Spin squeezing in nonlinear spin-coherent states

Xiaoguang Wang

Institute of Physics and Astronomy, Aarhus University, DK-8000, Denmark

Received 17 January 2001, in final form 21 March 2001

Abstract

We introduce the nonlinear spin-coherent state (SCS) via its ladder operator formalism and propose a type of nonlinear SCS by the nonlinear time evolution of the SCS. By a new version of spectroscopic squeezing criteria we study the spin squeezing in both the SCS and the nonlinear SCS. The results show that the SCS is not squeezed in the x, y and z directions, and the nonlinear SCS may be squeezed in the x and y directions.

Keywords: Spin squeezing, nonlinear spin-coherent states

1. Introduction

In nonlinear systems such as an optical Kerr medium [1], squeezed states of the radiation field [2] have been extensively studied. The spin-squeezed states are also studied in nonlinear systems [3] and have proved to be useful to enhance spectroscopic resolution [4], for example, in atomic clocks. The generation of the spin-squeezed states has been studied in many ways, such as by interaction of atoms with squeezed light [4–8], quantum nondemolition measurement of atomic spin states [9] and atomic collisional interactions [10].

The spin-squeezed states can be characterized in many different ways [3,4,11]. In our study we employ the criteria of spin squeezing recently proposed by Sørensen et al [12] as a new version of spectroscopic squeezing [4,13]. The squeezing parameter is defined as [12]

$$\xi^2_{\eta n} = \frac{2\langle J_{\eta n}^2 \rangle - \langle J_{\eta n} \rangle^2}{\langle J_{\eta n} \rangle^2 + \langle J_{\eta n}^2 \rangle^2},$$

where $J_{\eta n} = \vec{n} \cdot \vec{J}, \vec{n}_i (i = 1, 2, 3)$ are orthogonal unit vectors and $\vec{J}$ is the spin-$j$ angular momentum operator. The states with $\xi^2_{\eta n} < 1$ are spin squeezed in the $\vec{n}$ direction. We will show an interesting feature: that the squeezing parameters $\xi^2_x = \xi^2_y = \xi^2_z = 1$ for spin-coherent states (SCSs) [14], which indicates that the SCS is not squeezed in the $x, y$ and $z$ directions.

In this paper we introduce the nonlinear SCS and consider the spin squeezing in it. Section 2 gives the definition of the nonlinear SCS via its ladder operator formalism and proposes a type of nonlinear SCS by the time evolution of the SCS under the nonlinear Hamiltonian. In section 3 we first prove that the SCS exhibits no squeezing in the $x, y$ and $z$ directions and then study the spin squeezing in the nonlinear SCS. Conclusions are given in section 4.

2. Nonlinear SCS

We work in a $(2j+1)$-dimensional angular momentum Hilbert space $|j, m; m = -j, \ldots, +j\rangle$. It is convenient to define a number operator $\mathcal{N} = J_+J_- + J_0$, and the ‘number states’ $|n\rangle$ which satisfy

$$|n\rangle = |j, -j + n\rangle, \quad \mathcal{N}|n\rangle = n|n\rangle, \quad (2)$$

The SCS is given by [14]

$$|\eta\rangle = (1 + |\eta|^2)^{-j} \sum_{n=0}^{2j} \left(\frac{2j}{n}\right)^{1/2} \eta^n|n\rangle, \quad (3)$$

where $\eta$ is complex.

It is easy to check that the SCS satisfies the following equation:

$$J_-|\eta\rangle = \eta(2j - \mathcal{N})|\eta\rangle, \quad (4)$$

where the operators $J_{\pm} = J_x \pm iJ_y$. This is a ladder operator formalism of the SCS.

By recalling the definition of the bosonic nonlinear coherent state [15] and $su(1, 1)$ nonlinear coherent states [16], it is natural to define the nonlinear SCS as

$$f(\mathcal{N})J_-|\eta\rangle_{\eta} = \eta(2j - \mathcal{N})|\eta\rangle_{\eta}, \quad (5)$$

where $f(\mathcal{N})$ is a nonlinear function of the number operator $\mathcal{N}$. Equation (5) is of the ladder operator form.

Now we propose a type of nonlinear SCS. The SCS under the evolution of the nonlinear Hamiltonian $F(\mathcal{N})$ is directly given by

$$|\eta, t\rangle = e^{-itF(\mathcal{N})}|\eta\rangle$$

$$= (1 + |\eta|^2)^{-j} \sum_{n=0}^{2j} \left(\frac{2j}{n}\right)^{1/2} \eta^n e^{-itF(\mathcal{N})}|n\rangle, \quad (6)$$
From equations (4) and (6) we find the state $|\eta, t\rangle$ satisfies
\[ e^{i(F(N+1)-F(N))J_\xi}|\eta, t\rangle = \eta(2j - N)|\eta, t\rangle. \tag{7} \]

According to the definition of the nonlinear SCS, the above state is a nonlinear SCS with the nonlinear function $e^{i(F(N+1)-F(N))}$.

For $F(N) = N^2 - N$, equation (7) reduces to
\[ e^{i2N}\eta J_\xi|\eta, t\rangle = \eta(2j - N)|\eta, t\rangle. \tag{8} \]

Further let $t = \pi/2$; the above equation reduces to
\[ \Pi J_\xi|\eta, \pi/2\rangle = \eta(2j - N)|\eta, \pi/2\rangle, \tag{9} \]
where $\Pi = (-1)^N$ is the parity operator. Next we study the spin squeezing of the SCS $|\eta\rangle$ and the nonlinear SCS $|\eta, t\rangle$.

3. Spin squeezing

We first give an example that there is no spin squeezing in the SCS along the $x$, $y$ and $z$ directions, and then study the spin squeezing in the nonlinear SCS.

3.1. The SCS

In order to calculate the squeezing parameter (1) we need to know the expectation values $\langle N^2 \rangle$ and $\langle J^2 \rangle$ ($k = 1, 2$). It is convenient to calculate $\langle N^2 \rangle$ by the generation function method. The generation function of the SCS is given by
\[ G(\lambda) = \langle |\eta|^2 \rangle = \frac{(1 + \lambda |\eta|^2)^{2j}}{(1 + |\eta|^2)^{2j}}, \tag{10} \]
from which the factorial moments follow
\[ F(k) = \frac{d^k G(\lambda)}{d^k \lambda} \Bigg|_{\lambda = 0} = \frac{|\eta|^{2k}(2j)!}{(1 + |\eta|^2)^{2j}(2j - k)!}. \tag{11} \]

The factorial moments immediately give the expectation values of the operators $N$ and $N^2$ and the variance of $N$:
\[ \langle N \rangle = F(1) = \frac{2j|\eta|^2}{1 + |\eta|^2}, \tag{12} \]
\[ \langle N^2 \rangle = F(2) + F(1) = \frac{2j|\eta|^2 + 4j^2|\eta|^4}{(1 + |\eta|^2)^2}, \tag{13} \]
\[ (\Delta N)^2 = \frac{2j|\eta|^2}{1 + |\eta|^2} - \frac{4j^2|\eta|^4}{(1 + |\eta|^2)^2}. \tag{14} \]

From equation (3) the expectation values $\langle J^2 \rangle$ are obtained as
\[ \langle J^2 \rangle = \frac{|\eta|^2(2j)!}{(1 + |\eta|^2)^{2j}(2j - k)!}. \tag{15} \]

Now we calculate the squeezing parameter $\xi_t^2$, which is rewritten as
\[ \xi_t^2 = \frac{2j(\Delta N)^2}{(\langle J^2 \rangle)^2}. \tag{16} \]

Then substituting equations (14) and (15) into (16), we immediately obtain $\xi_t^2 = 1$.

To calculate $\xi_x^2$ and $\xi_y^2$ we need the identities
\[ J_x^2 = \frac{1}{4} \left[ 2j(2N + 1) - 2N^2 + J_y^2 + J_z^2 \right], \tag{17} \]
\[ J_y^2 = \frac{1}{4} \left[ 2j(2N + 1) - 2N^2 + J_x^2 + J_z^2 \right]. \tag{18} \]

which gives the expectation values of $J_x^2$ and $J_y^2$.

From equation (15) and the relation $J_z = N - j$, we obtain
\[ \langle J_z \rangle = j(\eta^* - \eta) \quad \text{and} \quad \langle J_z \rangle = \frac{j(|\eta|^2) - 1}{1 + |\eta|^2}. \tag{19} \]

From equations (12), (13), (15) and (17)–(19), the variances of $J_x$ and $J_y$ are expressed as
\[ (\Delta J_x)^2 = \frac{j(1 + |\eta|^2 - \eta^* - \eta^2)}{2} = \frac{(\langle J_x \rangle)^2 + (\langle J_y \rangle)^2}{2j}, \tag{20} \]
\[ (\Delta J_y)^2 = \frac{j(1 + |\eta|^2 + \eta^* + \eta^2)}{2} = \frac{(\langle J_x \rangle)^2 + (\langle J_y \rangle)^2}{2j}. \tag{21} \]

The above two equations lead directly to $\xi_x^2 = \xi_y^2 = 1$. Thus we have shown that the squeezing parameters $\xi_x^2 = \xi_y^2 = 1$ for the SCS. That is to say, the SCS exhibits no squeezing in the $x$, $y$ and $z$ directions, irrespective of the complex $\eta$. We expect that the spin squeezing exists in the nonlinear SCS.

3.2. The nonlinear SCS

We examine the spin squeezing in the nonlinear SCS $|\eta, t\rangle$. The expectation values $\langle N \rangle$, $\langle N^2 \rangle$ and the variance $(\Delta N)^2$ are time independent and given by equations (12)–(14), respectively.

From equation (6), we obtain the expectation value of $J_x^2$ on the state $|\eta, t\rangle$ as
\[ \langle J_x^2 \rangle = \eta^4(1 + |\eta|^2)^{-2j} \frac{(2j)!}{(2j - k)!} \times \sum_{n=0}^{2j-k} \left( \frac{2j - k}{n} \right) |\eta|^{2n} e^{i(F(n\eta) - F(n\eta+1))}. \tag{22} \]

Of course, equation (22) reduces to (15) at $t = 0$.

By substituting equations (14) and (22) into (16), the squeezing parameter $\xi_t^2$ is given by
\[ \xi_t^2 = \frac{1}{1 + (1+|\eta|^2)^{2j} \sum_{n=0}^{2j} \left( \frac{2j}{n} \right) |\eta|^{2n} e^{i(F(n\eta) - F(n\eta+1))}}. \tag{23} \]

For $M + 1$ complex quantities $c_i$ ($i = 0 \ldots M$), there is an inequality
\[ |c_0 + c_1 + \cdots + c_M| \leq |c_0| + |c_1| + \cdots + |c_M|. \tag{24} \]

Using this inequality in equation (23), we find that
\[ \xi_t^2 \geq 1. \tag{25} \]

So a general conclusion is made that no squeezing occurs in the $z$ direction for arbitrary nonlinear function $F(N)$. However, spin squeezing may exist in the $x$ or $y$ directions. Next we make numerical calculations to show the spin squeezing.

Figure 1 gives the squeezing parameters $\xi_x^2$ ($\alpha = x, y$) as a function of time $t$ for different nonlinear Hamiltonians $F(N) = N^k$ ($k = 2, 3, 4$). For small time $t$ we observe that...
the state is squeezed in the $x$ direction, not the $y$ direction. As $k$ increases, the frequency of occurrence of spin squeezing increases. Most of the time we also see that the spin squeezing appears alternatively in the $x$ and $y$ directions; i.e. when the state is squeezed in the $x(y)$ direction, it is not squeezed in the $y(x)$ direction. The state can show no spin squeezing in either the $x$ or $y$ directions in some small ranges of $t$, but it cannot show spin squeezing simultaneously in the two directions.

It is interesting to consider the squeezing in the nonlinear Hamiltonian $H = \sin(aN)$ which can be realized in physical systems [17]. The numerical results are shown in figure 2. For $\eta = 0.1$ we observe that the spin squeezing in the $x$ and $y$ directions appears alternatively at the beginning of the time evolution. For small time $t$, the state is squeezed in the $y$ direction, not the $x$ direction, in contrast to figure 1. We also observe that the time range of spin squeezing decreases as the parameter $\eta$ increases; i.e. the squeezing does not occur most of the time that $\eta$ is large.

4. Conclusions

In conclusion we have given the definition and proposed an example of the nonlinear SCS. We have studied the spin squeezing in both the SCS and the nonlinear SCS. The main results are as follows.

(1) The squeezing parameters $\xi_x^2 = \xi_y^2 = \xi_z^2 = 1$ for the SCS. That is to say, the SCS is not squeezed in the $x$, $y$ and $z$ directions, irrespective of the parameter $\eta$.
(2) The nonlinear SCS shows no spin squeezing in the $z$ direction for an arbitrary nonlinear Hamiltonian $F(N)$.
(3) The nonlinear SCS may be squeezed in the $x$ and $y$ directions. Most of the time the squeezing appears alternatively in the $x$ and $y$ directions as time goes on. In addition, we observe that the state cannot be squeezed simultaneously in the two directions.

The spin squeezing originates from the nonlinearity of the nonlinear SCS. Then we expect that the spin squeezing exists in other nonlinear SCSs with different nonlinear functions.

Acknowledgments

The author is grateful for many helpful discussions with Klaus Mølmer, Anders Sørensen and Bin Shu. This work is supported by the Information Society Technologies Programme IST-1999-11053, EQUIP, action line 6-2-1.

References

