Spin squeezing in the Ising model

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We analyze the collective spin noise in interacting spin systems. General expressions are derived for the short-time behavior of spin systems with general spin-spin interactions, and we suggest optimum experimental conditions for the detection of spin squeezing. For Ising models with site-dependent nearest-neighbor interactions, general expressions are presented for the spin squeezing parameter for all times. The reduction of collective spin noise can be used to verify the entangling powers of quantum computer architectures based on interacting spins.

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1. INTRODUCTION

Parallel to the study of nonclassical squeezed states of electromagnetic radiation [1], increasing attention has been devoted to the study of atomic spin squeezed states [2–14]. In Ramsey spectroscopy, a sample of \( N \) two-level atoms is represented by the collective operators \( J_\alpha = \sum_{i=1}^{N} j_{\alpha,i} \) (\( \alpha = x, y, \) or \( z \)), where \( j_{\alpha,i} = \sigma_{\alpha,i}/2 \) and \( \sigma_{\alpha,i} \) are the Pauli operators for the \( i \)th atom. Atomic spin squeezed states are quantum correlated states with reduced fluctuations in one of the collective spin components, and they have possible applications in atomic interferometers and high-precision atomic clocks. Wineland et al. [3] have shown that the frequency resolution in spectroscopy depends on the spin squeezing parameter

\[
\xi^2 = \frac{N(\Delta J_x)^2}{\langle J_x \rangle^2},
\]

where \( \perp \) denotes a direction perpendicular to the mean spin. The inequality \( \xi^2 \leq 1 \) indicates that the system is spin squeezed, and it has been shown that any state with \( \xi^2 \leq 1 \) is an entangled state [10,11].

In 1993, Kitagawa and Ueda showed that spin squeezing is produced by simple nonlinear spin Hamiltonians in analogy with the nonlinear Hamiltonians leading to squeezed light [2]. A number of experimental proposals for atomic spin squeezing have appeared involving interaction of atoms with squeezed light [3–7], quantum nondemolition measurement of atomic spin states [8], and atomic collisional interactions [9,10], and recently the first experimental realizations of spin squeezing have been achieved [12–14].

There is a link between the theory of quantum computing and the theory of spin squeezing through the observation that the register of quantum bits in a quantum computer constitutes an ensemble of effective spins, and a general purpose quantum computer can obviously produce a spin squeezed state of its register qubits. Moreover, proposals for quantum computing may be partly implemented, e.g., with only a restricted class of operations, which do not suffice for general computing purposes, but which do lead to massive entanglement and possibly to spin squeezing. We have previously considered such reduced instruction set (RISQ) quantum computers with ions and atoms [15], and we have pointed out that apart from their ability to address particular physics problem such as antiferromagnetism, they can also be used to synthesize useful quantum states. In this paper, we present a theory for the spin squeezing expected for different models of interacting spins. These models encompass a number of theoretical proposals for quantum computing, and although spin squeezing may not be a particularly relevant property in, e.g., a spintronic or a quantum dot realization of a quantum computer, we wish to point out the possibility of verifying the entangling powers of the permanent or controllable interactions in these systems by simple measurements on the entire system. The detection of spin squeezing may constitute a useful diagnostic tool in the early stages of the construction of a quantum computer.

The treatment of the most general case of interacting spin systems is a formidable task, dealt with by a number of ingenious approximations in the theory of magnetism in solid-state physics. In Sec. II of the paper, we consider the case of an initially known state of the spins, subject to an arbitrary interaction Hamiltonian. By application of Ehrenfest’s theorem, we can determine the short-time behavior of the system analytically, we can identify the spin squeezing signal, and we can devise the optimum conditions for detecting this signal. In Sec. III, we consider a general Ising-type Hamiltonian with the spins constituting a chain with only nearest-neighbor interactions of varying magnitude. We obtain an analytical expression for the spin squeezing of the system at any future time, and we provide different examples of spin chains with constant, alternating, and random couplings.

II. SHORT-TIME BEHAVIOR FOR GENERAL PAIRWISE INTERACTIONS

Consider a collection of \( \frac{1}{2} \) particles which only interacts through pairwise interactions. The most general Hamiltonian describing this situation is given by

\[
H = \sum_{k \neq l} \tilde{j}_k^T \cdot m_{kl} \cdot \tilde{j}_l,
\]

where superscript \( T \) denotes the transpose \( \tilde{j}_k^T = (j_{x,k}, j_{y,k}, j_{z,k}) \), and where \( m_{kl} \) are real \( 3 \times 3 \) matrices. We assume that the particles are initially prepared in a state where all spins are pointing in a given direction prepared, for
instance, by optical pumping techniques, and we compute the time evolution of the noise in a component perpendicular
to the direction of the mean spin. A convenient representation
of this situation is by a collective spin state
\( \hat{R}(\alpha, \beta, \gamma)|J, J\rangle \), where \( \hat{R} \) is a rotation operator given by the
three Euler angles \( \alpha, \beta, \) and \( \gamma \), and where \( |JM\rangle \) denotes an
eigenstate of \( \hat{J}^2 \) and \( J_z \) with eigenvalues \( J(J+1) \) and \( M \).

The noise in a direction perpendicular to the mean spin can be
found by calculating the square of the rotated \( J_x \) operator
\( \hat{J}_x = \hat{R} J_x \hat{R}^\dagger \).

With the general Hamiltonian (2), it is not possible to
calculate the full time evolution of the noise. It is, however,
straightforward by Ehrenfest’s theorem to determine the
short-time evolution,
\[
\frac{d}{dt}(\Delta J_x)^2 = \frac{i}{\hbar} [J J | [\hat{H}, J_x^2] | J J],
\]
where we have introduced the transformed Hamiltonian
\( \tilde{H} = \hat{R}^\dagger \hat{H} \hat{R} \) and we have used \( \langle J_x \rangle = 0 \). By using
\( \hat{R}^\dagger \hat{R} = \hat{R}^\dagger \hat{R} \), where \( R \) is the \( 3 \times 3 \) matrix representation of the rotation in
coordinate space, we may express the transformed Hamiltonian as
\( \tilde{H} = \sum_{k+l} m_{kl} \tilde{j}_k \tilde{j}_j \), where \( m_{kl} = R^T, m_{kl} = R \). By
calculating the commutator and evaluating the expectation value, we find
\[
\frac{d}{dt}(\Delta J_x)^2 = \frac{1}{2\hbar} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & M_x & 0 \\
0 & 0 & M_z \\
\end{array} \right)
\]
In this expression we have introduced a matrix \( M = \sum_{k+l} m_{kl} \), and transformed it with the rotation matrix \( \tilde{M} = R^T, \tilde{M} = R \).

In Eq. (2), the terms involving the \( k \)th and \( l \)th particles are
\( \tilde{j}_k \tilde{j}_l \), where \( \tilde{j}_k = m_{kl} \tilde{j}_j \). The sum of these terms can also
be written as \( \tilde{M}^T (\tilde{M}^T + m_{kl}) \tilde{j}_j \).

From this expression we see that without loss of generality we can assume the matrix \( M \) to be symmetric and hence
diagonalizable and in a convenient coordinate system we have
\[
\tilde{M} = \left( \begin{array}{ccc}
M_x & 0 & 0 \\
0 & M_y & 0 \\
0 & 0 & M_z \\
\end{array} \right)
\]
By calculating \( \tilde{M} \) using the form (5) for the matrix \( M \) and the
well-known rotation matrices \( R \), we can find the time
derivative of the noise perpendicular to the mean spin for
any orientation of the spin by using Eq. (4). Since we start
out in a state with \( \xi^2 = 1 \) and since \( d/dt(\tilde{J}_x)^2 = 0 \) at \( t = 0 \), the
interaction produces spin squeezing if we can find any set of
Euler angles \( \alpha, \beta, \) and \( \gamma \) which gives a negative derivative in
Eq. (4). The optimal orientation of the spin is found by
minimizing the derivative (4) with respect to the angles \( \alpha, \beta, \) and \( \gamma \). We find that the extrema of Eq. (4) are always with
the mean spin along one of the eigenvectors of \( M \). If the spin
is polarized along the \( z \) axis, the change in the noise is maxi-
mal for the perpendicular components \( J_x = 1/\sqrt{2}(J_x \pm J_y) \)
and we find
\[
\frac{d}{dt}(\Delta J_x)^2 = \frac{1}{4\hbar}(M_x - M_z).
\]

Similar expressions are found if the spin is oriented along the
\( x \) or \( y \) axes.

Note that no assumptions about the values of the coupling
matrices are made; they may vary randomly for any pair \( (k,l) \) of spins. Only the sum needs to be specified to deter-
mine the short-time spin squeezing. The above argument
only states that some squeezing will be produced, it does not
say anything about the maximum squeezing. In the following
section, we shall analyze special cases of the interaction (2)
where the squeezing can be calculated exactly for all times.

III. ISING CHAIN WITH NEAREST-NEIGHBOR
COUPLING

Consider now a general model with arbitrary coupling
constants between nearest-neighbors in an Ising spin chain.
The chain consists of \( N \) spins with the Hamiltonian
\[
H = \hbar \sum_{i=1}^N \chi_{ij} j_i j_{i+1},
\]
where we identify the \( (N+1) \)st spin with the first one in the
chain. Depending on the value of \( \chi_N \), the chain can be
equipped with open or closed boundary conditions.

This Hamiltonian arises in recent proposals for quantum
computation with atoms in optical lattices [16,17]. In these
proposals, the atoms interact with the nearest neighbors and
it has been shown that the interaction can be put in the form
(7) and that this interaction produces spin squeezing [9].
Other application of this Hamiltonian in optical lattices can be
found in [18]. Spin squeezing is much less experimentally
challenging to produce and to verify than the full quantum
computer, and squeezing can provide a demonstration of the
entangling capabilities of the setup as well as having practi-
cal application in atomic clocks. From the discussion in Sec.
II it follows that for short times the optimal squeezing is
produced by having the spin initially polarized along \( z \) and
by looking at the noise in one of the components \( J_x = 1/\sqrt{2}(J_x \pm J_y) \), where
\( d/dt(\Delta J_x)^2 = \frac{1}{2\hbar} \Sigma_{i \neq j} \chi_i \).
For longer time intervals, the optimal noise reduction is in a
component \( \chi_j = \cos(\theta J_i) + \sin(\theta J_i) \) with \( \theta \neq \pm \pi/4 \) but for
simplicity we shall only consider \( \theta = \pm \pi/4 \).

The unitary time evolution operator for the Hamiltonian \( H \)
can be written
\[
U(t) = e^{-iHt/\hbar} = \prod_{i=1}^N e^{-i\chi_i j_i j_{i+1}},
\]
and the operators \( \tilde{j}_i \) in the Heisenberg picture are readily
obtained:
identical coupling coefficients which is a general expression for any set of values for the \( \chi_i \) and \( J_i \). We assume an initial state where all spins are pointing up, i.e., the \( j_{z,i} \) eigenstate for all \( i \), and Eq. (9) then provides the mean values of the collective spin components at later times:

\[
\langle J_z \rangle = \langle J_y \rangle = 0,
\]

\[
\langle J_x \rangle = \frac{1}{2} \sum_{i=1}^{N} \cos \left( \frac{\Delta t}{2} \right) \cos \left( \frac{\chi_i \Delta t}{2} \right).
\]  

(10)

Since the collective operator \( J_z \) commutes with \( H \), it follows that \( \langle J_z^2 \rangle \) retains its original value at \( t = 0 \): \( \langle J_z^2 \rangle = N/4 \). In order to calculate \( \langle J_x^2 \rangle \), we need to know the expectation values \( \langle j_{x,i}(t)j_{y,i+k}(t) \rangle \). We always have \( \langle j_{y,i}(t)^2 \rangle = \frac{1}{4} \), and by direct calculation we see that \( \langle j_{x,i}(t)j_{y,i+1}(t) \rangle = 0 \), whereas for \( k = 2 \) we obtain

\[
\langle j_{y,i}(t)j_{y,i+2}(t) \rangle = \frac{1}{4} \cos \left( \frac{\Delta t}{2} \right) \sin \left( \frac{\chi_i \Delta t}{2} \right) \sin \left( \frac{\chi_i + \Delta t}{2} \right) \cos \left( \frac{\chi_i + 2\Delta t}{2} \right).
\]  

(11)

From Eq. (9) we see that \( \langle j_{y,i}(t)j_{y,i+k}(t) \rangle \) vanishes for \( k \gg 3 \), and we thus obtain

\[
\langle J_y \rangle = \frac{N^2 + N \sum_{i=1}^{N} \cos \left( \frac{\chi_i \Delta t}{2} \right) \sin \left( \frac{\chi_i + \Delta t}{2} \right) \sin \left( \frac{\chi_i + 2\Delta t}{2} \right) \cos \left( \frac{\chi_i + 3\Delta t}{2} \right) - N \sum_{i=1}^{N} \sin \left( \frac{\chi_i \Delta t}{2} \right) \cos ^3 \left( \frac{\chi_i + \Delta t}{2} \right) \right)^2}{N \sum_{i=1}^{N} \cos \left( \frac{\chi_i \Delta t}{2} \right) \cos \left( \frac{\chi_i + \Delta t}{2} \right) \cos \left( \frac{\chi_i + 2\Delta t}{2} \right) \cos \left( \frac{\chi_i + 3\Delta t}{2} \right) - N \sum_{i=1}^{N} \sin \left( \frac{\chi_i \Delta t}{2} \right) \cos ^3 \left( \frac{\chi_i + \Delta t}{2} \right) \right)^2},
\]  

(15)

which is a general expression for any set of values for the coupling coefficients \( \chi_i \). For a uniform closed chain (all \( \chi_i \) identical), the above equation reduces to

\[
\xi^2_{\chi/\Delta t} = \frac{1 + 0.25 \sin^2(\chi \Delta t) - \sin(\chi \Delta t)}{\cos^4(\chi \Delta t/2)}
\]  

(16)

as found in [9].

### A. Ising model with few spins

In the above expressions, coefficients \( \chi_{i+k} \) with \( i+k \) > 4 assume in a cyclic manner the values of the coupling among the first spins in the chain. It should accordingly be noted that Eq. (15) is not valid if \( N \equiv 4 \), for which special expressions apply.

In the Ising model with only two spins \( H_2 = 2H \chi_{j,i}j_{j,i+2} \), the related expectation values are obtained as \( \langle J_0 \rangle = \langle J_1 \rangle = 0 \), \( \langle J_y \rangle = \cos(\chi \Delta t) \), \( \langle J_y^2 \rangle = \frac{1}{2} \), and \( \langle J_x J_y + J_y J_x \rangle = -\sin(\chi \Delta t) \), from which we obtain the squeezing parameter

\[
\xi^2_{\chi/\Delta t} = \frac{1 - \sin(\chi \Delta t)}{\cos^3(\chi \Delta t)}.
\]  

(17)

The system can be squeezed, and the maximum squeezing \( \xi^2_{\chi/\Delta t} = 0.5 \) occurs when \( \Delta t = \pi/(2 \chi) \).
For the Ising model with three spins $H_3 = \hbar \sum_{i=1}^{3} \chi_{i} J_{x,i} \sigma_{i}^{x} \sigma_{i+1}^{x}$, the squeezing parameter is
\[
\xi^2 = \frac{9 + 3 \sum_{i=1}^{3} \sin(\frac{x_i}{2}) \sin(\frac{x_{i+1}}{2}) - 3 \sum_{i=1}^{3} \sin(\frac{x_{i}+x_{i+1}}{2})}{\left( \sum_{i=1}^{3} \cos(\frac{x_{i}}{2}) \cos(\frac{x_{i+1}}{2}) \right)^2},
\]
and for a uniform closed chain this equation reduces to
\[
\xi^2 = \frac{1 + \sin^2(\frac{x_{i}}{2}) - \sin(\chi t)}{\cos^4(\frac{x_{i}}{2})},
\]
which is different from Eq. (16).

Finally, in the Ising model with four spins $H_4 = \hbar \sum_{i=1}^{4} \chi_{i} J_{x,i} \sigma_{i}^{x} \sigma_{i+1}^{x}$. The squeezing parameter is given by
\[
\xi^2 = \frac{8(\langle J_{x}^2 \rangle + \langle J_{y}^2 \rangle) - 4 \sum_{i=1}^{4} \sin(\frac{x_i+x_{i+1}}{2} t)}{\left( \sum_{i=1}^{4} \cos(\frac{x_{i}}{2}) \cos(\frac{x_{i+1}}{2}) \right)^2},
\]
where
\[
\langle J_{x}^2 \rangle + \langle J_{y}^2 \rangle = 2 + \sum_{i=1}^{4} \sin(\frac{x_i}{2}) \sin(\frac{x_{i+1}}{2}) \cos(\frac{x_{i}+x_{i+1}}{2}) \times \cos(\frac{x_{i+1}+x_{i}}{2}).
\]

For a uniform chain Eq. (20) actually reduces to Eq. (16), but for general coupling constants $\chi_i$, Eq. (15) is only valid for $N \geq 5$.

Figure 1 is a plot of the squeezing parameter as a function of time for few spins in a uniform chain. It shows that maximum squeezing occurs for $N=2$ and that the degree and the temporal range of squeezing are both small for $N=3$ in comparison with the other cases.

**B. Dimerized and random chains**

It is interesting to study a dimerized Ising chain with an even number of spins $N=2M$, where the coupling constants are chosen as $\chi_i = \chi(1+(-1)^{i+1} \delta)$. We have two values of the couplings $\chi_0 = \chi(1+\delta)$ for odd $i$ and $\chi_0 = \chi(1-\delta)$ for even $i$. Such systems appear naturally in heterogeneous structures where every second site is occupied with one type of qubit/particle, and their mutual communication is provided through intermediate particles acting as short-range data-bus elements in, e.g., the quantum computer. Two overlapping optical lattices with two different kinds of atoms or atoms in two different ground states may be moved relative to each other in order to establish the dimerized chain Hamiltonian.

For the dimerized Ising chain, Eq. (15) reduces to
\[
\xi^2 = \frac{1 + 0.25 \sin(\chi t) \sin(\chi t) - \sin(\chi t)}{\cos^2(\chi t) \cos^2(\chi t)},
\]
and we see that the spin squeezing does not depend on the number of atoms. Obviously Eq. (22) reduces to Eq. (16) in the limit $\delta \to 0$, and in the limit $\delta \to 1$ it gives Eq. (17).

Figure 2 is a plot of the squeezing parameter as a function of the parameter $\delta$. We see that both the squeezing range and degree of squeezing increase as $\delta$ approaches unity, i.e., the introduction of a dimerized coupling makes the squeezing better. For $\delta$ larger than unity, the denominator vanishes within the time interval displayed in the figure, as shown for $\delta = 1.1$ which causes a divergence of the squeezing parameter at $\chi t = \pi/2.1 = 1.496$.

Let us finally study a random chain model in which spins interact with their neighbors with a fixed coupling constant $\chi$, but only with probability $p$, i.e., any coupling constant is $\chi$ with probability $p$ and zero with probability $1-p$. This model corresponds, e.g., to the atomic lattice system with a
nonunit filling fraction, so that empty lattice sites appear. Different assumptions may be made about the correlations between different lattice sites, but let us for simplicity consider the case where the occupancies are completely uncorrelated. Simulations for such a model were presented in Ref. [9], but we can now present analytical results (for simplicity presented only for the \( \theta = \pi/4 \) direction).

We introduce \( \mu = \chi t/2 \), in terms of which we can write the mean values of trigonometric functions,

\[
\cos \left( \frac{\chi t}{2} \right) = p \cos \mu + (1 - p),
\]

\[
\sin \left( \frac{\chi t}{2} \right) = p \sin \mu,
\]

As shown in Fig. 3 and observed already in [9], spin squeezing is obtained also in the case of a random filling.

IV. CONCLUSION

We have obtained general expressions for squeezing in different models of interacting spins. In the general model considered in Sec. II, we showed that almost any kind of spin-spin interaction can be used to create squeezed states. As long as the coupling matrix in Eq. (5) is not proportional to the identity, spin squeezing can be produced. In the one-dimensional Ising model with arbitrary nearest-neighbor coupling constants, we obtained analytical results for all times, and we addressed the case of constant couplings and different kinds of departure from constant couplings of the spins. We found that the amount of spin squeezing was larger in a dimerized model with periodically varying coupling coefficients than in a homogeneous model.

Among many possible generalizations of the present work, we wish to mention long-range interactions and interactions in higher dimensions, and, e.g., the question whether extensions of the dimerized model to these cases are also superior to homogeneous couplings.

As mentioned in the Introduction, spin squeezing is both a useful property in itself [3] and a signature of entanglement [10,11]. It can be accomplished and detected in different physical systems without the need for experimental access to individual spins, and as such it can be used to test, e.g., spintronics proposals for quantum information processing. Studies of the Ising model in condensed-matter physics have so far focused on the identification of energy spectra and eigenstates and on the temperature dependence of steady-state properties. Unitary dynamics as presented in this work may, however, become the subject of experimental research, and then our computational methods and results may find application in condensed-matter physics.

When applied to a specific physical system, our theory should be supplemented with an analysis of decoherence and loss mechanisms. Inhomogeneities and noisy bias fields may deteriorate the spin squeezing, and since squeezing is intimately related to entanglement of the particles, loss of one or a few particles will leave the remaining spins in a state with reduced squeezing. An analysis will depend on explicit details of the noise model, e.g., dephasing will not alter the variance of the \( \varepsilon \) component of the spin, but it will reduce the orthogonal components and hence increase the corresponding parameter \( \xi^2 \), possibly by a different amount from the increase of the squeezing parameter for a state squeezed in the \( y \) direction. The effect of particle loss on spin squeezing
has been studied in a different model [19], but the results are expected to be similar here. Indeed, for a uniform string the variance of the collective spin component of the remaining $N-1$ spins after loss of a single particle is given in terms of the variance before the loss: $\text{var}(J_{\pi/4})_{N-1} = \text{var}(J_{\pi/4})_{N}(1 - 2/N) + 1/4$, as found in [19]. For many atoms and moderate spin squeezing obtained in the Ising model, the effect of a few losses is small.