where $t_1$ is the time for performing $R$, $t_2$ is for performing $H_{ij}$ three times, and the time for undergoing $P_{34}$ and $P_{34}^{-1}$ is neglected. If we suppose $\nu \approx \delta$, we have

$$t = \frac{3\pi}{4} \frac{2n+2}{2n+1} \frac{\delta}{(\Omega \eta)^2}.$$  

From Refs. [8, 9], where $\eta = 0.23/\sqrt{N}$ ($N = 4$ in our scheme) and $\Omega = 2\pi \times 500$ kHz = 0.1 $\nu$, we obtain $t = 5.9 \times (2n+2)/(2n+1) \times 10^{-4}$ sec. So mathematically even if $n$ is extremely large, the time for performing a controlled-NOT operation in our scheme is still much shorter than the lifetime (1–10 sec) of the metastable level of a trapped ion, which acts as the excited state of a qubit. It means that our controlled-NOT gate will work very well. But if we want to transfer the message from the accumulator region to the storage region, we must take time to move ion pairs adiabatically from the storage region to the accumulator region and cool the moved ion pairs after they get to the accumulator region. This time should not be larger than 1–10 sec. However, how to effectively move trapped ions from one place to another place with minimum heating is still an open question.

In summary, a robust scheme for performing a perfect quantum computing has been proposed in a trapped-ion system. The entanglement of four trapped ions in our scheme is of a different form from that achieved by Kielpinski et al., but our entanglement is more important because it corresponds to DFS. Although transitions between different four-ion states are formally dependent on the vibrational state of ions, as the vibrational state is only virtually excited, it is not strictly required to cool the ions to vibrational ground states for achieving our scheme, whereas the quantum number of the vibrational state of ions cannot be too large. For avoiding any possible decoherence, it is better to cool the ions to the vibrational state close to ground state before implementing our scheme. Moreover, we benefit from DFS not only in the phase stability during the period of quantum computing and teleportation, but also in the safe storage of information.

The work is partly supported by the National Natural Science Foundation of China.
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