Quantum entanglement of unitary operators on bipartite systems

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We study the entanglement of unitary operators on $d_1 \times d_2$ quantum systems. This quantity is closely related to the entangling power of the associated quantum evolutions. The entanglement of a class of unitary operators is quantified by the concept of concurrence.

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\begin{center}
\textbf{I. INTRODUCTION}
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Quantum entanglement plays a key role in quantum information theory. In recent years, there has been a lot of effort to characterize the entanglement of quantum states both qualitatively and quantitatively. Entangled states can be generated from disentangled states by the action of nonlocal Hamiltonians. That means these Hamiltonians have the ability to entangle quantum states. It is therefore natural to investigate the entangling abilities of nonlocal Hamiltonians and the corresponding unitary evolution operators. The first steps along this direction have been initiated [1,2] recently.

In Ref. [1] the entangling capabilities of unitary operators on a $d_1 \times d_2$ system have been analyzed and an entangling power measure has been introduced, given by the mean linear entropy produced by applying with the unitary operator on a given distribution of product states. Dürr et al. [2] investigated the entangling capability of an arbitrary two-qubit nonlocal Hamiltonian and designed an optimal strategy for entanglement production. Cirac et al. [3] studied which physical operations acting on two spatially separated systems are capable of producing entanglement, and show how one can implement certain nonlocal operations if one shares a small amount of entanglement and is allowed to perform local operations and classical communications.

The notion of entanglement of quantum evolutions e.g., unitary operators, has been introduced and quantified by linear entropy in Ref. [4]. As discussed in that paper, this notion arises in a very natural way once one recalls that unitary operators of a multipartite system belong to a multipartite state space as well, the so-called Hilbert-Schmidt space. It follows that one can lift all the notions developed for entanglement of quantum states to that of quantum evolutions. In this report we shall make a further step by studying the entanglement of a class of useful unitary operators, e.g., quantum gates, on general, i.e., $d_1 \times d_2$ bipartite quantum systems.

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\textbf{II. OPERATOR ENTANGLEMENT}
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We shall denote the $d$-dimensional Hilbert state space by $\mathcal{H}_d$. The linear operators over $\mathcal{H}_d$ also form a $d^2$-dimensional Hilbert space and the corresponding scalar product between two operators is given by the Hilbert-Schmidt product $\langle A, B \rangle := \text{tr}(A^\dagger B)$, and $||A||_{HS} = \sqrt{\text{tr}(A^2)}$.

We denote this $d^2$-dimensional Hilbert space as $\mathcal{H}_d^{HS}$ and the bra-ket notations will be used for operators. Let $T_{ij}$ be the permutation (swap) operator between the Hilbert space $H_{d_i} \otimes H_{d_j}$ ($d_i = d_j$) and denote its adjoint action, by $\hat{T}_{ij}$ i.e., $\hat{T}_{ij}(X) = T_{ij} X T_{ij}$. We also define the projectors $P_{ij} = 2^{-1}(1 \pm T_{ij})$ over the totally symmetric (antisymmetric) subspaces of $H_{d_i} \otimes H_{d_j}$.

In this section, following Ref. [1], we shall adopt the linear entropy as an entanglement measure of (normalized) the unitary operator $|U\rangle \in \mathcal{H}_d^{HS} \otimes \mathcal{H}_d^{HS}$,

$$E(U) = 1 - \text{Tr}_1 [\rho_U^2], \rho_U = \text{Tr}_2 |U\rangle \langle U|.$$  \hfill (1)

Note that Tr and tr denote traces over $\mathcal{H}_d^{HS}$ and $\mathcal{H}_d$, respectively. Using the identity $\text{tr}_1(U \otimes B) T_{13} = \text{tr}_1(AB)$, one can easily obtain [4]

$$E(U) = 1 - \frac{1}{d_1^2 d_2^2} \text{Tr}_{13} [(\rho_U \otimes \rho_U) \hat{T}_{13}]$$

$$= 1 - \frac{1}{d_1^2 d_2^2} \langle U^\otimes 2, \hat{T}_{13}(U^\otimes 2) \rangle$$

$$= 1 - \frac{1}{d_1^2 d_2^2} \text{tr}(U^\otimes 2 T_{13} U^\otimes 2 T_{13}).$$  \hfill (2)

The term $1/(d_1^2 d_2^2)$ is nothing but the normalization factor of $U^\otimes 2$. Now we give a relation between the entanglement of unitaries and the entangling power introduced in Ref. [1] that extends the analogous one given in Ref. [4].

The entangling power $e_p(U)$ of a unitary $U$ is defined over $\mathcal{H}_d^{\otimes 2}$ as the average of the entanglement $E(U|\Psi\rangle)$, where the $|\Psi\rangle$’s are product states generated according to some given probability distribution $p$. By choosing for $E$ the linear entropy and using the uniform distribution $p_0$, one finds [1]

$$e_{p_0}(U) = 1 - \frac{4}{d_1(d_1+1)d_2(d_2+1)} \text{tr}(U^\otimes 2 P_{13} P_{24} U^\otimes 2 T_{13}).$$  \hfill (3)
By comparing Eqs. (3) and (2), after some straightforward algebra, we find
\[ e_{p_0}(U) = \frac{d_1 d_2}{(d_1 + 1)(d_2 + 1)} \left( E(U) + \bar{E}(U) + \frac{1}{d_1 d_2} - 1 \right), \] (4)
where
\[ \bar{E}(U) = 1 - \frac{1}{d_1^2 d_2^2} \text{tr}(U^{\otimes 2} T_{2A} U^{\dagger \otimes 2} T_{1B}). \]
Equation (4) shows that the entangling power of a unitary is directly related to its entanglement. When \( d_1 = d_2 \), Eq. (4) reduces to [1]
\[ e_{p_0}(U) = \frac{d^2}{(d + 1)^2} \left( E(U) + E(U_S) - E(S) \right), \] (5)
where \( S \) is the swap operator.

**III. CONCURRENCE FOR BIPARTITE OPERATORS**

From now on we will focus on the class of unitary operators given by
\[ U = \mu A_1 \otimes A_2 + \nu B_1 \otimes B_2, \] (6)
where \( A_1 \) and \( B_1 \) are operators on the \( d_1 \)-dimensional system and similarly \( A_2 \) and \( B_2 \) are operators on the \( d_2 \)-dimensional system with complex values \( \mu \) and \( \nu \). Several interesting unitary operators and quantum gates, as we will see later on, are of the form (6).

After normalization with respect to the Hilbert-Schmidt norm, the unitary operator \( U \) is given by
\[ |U\rangle = \tilde{\mu} |A_1\rangle \otimes |A_2\rangle + \tilde{\nu} |B_1\rangle \otimes |B_2\rangle, \] (7)
where
\[ \tilde{\mu} = \mu \prod_{k=1}^2 d_k^{-1/2} \|A_k\|_{HS}, \quad \tilde{\nu} = \nu \prod_{k=1}^2 d_k^{-1/2} \|B_k\|_{HS}. \] (8)
Here the normalized operator corresponding to the operator \( O \) is denoted by \( |O\rangle \), i.e.,
\[ |O\rangle = |O\rangle_{HS}, \quad \langle O|O\rangle = 1. \] (9)

The two operators \( |A_i\rangle \) and \( |B_i\rangle \) \((i = 1, 2)\) are assumed to be linearly independent and span a two-dimensional subspace in the Hilbert space \( \mathcal{H}^2_{d_i} \). We choose an orthogonal basis \( \{|0\rangle, |1\rangle\} \) as in Ref. [5],
\[ |0\rangle_1 = |A_1\rangle, \]
\[ |1\rangle_1 = \frac{1}{\mathcal{M}_1}(|B_1\rangle - \langle A_1|B_1\rangle |A_1\rangle), \] (10a)
\[ |0\rangle_2 = |B_2\rangle, \]
\[ |1\rangle_2 = \frac{1}{\mathcal{M}_2}(|A_2\rangle - \langle B_2|A_2\rangle |B_2\rangle), \] (10b)
where
\[ \mathcal{M}_k^2 = 1 - \frac{\langle A_k, B_k\rangle^2}{\|A_k\|_{HS}^2 \|B_k\|_{HS}^2}, \]
\[ k = 1, 2. \] (11)
Using this basis, the entangled state \( |U\rangle \) can be rewritten as
\[ |U\rangle = \tilde{\mu} |0\rangle_1 \otimes (|B_2\rangle|A_2\rangle |0\rangle_2 + \mathcal{M}_2 |1\rangle_2) \]
\[ + \tilde{\nu} (\langle A_1|B_1\rangle |0\rangle_1 + \mathcal{M}_1 |1\rangle_1) \otimes |0\rangle_2. \] (12)
After the “encoding,” the above “state” can be considered as a two-qubit state. Therefore it is very convenient to use one simple entanglement measure, the concurrence [6], to quantify the entanglement in state \( |U\rangle \). The concurrence for a pure state \( |\psi\rangle \) is defined as
\[ C = |\langle \psi|\sigma_y \otimes \sigma_y|\psi\rangle|. \] (13)
Here \( \sigma_y = -i |1\rangle \langle 0| - |0\rangle \langle 1| \). A direct calculation shows that the concurrence of the unitary operator \( |U\rangle \) is given by
\[ C(U) = 2 |\mathcal{M}_1 \mathcal{M}_2 \tilde{\mu} \tilde{\nu}| \]
\[ = 2 |\mu \nu| \prod_{k=1}^2 d_k^{-1/2} \|A_k\|_{HS} \|B_k\|_{HS} (|\langle A_k, B_k\rangle|)^2. \] (14)

In the derivation of the above equation we have used Eqs. (10a)–(13).

Equation (14) defines a (non-negative) real-valued functional over the operatorial family (6) The relevant properties of \( C \) are summarized in the following.

(a) The entanglement of the operator \( |U\rangle \) is independent on the phases of the complex values \( \mu \) and \( \nu \).

(b) From the Cauchy-Schwarz inequality, it follows that \( C = 0 \) iff either \( A_1 = \lambda_1 B_1 \) or \( A_2 = \lambda_2 B_2 (\lambda_k \in \mathbb{C}) \). This is just the separable case.

(c) For arbitrary unitary operator of the form \( U_1 \otimes U_2, \quad C(U_1 \otimes U_2) = C(U_1) C(U_2) \), which is nothing but the invariance of entanglement measure under the local unitary transformations.

(d) The Hermitian conjugation of the unitary operator \( U \) is given by \( U^\dagger = \mu^* A_1^\dagger \otimes A_2^\dagger + \nu^* B_1^\dagger \otimes B_2^\dagger \). From Eq. (14) it is straightforward to check that \( C(U^\dagger) = C(U) \).

This claim immediately follows from Eq. (14) and the properties of the Hilbert-Schmidt scalar product. Notice also that for the orthogonal case, i.e., \( \langle A_k, B_k\rangle = 0 \ (k = 1, 2) \), Eq. (14) reduces to
\[ C(U) = 2 |\mu \nu| \prod_{k=1}^2 d_k^{-1/2} \|A_k\|_{HS} \|B_k\|_{HS}. \] (15)
Furthermore, if the operators $A_i$ and $B_i$ are Hermitian and self-inverse, i.e., $A_i^2 = B_i^2 = 1$, we have $\|A_i\|_{HS} = \|B_i\|_{HS} = d_i^{1/2}$ and $\|A_2\|_{HS} = \|B_2\|_{HS} = d_2^{1/2}$. Therefore Eq. (15) reduces to

$$C(U) = 2|\mu||\nu|,$$

which will be used for later discussions.

IV. EXAMPLES

In the following we consider several explicit examples of the unitary operators $U$.

**Example 1.** We consider the unitary operator acting on $2 \times 2$ systems, which is defined as

$$U_{2 \times 2} = e^{-i2\theta \sigma_z} = I \otimes \cos(2J_z \theta) - i \sigma_z \otimes \sin(2J_z \theta),$$

where $d = 2j + 1$, $\sigma_z$ is the Pauli matrix, and $J_z$ is the $z$ component of the spin-$j$ angular-momentum operator $J$. This unitary operator describes the interaction between a spin $\frac{1}{2}$ and spin $j$. After normalization $U_{2 \times 2}$ is written as

$$|U_{2 \times 2} = \sqrt{\frac{c}{2j+1}} I \otimes |\cos(2J_z \theta)| - i \sqrt{\frac{s}{2j+1}} |\sigma_z| \otimes |\sin(2J_z \theta)|,$$

where

$$c = \text{Tr}(\cos^2(2J_z \theta)) = \frac{2j+1}{2} + \frac{x}{2},$$

$$s = \text{Tr}(\sin^2(2J_z \theta)) = \frac{2j+1}{2} - \frac{x}{2},$$

$$x = \sum_{k=0}^{j} \cos(4k \theta) = \frac{\sin[2(2j+1)\theta]}{\sin(2\theta)}.$$  

From Eq. (14), the concurrence is obtained as

$$C(U_{2 \times 2}) = \sqrt{1 - \frac{\sin^2[2(2j+1)\theta]}{(2j+1)^2 \sin^2(2\theta)}}.$$  

In Fig. 1 we give a plot of the concurrence against $\theta$. The period of the above function with respect to $\theta$ is $\pi/2$. So we just plot the figure and make discussions within one period. We see that there is $2j$ maximal points in one period at which the operator is maximally entangled.

**Example 2.** The $d_1 \times d_2$ unitary operator [7]

$$U_{d_1 \times d_2} = e^{-i \pi N_i \otimes N_2},$$

$$= \frac{1}{2}[1 + \Pi_1] \otimes I + \frac{1}{2}[1 - \Pi_1] \otimes \Pi_2,$$  

where $N_i = J_i + j_i$ is the number operator, $\Pi_i = (-1)^{N_i}$ ($i = 1, 2$) is the parity operator of system $i$, and $d_i = 2j_i + 1$.

FIG. 1. The concurrence against $\theta$ for different values of $j$: $j = 1/2$ (solid line), $j = 1$ (dashed line), and $j = 5/2$ (dotted line).

This operator describes the interaction between spin $j_1$ and spin $j_2$ and it can be used to generate entangled SU(2) coherent states [7].

By comparing Eqs. (6) and (21), we find

$$\mu = \nu = 1,$$

$$A_1 = \frac{1}{2}[1 + \Pi_1], \quad B_1 = \frac{1}{2}[1 - \Pi_1],$$

$$A_2 = I, B_2 = \Pi_2,$$  

which have the following properties:

$$A_1^2 = A_2, \quad B_1^2 = B_2, \quad A_1 B_1 = 0,$$

$$A_2^2 = B_2^2 = I, \quad A_2 B_2 = \Pi_2.$$  

From Eqs. (14) and (21)–(23), the concurrence is obtained as

$$C(U_{d_1 \times d_2}) = \frac{2}{d_1 d_2} \sqrt{\text{Tr}(A_1) \text{Tr}(B_1)} \sqrt{d_2 - |\text{Tr}(B_2)|^2}$$

$$= \sqrt{\left(1 - \frac{|\text{Tr}(\Pi_1)|^2}{d_1^2}\right) \left(1 - \frac{|\text{Tr}(\Pi_2)|^2}{d_2^2}\right)}$$

$$= \sqrt{\left(1 - \frac{1 - (-1)^{d_1}}{2d_1^2}\right) \left(1 - \frac{1 - (-1)^{d_2}}{2d_2^2}\right)},$$  

where we have used the identity $\text{Tr}(\Pi_i) = \frac{1}{2}[1 - (-1)^{d_i}]$. We see that $U_{d_1 \times d_2}$ is a maximally entangled operator for even $d_1$ and even $d_2$. The concurrence $C(U_{d_1 \times d_2})$ becomes $\sqrt{d_2 - 1/d_2}$ for even $d_1$ and odd $d_2$, $\sqrt{d_1 - 1/d_1}$ for odd $d_1$ and even $d_2$, and $\sqrt{(d_1^2 - 1)(d_2^2 - 1)/(d_1 d_2)}$ for odd $d_1$ and odd $d_2$. In the limit of $d_i \to \infty$, the operator $U_{d_1 \times d_2}$ is a maximally entangled operator.
Example 3. Now we consider the entanglement of quantum gates. Let us see the controlled\(^N\) NOT gate \([8]\) that includes the controlled-NOT gate \([9]\) \((N=1)\) as a special case. It is defined as
\[
C_{\text{NOT}}^{N} = e^{-i\pi 2^{N+1}} (1 - \sigma_z) \otimes \cdots \otimes (1 - \sigma_z) \\
= I \otimes \cdots \otimes I \otimes (1 - \sigma_z) \otimes \cdots \otimes (1 - \sigma_z) \\
= I \otimes \cdots \otimes I \otimes P_{z}^{N-k} \otimes P_{x},
\]
where \(P_{a} = (1 - \sigma_a)/2\) \((a=x,y,z)\) are the projectors satisfying \(P_a^2 = P_a\). The above equation implies that the \((N+1)\)-th qubit flips if and only if all the other \(N\) qubits are in the state \(|1\rangle^\otimes N\). We split the whole system into two subsystems, one \(2^k\)-dimensional subsystem and another \(2^{N+1-k}\)-dimensional subsystem. So, in fact, we are studying the entanglement of an operator on \(2^k \times 2^{N+1-k}\) systems. Using the identity \(\text{tr}(P_a) = 1\) and Eq. (14), the concurrence is obtained as
\[
C = 2^{1-N} \sqrt{ (2^k-1)(2^{N+1-k}-1) }.
\]
For \(N=1, k=1\), the concurrence \(C=1\). So the controlled-NOT gate is a maximally entangled operator. It is interesting to see that we can use this maximally entangled gate to generate a maximally entangled two-qubit state. As the local unitary operation does not change the amount of entanglement, we know that the concurrence for the controlled-NOT gate is the same as the unitary operator \(e^{-i(\pi/4)\sigma_z \otimes \sigma_z}\), whose concurrence is also 1.

Example 4. As a final example we investigate the following operator that can be generated by the many-body Hamiltonian \(\sigma_z \otimes \cdots \otimes \sigma_z\):
\[
V(\theta) = e^{-i \theta \sigma_z} \otimes \cdots \otimes \cos \theta I \otimes \cdots \otimes \sin \theta \sigma_z.
\]
The operator \(V(\pi/4)\) can be used to generate Greenberger-Horne-Zeilinger state. Like the above discussions, we split the whole system into two subsystems, one \(2^k\)-dimensional subsystem contains \(k(1 < k < N)\) parties and another \(2^{N-k}\)-dimensional subsystem contains \(N-k\) parties. We have
\[
\mu = \cos \theta, \quad \nu = -i \sin \theta,
\]
\[
A_1 = I \otimes \cdots \otimes I \otimes P_{z}^{N-k}, \quad B_1 = \sigma_z \otimes \cdots \otimes \sigma_z.
\]
This unitary operator belongs to the orthogonal case and the operators \(A_1\) and \(B_1\) are obviously self-inverse, so the concurrence is simply given by Eq. (16), \(C = 2|\mu \nu| = |\sin(2\theta)|\), which clearly displays the disentangled (maximally entangled) character of the unitary operator for \(\theta = 0, \pi/2\) \((\theta = \pi/4)\).

V. CONCLUSIONS

We have studied the entanglement of unitary operators acting on the state space of \(d_1 \times d_2\) quantum systems. We have used as entanglement measures, linear entropy and concurrence. The former measure allows us to make an explicit connection [Eq. (4)] between the entanglement of an operator and its entangling power. Whereas the latter measure has been exploited in order to study the bipartite entanglement of a class of interesting unitary operators. The existence of some, more or less a direct, relation between operator entanglement quantified by concurrence and entangling capabilities is an open issue.

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