Symmetry Groups in Physics

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This article is a résumé, intended for experimentalists and other nonspecialists who have a need to know of that part of group representation theory which is employed in the spectroscopic classification of energy levels. A summary of the general properties of semisimple Lie groups, sufficient to make the treatment self-contained, is included. Attention is then focused on Racah’s important invariant operators (such as $J^2=J_x^2+J_y^2+J_z^2$ for the rotation group), whose eigenvalues provide the essential connection between group theory and physics by serving as the “good quantum numbers” which label the multiplets and energy levels of the system. The rotation and isospin groups are used as examples throughout the discussion. The final section describes the way in which, in entirely analogous fashion, $SU_3$ and larger groups are being employed at present in the classification of strong-interaction multiplets.

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I. INTRODUCTION

Three of the principal reasons group theory has been useful in physics are these:

(1) It enables one to understand the degenerate energy eigenstates of a system (“mass multiplets”) in terms of the symmetry properties possessed by its interactions.

(2) It produces from these symmetry properties a complete set of commuting observables, whose eigenvalues (“good quantum numbers”) serve to label the multiplets uniquely (e.g., as $J^2=\frac{1}{4}$, etc.).

(3) These same commuting observables identify conservation laws obeyed by the system, and so establish selection rules for transitions, decay schemes, and reactions.

This is familiar ground if the symmetry group merely involves translations of geometric coordinates: the rotations, inversions, translations, and Lorentz transformations. Transformations on discrete coordinates, such as charge, baryon number or strangeness, are somewhat less familiar. Because of the current interest in symmetry groups on these discrete coordinates, and because of recent developments in group theory itself, this would seem to be an appropriate time for a concise review of the traditional role played by symmetry groups. This is especially so because from such a vantage point one sees immediately that the isotopic-spin group and SU_3 in their application to elementary particles and resonances are being employed to define and classify multiplets in exactly the same way that the geometric groups have long been used, for instance, in atomic spectroscopy.

More explicitly, we take the point of view that the theory of Lie groups, used properly with all the most powerful theorems, is both difficult and time-consuming. However, the key theorems are perfectly understandable without a detailed knowledge of the trickery involved in their proofs, and once these are accepted there is a short, physical and direct route to a complete understanding of symmetry groups and their multiplets. It is a route which, in particular, avoids the concept of matrix representations and concentrates instead on the multiplets of physical states which provide the bases for representations. Our purpose is to describe this route in a self-contained article, and par-
ticularly in terms which make it accessible to interested experimentalists without extensive background reading in the mathematical literature.

We emphasize that this is not a handbook of practical manipulations for evaluating Clebsch–Gordan coefficients or other distinctive numbers for any particular group, since these are already in good supply. The introductory material in the earlier sections includes all the necessary standard definitions merely as a convenience to the reader. The essential point of the article is made in Sec. IV, where Racah’s group invariants \( (l \) of them for a Lie group of rank \( l \)) are identified as the complete set of commuting observables of the full symmetry group, whose eigenvalues are the good quantum numbers of the system at hand and label its multiplets uniquely. Direct product groups and the construction of composite particles as products of multiplets are reviewed briefly in Secs. V and VI and a consideration of the physical significance of the \( I \)-spin and \( SU_3 \) groups is included in Sec. IV.

II. ESSENTIAL PROPERTIES OF GROUPS

A. Symmetry-Group Operators

The spherical symmetry of, say, a ball bearing, is checked by measuring a diameter and then rotating the bearing through a series of angles, measuring it each time to see if it looks the same from all directions. In exactly the same way, one investigates a certain symmetry property of any physical system (represented in the usual formalism by its Hamiltonian or Lagrangian) by “rotating” the appropriate coordinate (which may be an angular coordinate, or a charge coordinate, or a strangeness coordinate, etc.) and checking to see if the Hamiltonian still looks the same from the new point of view.

The tool needed to carry out this procedure is a set of “rotators,” i.e., operators which change certain parameters or coordinates of the system by a specified amount. The entire set of such operators on a certain coordinate conveniently forms a mathematical group of operators, called a symmetry group; the symmetry groups which operate on the geometric coordinates, e.g., are the translation group, the rotation group, and the inversion group.

Although the operators act on the coordinates of the system, these coordinates appear in the quantum-mechanical formalism as the arguments of wave functions, so the “rotators” are most conveniently thought of as transformations on the wave functions themselves.

For instance, the translation operator takes the familiar form,

\[
T(a)\psi(x) = \psi(x+a) = \sum (a^n/n!) (d^n/dx^n) \psi(x)
\]

\[
= \sum \frac{(ia)^n}{n!} \left( i \frac{d}{dx} \right)^n \psi(x) = \sum \frac{(ia\partial_x)^n}{n!} \psi(x),
\]

i.e., the operator for translation through the distance \( a \) can be written in terms of the conjugate variable \( \partial_x = -i \frac{d}{dx} \) as

\[
T(a) = \exp (ia\partial_x).
\]

The set \( \{T(a)\} \), for all values of \( a \), is the translation group on a line.

In order that these coordinate transformations preserve normalization and hence total probability, they must be unitary (which means the displacement \( a \) is real in the above example); for this reason we restrict ourselves to unitary group operators throughout this article.

B. Symmetry Groups

The groups, then, which are of interest for the investigation of symmetry properties are groups of operations or transformations. A set of transformations \( \{T_n\} \) is said to form a mathematical group if an associative “multiplication” is defined for the operators \( (T_mT_n) \) means first \( T_n \), then \( T_m \), and the set has the properties:

(a) It is closed under multiplication, i.e., \( T_mT_n \) is a member of the set if \( T_m \) and \( T_n \) are.

(b) The unit operator \( I \) is in the set.

(c) Every operator \( T \) in the set has an inverse \( T^{-1} \) which is also in the set.

Probably the most essential property from the physical point of view is closure, since it assures us that all the operators of a given type must be present if any are. For instance, the set of translations \( \{T(a)\} \) is not a group for a particle confined to a finite line segment (unless the segment is closed on itself by imposing periodic boundary conditions on the wave function), for even though \( T(a) \) and \( T(b) \) may be “allowed” translations which keep the particle on the segment, \( T(a)T(b) \) may push it past the end and so not be allowed.

A finite group is one with a finite number of elements, such as the rotations of a cube by 90°, 180°, etc., which return it to its original position. The inversion (parity) group, consisting of just \( P = P^{-1} \) and \( P^2 = 1 \), is one which is of general interest because of the “natural” limit on the number of operators, but most finite

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1 Whenever \( \hat{p} \) is the derivative with respect to the coordinate \( \hat{x} \), the two operators are canonically conjugate (because their commutator is \([\hat{q}, \hat{p}] = i\hbar\)), and the corresponding displacement operator (by the increment \( \Delta \)) is \( T(\Delta) = \exp (i\Delta \hat{\mathbf{p}}) \). This is clearly true for any continuous coordinate \( \hat{q} \), be it distance, angle, or a more esoteric one such as angle in “charge space.”
groups are of use only for the study of such systems as crystals and simple molecules like NH₃, which are invariant only under rotations through very special angles.

To investigate invariance under an arbitrary or continuous range of coordinate transformations, we need a continuous group of operators, operators like \( T'(a) \) which are functions of one or more parameters which are allowed to vary over a continuous range of values. A familiar example is the rotation group, which we shall employ extensively to illustrate general results. Its operators, which operate on the angular coordinates of wave functions \( \psi(r) \), depend on three parameters \( (\theta_1, \theta_2, \theta_3) = 0 \) and have the form

\[
R(\theta) = \exp \left( i\theta \cdot J \right) = \sum_{n=0}^{\infty} \frac{(i\theta \cdot J)^n}{n!},
\]

in which \( J \) are the three Cartesian angular momentum operators and the \( \theta \)'s are the angles of rotation about the three axes. Because the \( \theta \)'s have a continuous range, the number of rotation operators is infinite—i.e., an infinite number of different functions of the same three operators \( J \). Because the \( J \)'s are Hermitian, these rotation operators are unitary if \( \theta \) is real.

The rotations about a single axis, e.g.,

\[
R(0, 0, \theta) \equiv R(\theta) = \exp \left( i\theta J_3 \right)
\]

themselves form a 1-parameter subgroup: \( R(\theta_1)R(\theta_1') = R(\theta_1 + \theta_1') \) is a rotation about that axis, \( R(0) = 1 \) as well, and \( R(-\theta) = R(\theta)^{-1} \) is also. In fact it, like any other one-parameter group, is Abelian, meaning that its members all commute with one another (because they are all just different functions of the single operator \( J_3 \)).

The three-dimensional rotation group is a set of operators \( R(\theta_1, \theta_2, \theta_3) \) which depend on three parameters. In general a continuous group is a set of operators \( U(\alpha_1, \ldots, \alpha_n) = U(\alpha) \), which depend on \( n \) continuous parameters; in physical applications they will be coordinates such as \( \theta, r \), or "orientation angles in charge space" (isotopic spin group, three parameters) or in "charge and isotopic spin space" (SU₃ group, eight parameters).

C. Multiplets

A concept of fundamental importance for physical applications is that of an invariant subspace of the entire Hilbert space of states on which the operators of a symmetry group act. The invariance is with respect to the group under consideration and means that every state in this subspace, when acted upon by any operator of the group, produces another vector in the subspace; the group operators merely transform the vectors of an invariant subspace among themselves, and have no matrix elements connecting a state within the subspace to one outside it. For instance, the space spanned by the orthogonal vectors \( (Y_{00}, Y_{10}, Y_{10}, Y_{1,-1}) \)

is a four-dimensional invariant subspace of the rotation group—for each of the \( J \)'s (which can change \( m \) but not \( l \)) transforms any vector in this space into another vector in the space, and any function \( R(\theta) \) of the \( J \)'s will do the same.

A multiplet is an irreducible invariant subspace of a group, i.e., one which contains no smaller invariant subspaces. For instance, the above four-dimensional example is a reducible invariant space, for it contains two smaller invariant spaces which are multiplets, one a triplet \((Y_{11}, Y_{10}, Y_{1,-1})\), and one a singlet \((Y_{00})\). (Since, e.g., \( J_1 Y_{1M} = a Y_{1M} + b Y_{1 M-1} \), the triplet clearly cannot be decomposed further.) This means that \( <Y_{00} | R(\theta) | Y_{1m}> = 0 \) for every \( R(\theta) \) in the group; no group operator has matrix elements between the singlet and the triplet. Their complete disjointedness is indicated by writing \( (Y_{00}, Y_{11}, Y_{10}, Y_{1,-1}) = (Y_{00}) \oplus (Y_{11}, Y_{10}, Y_{1,-1}) \).

From this definition one suspects that the members of a multiplet must bear a strong family resemblance to each other. They do, and the kinship can be made explicit by the following alternative construction of a multiplet. Starting with any normalized vector \( \psi_0 \) which lies entirely within a multiplet of the group, construct all the vectors \( \psi_a = U(\alpha) \psi_0 \) which can be reached from it by all the unitary operators of the group. This "unit sphere" of vectors is transformed into itself by any operator \( U(\delta) \) (by the closure property of the group), but it is not a vector space. However, all linear combinations of the \( \psi_a \) do form a vector space, which is clearly also invariant under the group. It is exactly the multiplet of the group which contains \( \psi_0 \).

One of the most important physical consequences of this close relationship between the states of a multiplet appears when the Hamiltonian of the system under consideration commutes with all the operators \( U(\alpha) \) of the symmetry group.\(^4\) In this case \( H \) has no matrix elements between different multiplets of the group [this is proved in Eq. (12), Sec. IV]. It is thus "confined" entirely to the multiplet and so must have at least one eigenvector in any multiplet. If we call it \( \psi_0 \), it follows immediately from \( HU(\alpha) = U(\alpha)H \) that every \( \psi_a \)—and hence every vector in the multiplet—is also an eigenvector of \( H \), all with the same energy eigenvalue (Schur's Lemma). In other words, whenever a symmetry group can be found which commutes with the Hamiltonian of the system, the multiplets of the group are guaranteed to be sets of degenerate

\(^4\) Because the \( U(\alpha) \)'s have no matrix elements between \( \psi_0 \) and states outside the multiplet \( \mathcal{M} \), the space cannot be larger than \( \mathcal{M} \). It must therefore be contained in \( \mathcal{M} \), but it also cannot be smaller, for if it were \( \mathcal{M} \) would not be irreducible. Hence the two spaces are identical.

\(^5\) If the system possesses a Hamiltonian, the \( S \) matrix is a function of \( H \), so commutation with the Hamiltonian implies commutation with the \( S \) matrix, which is really the essential operator. If there is no \( H \), one simply requires commutation with \( S \), and statements about bound states and resonances become statements about poles of \( S \). For simplicity, we state all such arguments in terms of \( H \) rather than \( S \).
energy eigenstates of the system. This often provides a means of learning a great deal about the eigenvectors (e.g., the angular dependence of the eigenvectors of any spherically symmetric system) even though the wave equation may not be explicitly soluble.

The term multiplet is a carry-over from atomic spectroscopy in which, in the absence of spin-orbit coupling, the invariant subspaces of a certain extension of the rotation group are labelled by \( L \) and \( S \), e.g., as \( ^2D \). The presence of the “symmetry-breaking” \( L \cdot S \) interaction splits this large set of degenerate states (multiplet) into several smaller sets, each corresponding to a fine-structure energy level and each labelled by \( J \). (For example, \( ^2D \) splits into \( J=1, 2, 3 \).) It is this set of nearby energy levels which produces a “multiplet” of lines in the spectrum of the atom, such as the sodium doublet.

In a group-theoretic context, it is the set of degenerate states which is called a multiplet. \( Y_{00} \), for example, is a singlet with respect to the rotation group, i.e., an angular momentum or “spin” singlet, while the \( \Delta \) particle is an isotopic spin singlet. \((n, \ell)\) is an \( I \)-spin doublet, as are the ground states of \( H^2 \) and \( He^2 \), and \((\pi^+, \pi^0, \pi^-)\) an \( I \)-spin triplet.

The essential point to recognize at this stage is that each group possesses a definite, unambiguous (and to some extent distinctive) set of multiplets. Although they are determined by the structure of the group, there is no general procedure available for finding them for an arbitrary continuous group. That is one of the reasons we shall restrict ourselves below to a special class of continuous groups, whose multiplets can be identified in a straightforward way, and labeled by a set of “quantum numbers.”

The rotation group is one member of this class; its multiplets are labeled by the single quantum number \( j \), since as we shall see they consist of just the \((2j+1)\) degenerate eigenstates of \( J^2 \). Thus for \( j=0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \), we get, respectively, a singlet, a doublet, a triplet, a quartet, etc. of the rotation group. SU\(_3\) is another group in this class, and its multiplets are labeled by two quantum numbers; it turns out to have one singlet, two triplets, two sextets, one octet, etc. Still another is \( G_8 \), which has one seven-dimensional multiplet, one 14-dimensional, one 27-dimensional, etc. Empirically, one of the ways we have of recognizing the occurrence of such a group in the symmetry of a system is by the occurrence in its multiplets. This is the reason they play such an important role in the investigation of an unfamiliar symmetry property, as we shall see in more detail in Sec. VII.

Increasing the size of a group \( G \) by adding operators to it to form a new group \( G' \) cannot decrease the size of its multiplets—for any multiplet of \( G' \) (a space invariant under all the operators of \( G' \)) is surely invariant under \( G \) (which is now imbedded as a subgroup in \( G' \)), and so must include at least one multiplet of \( G \). It will actually include more than one if the added operators mix the original multiplets, forcing us to put two or more of them together to make a “supermultiplet” of the larger group. \((Y_{11})\), for example, is a one-dimensional multiplet (eigenstate in this case) of the \( z \)-axis rotation subgroup of Eq. (2). If we increase the group to the entire rotation group by adding rotations about the other axes (i.e., functions of \( J_1 \) and \( J_2 \)) the original multiplets \((Y_{11}), (Y_{00}), (Y_{-1})\), which are mixed by \( J_1 \) and \( J_2 \), coalesce into a single supermultiplet of this larger group. Similarly, enlarging the \( I \)-spin group to \( SU_3 \) by adding strangeness-changing operators (thus requiring more symmetry of the Hamiltonian) causes, e.g., the singlet \( \Lambda \), the doublets \((n, \ell)\) and \((\Xi^0, \Xi^-)\), and the triplet \((\Sigma^+, \Sigma^0, \Sigma^-)\) all to coalesce into a “super octet.”

D. Lie Groups

Returning to the groups themselves, we recall that our example, the rotation group, contains an infinite number of operators, and that the essential reason it is manageable at all is that these are merely an infinite number of functions [each one specified by the numbers \((\theta_0, \theta_\theta, \theta_\phi)\)] of just three basic operators, \( J_3 \), \( J_0 \), and \( J_\Phi \). These basic operators can be recovered from the group operators \( R(\theta) \) by differentiation:

\[
i J_\ell = \{ \partial R(\theta)/\partial \theta_\ell \} \mid _{\theta=0}, \tag{3}\]

which could not have been accomplished unless the \( R(\theta) \) depended \textit{analytically} on \( \theta \) in the vicinity of the identity operator \( [R(0) = I] \).

The rotation group does not have enough structure to describe all the known symmetry properties of physical systems, but it does contain all the essential ideas. The simplest generalization which has so far proved adequate is the class of groups known as \textit{Lie groups}, after the Norwegian mathematician S. Lie.

These are continuous groups, composed of operators \( U(a) = U(a_1, \cdots, a_n) \)(which depend on \( n \) parameters or coordinates. Their distinguishing characteristic is that their members depend \textit{analytically} on all \( n \) parameters. If the general operator is \( U(a) \), one chooses the parameterization conventionally so that \( U(0) = I \). Then exactly as in (3), \( n \) basic operators, called the \textit{generators} of the group, are defined by

\[
L_i = \{ \partial U/\partial a_i \} \mid _{a=0}. \tag{4}\]

Whether the \( U(a) \) are then really functions of the generators will become clear shortly. In any case, from the analyticity assumption and Eq. (4) it follows that

\[
U(a) \approx 1 + \sum a_i L_i. \tag{4a}\]

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6To speak carefully, all such operations should be carried out on a matrix representation of the group, but we wish to avoid representations altogether. There seems to be little chance of confusion in this shortcut notation.

7The generators \( J_i \) of the rotation group are actually defined as \( J_i = -i \partial R/\partial \theta_i \) to make them Hermitian. It is well to note that the generators themselves are not members of the group; e.g., the \( J_\Phi \) are Hermitian, while the \( R(\theta) \) are unitary.
for all $\alpha_i$ small, and from this it is clear that the generators must be linearly independent operators ($\sum \alpha_i L_i = 0$ only if all $\alpha_i = 0$) if $\alpha = 0$ is to be the unique value of $\alpha$ which gives the identity operator. Also, since $U(\alpha)$ is assumed unitary, it is customary to choose the parameters pure imaginary ($i\alpha$ instead of $\alpha$) to make the generators Hermitian.

The commutators of the generators of such a group turn out to be essential; by using $\partial U/\partial \alpha_\xi \partial \alpha_\eta = \partial^2 U/\partial \alpha_\xi \partial \alpha_\eta$, one finds that they necessarily have the form

$$[L_a, L_b] = \sum_{\xi=1}^n C_{ab}^\xi L_\xi, \quad (5)$$

i.e., the generators form a set which is "closed under commutation." The familiar commutation rules for the generators of the rotation group,

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad \text{(cyclic)}, \quad (6)$$

are clearly a special case; we shall generally find them more useful in the form

$$[J_+ J_-] = \pm J_3, \quad [J_+, J_-] = 2 J_3, \quad (7)$$

where $J_\pm = J_1 \pm i J_2$.

The set of numbers $C_{ab}^\xi$ are called the structure constants of the Lie group. They are clearly not invariant under linear transformations on the generators [such as replacing $J_1$ and $J_2$ by $J_1$ and $J_3$; compare Eqs. (6) and (7)], but taken together, remarkably enough, they contain all the information we need about the group.

The commutation relations (5) can be looked upon as a direct generalization of the vector or cross product of two vectors, which we might write as $L_a \times L_b = \sum C_{ab}^\xi L_\xi$. The three-dimensional case with $J_1$, $J_2$, and $J_3$ as unit orthogonal vectors is given by Eq. (6) ($\delta \times \gamma = \epsilon$). Equation (5) is the more general case of skew axes and higher dimension, but its essential point is that the cross product of any two "basis vectors" is a vector in the space, and so linear combination of the basis vectors. The algebra of sums and "vector products" of the generators is called the Lie algebra of the group, and is one of the tools used to prove some of the theorems stated below. From this point of view a rank-2 Lie algebra (see definition below) is a space with the remarkable property of possessing two vectors whose cross product $[L_a, L_b]$ vanishes even though they are linearly independent vectors.

In other words, starting from a Lie group, one can define the generators and determine the structure constants by evaluating their commutators. But the procedure is also nearly reversible, i.e., given any set of operators $L_i$ which are closed under commutation as in Eq. (5), they almost uniquely determine a Lie group which has the $L_i$ as its generators. This is the content of the celebrated Lie theorem, which we state as:

**Theorem 1.** Given any set of $n$ operators satisfying the commutation relations (5), there exists a Lie group which has these operators as its generators. Its multiplets are uniquely determined by the structure constants, as are the matrix elements of the generators between all states of a multiplet.

The structure constants, in other words, are the real mub of any Lie group. Given the generator commutators, and nothing else, we can directly work out the multiplets of the group and all their properties, with no further assistance. This is exactly what one does, for instance, in the usual treatment of the rotation group. From the angular momentum commutators (7) above, one determines the existence of simultaneous eigenstates $|jm\rangle$ of $J^3$ and $J_3$, the matrix elements of the generators with respect to these states,

$$\langle jm \mid J_\pm | j'm'\rangle = \epsilon_{\pm jjm'} (j+m) (j-m+1) \delta_{mm'} \delta_{jj'}, \quad (8)$$

the fact that $J^3$ has eigenvalues $j(j+1)$ for $j = 0, \frac{1}{2}, 1, \text{etc}$., with degeneracy $2j+1$, the values of the Clebsch–Gordan coefficients, and all the rest of the awe-inspiring Chapter 3 of Condon and Shortley.

This is an impressive amount of information to extract from three generators, but the list does not in fact exhaust their virtues, for of course the Hermitian operators $J$ represent important physical observables in their own right, independently of any group structure built on them. Nor is this a peculiarity of the rotation group. A very satisfying feature of all the symmetry groups of interest to physics is that in every case the group generators are Hermitian observables directly related to the symmetry described by the group; the group operators are simply unitary functions of these basic transformations on the relevant coordinates of the wave functions. Thus the translation group is the set of operators $\exp (ix \cdot p)$ with generators $p$. The "time-translation" group is $\exp (-i P t)$ with the single generator $H = i \partial/\partial t$. Together these four operators $p_\mu$ generate a four-dimensional translation group. The charge operators $T$ generate the isotopic-spin group, and $T$ together with five more operators on charge and strangeness generate the group SU$_3$, which is considered in detail in Sec. VII.

Finally, we define a characteristic of a Lie group

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8 A concise proof is given in Gubarev, Ref. 1, pp. 181, 185.

9 This is the sense in which the group operators are functions of the generators. The concise statement of the theorem is that the representations of the group are uniquely determined by the structure constants. Because the generators are only determined by the properties of the group operators in the immediate vicinity of the identity (i.e., for the $\alpha_i$ all small), it is perhaps not surprising that the structure constants do not uniquely determine the group operators for arbitrary $\alpha$. Fortunately, however, the representations are uniquely determined for all $\alpha$, and they are all a physicist uses.
which, as we shall see shortly, is of great importance. **Definition:** The rank of a Lie group is defined as the number of mutually commuting generators it possesses.

In an Abelian group like the translation group in which all the generators \((p_1, p_2, p_3)\) commute with each other, the rank is just the total number of generators. However in this case the commutativity allows us to write \(\exp(\mathbf{i}x \cdot \mathbf{p}) = \exp(i x p_1) \exp(i x p_2) \exp(i x p_3)\), and this rank-3 group is really just the product of three rank-1 groups. The more interesting situation is represented by the rotation group in which none of the generators commute (this is called rank 1), or \(SU_3\), which is of rank 2 \((T_3\) and strangeness \(S\) commute).

### E. Semisimple Lie Groups

Not all Lie groups, unfortunately, are good groups from the point of view of a physicist. Many have such unruly properties as possessing no finite-dimensional multiplets at all, or if they do, not permitting us to decompose an arbitrary invariant space into nonoverlapping multiplets. (The product of two doublets, for example, is a four-dimensional space. If the group is the rotation group, this decomposes into a singlet and a triplet—but an arbitrary group might not even have a singlet and triplet.)

Even the Lie groups, in other words, are too general for our purposes, and we need to specialize further to eliminate the unmanageable ones. The path which at the moment appears most reasonable is to restrict ourselves to the so-called “semisimple” Lie groups, since it is a set of groups whose properties are well studied mathematically, and which is large enough to include all symmetry groups studied so far.

Suppose that a group possesses a subgroup which is Abelian (its members all commute with each other), and whose members \(A\) have the property that \(BA, B^{-1} = A B\), another element in the subgroup, if \(B\) is any element of the full group. If such a subgroup (called an Abelian-invariant subgroup) exists, the full group is not semisimple; if it possesses no such subgroup (other than the identity element), it is a semisimple group. For instance, the group of all translations and rotations, a Lie group with the six generators \((p_1, p_2, p_3, J_3, J_\lambda, J_\sigma)\) called the translation–rotation group, has the translation group as a subgroup, and it is Abelian. Furthermore a bit of reflection shows that \(RTR^{-1}\) (\(R\) is a rotation and \(T\) a translation) is itself a pure translation—which means that the Abelian subgroup is invariant, and the translation–rotation group is not semisimple. (A tighter restriction is that the group be simple, which means that it possess no invariant subgroups at all, Abelian or otherwise. Only Theorem 4 requires simplicity, so we state the others on the basis of the less restrictive hypothesis of semisimplicity. The groups we use, such as \(SU_3\) and \(SU_2\), are actually simple.)

Just what one gains by considering only the semisimple Lie groups is surely not evident to the naked eye, but it is an important restriction and we indicate its main advantages by stating the important theorems it engenders.

**Theorem 2.** The number of semisimple Lie groups of a given rank is finite.

In fact, even small for rank 3. There is only one of rank 1, the group of unimodular (determinant = +1) unitary \(2 \times 2\) matrices (whose generators are the Pauli matrices). It is essentially the same group as \((\mathfrak{2} \times \mathfrak{1})\) homomorphic to, to be exact) our prototype, the rotation group. There are three semisimple Lie groups of rank 2, of which \(SU_3\) is one.

**Theorem 3 (Peter–Weyl Theorem).** Every finite-dimensional invariant space of a semisimple Lie group is completely decomposable into nonoverlapping multiplets.

This guarantees the existence of Clebsch–Gordan expansions. The result of multiplying two multiplets of a group together is a space which is invariant under the group, and the theorem assures that such a space is simply a sum of multiplets. For example, the product of two three-dimensional multiplets of \(SU_3\) is a nine-dimensional space which is decomposable into a sextet plus a triplet, or an octet plus a singlet, depending on whether the two triplets are identical or different.

In this context we also quote an important related theorem due to Cartan (see, e.g., Racah, Ref. 1, p. 39):

**Theorem 4.** Every simple Lie group of rank 1 possesses 1 “fundamental multiplet,” which have the property that every multiplet of the group can be obtained from the products of these multiplets with themselves.

Details of multiplet multiplication are given in Sec. V. We merely remark here that the fundamental multiplet of \(SU_3\) is the doublet; the isospin group is also \(SU_3\), and nuclei, to which the \(I\)-spin group was originally applied, are indeed all made of isotopic doublets. The fundamental multiplets of \(SU_3\) are the two triplets, and the essence of most current applications of \(SU_3\) to the classification of elementary particles is the hope that all the observed elementary particle multiplets can be constructed from these fundamental triplets.

### III. The Invariant Operators

The spherical harmonics \(Y_{lm}\) are labeled by two “quantum numbers” \(l\) and \(m\), which means that they are simultaneously eigenvectors of the two operators, \(J_3\) a generator of \(SU_3\) and \(J^3\). \(J^3\) is not a generator of the group. Rather, it is a function of all the generators which has the important property of commuting with each of them \(\{\text{and hence with all the members}\}

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10 L. Pontrjagin, see Ref. 1.
11 The original theorem [F. Peter and H. Weyl, Math. Ann. 97, 737 (1927)] assumes more (e.g., compactness of the group) in its hypothesis, and consequently is able to prove more (e.g., that all the multiplets of the group have finite dimension). The more restricted version quoted here, which is applicable to a larger class of physically interesting groups, is discussed in detail in the notes Continuous Groups in Quantum Mechanics, by W. Pauli (CERN, Geneva, Switzerland), CERN 56–31, pp. 10 and 11.
The importance of the operator $J^2$ from this point of view is that its degenerate eigenvectors are exactly the multiplets of the rotation group. In this sense they are "eigenmultiplets" of $J^2$ and are uniquely labeled by the eigenvalues $j(j+1)$. ($j=0$ is the singlet, $j=\frac{1}{2}$ the doublet etc.) This important property is not an idiosyncrasy of the rotation group, but, slightly generalized, is a characteristic of any semisimple Lie group.

In his Princeton lecture notes Racah (Ref. 1, p. 50) states the following basic theorem:

**Theorem 5.** For every semisimple Lie group of rank 1 there exists a set of $l$ "invariant operators." These are functions $C_\lambda(L_0, L_1, \cdots, L_l) (\lambda = 2, 3, \cdots, l+1)$ of the generators which commute with every operator of the group, and whose eigenvalues uniquely characterize the multiplets of the group.

We saw earlier that any operator which commutes with the entire group (the Hamiltonian was considered as an example) has every state in a multiplet of the group as an eigenstate, and all the states of a multiplet have the same eigenvalue—i.e., the operator is degenerate on every multiplet. This is true for all $l$ commuting operators $C_\lambda$ simultaneously, so that on a given multiplet these operators have just one set of eigenvalues $(c_\lambda, c_{\lambda+1}, \cdots, c_{\lambda+l})$. The importance of Racah's theorem is that it guarantees that one and only one multiplet will have such a set of eigenvalues, so that each multiplet of an arbitrary semisimple Lie group can be labeled uniquely by them, just as each multiplet of the rotation group (rank 1) is labeled by $j$.

In Sec. IIC it was found that a multiplet of a group is a set of closely related states which have the property that any one of them, $\psi$, can be obtained from any other, $\phi$, by an appropriate combination of operators of the group, $\psi = \sum a^\alpha \psi^\alpha \phi^\alpha$. Racah's theorem provides an alternative way of looking at this relationship: a multiplet is a set of simultaneous eigenstates of the $l$ invariant operators $(C_\lambda)$ of the group, all of which have the same set of eigenvalues, $(c_\lambda, c_{\lambda+1}, \cdots, c_{\lambda+l})$.

More importantly, this set of eigenvalues summarizes the symmetry properties of the entire multiplet (with respect to the symmetry group in question), in exactly the way the single eigenvalue $j$ characterizes the rotational symmetry properties of a multiplet of the rotation group. In fact, as we shall see from Theorem 7, we are guaranteed that the multiplets cannot have any further symmetry properties with respect to this group. Consequently, from this point of view, the invariant operators of a symmetry group take on a very fundamental significance: they represent all the essential symmetry properties of the group, and their eigenvalues $(c_\lambda, c_{\lambda+1}, \cdots, c_{\lambda+l})$ are the quantum numbers which uniquely identify and classify the multiplets of the group in terms of these symmetry properties.

The rotation group, then, is special only in having rank 1, so that one invariant operator suffices. For a rank-2 group, like $SU_3$, the single invariant corresponding to $J^2$ (i.e., the Casimir invariant) may have the same eigenvalue on different $SU_3$ multiplets, so that more operators are necessary to distinguish between such multiplets; Racah's theorem states that one more is sufficient. The 3 triplet, for example, is often designated as $D^3 (0, 1)$, the octet as $D^8 (1, 1)$, and one of the 15-dimensional multiplets as $D^{15} (2, 1)$. The numbers in parentheses are the eigenvalues of the two invariant operators of the group; all three multiplets have the same eigenvalue for the second one, but are successfully distinguished by the eigenvalue of the first one.

Biedenharn's recent work (Ref. 13) has recently shown that at least for the groups $SU_3$ (unitary unimodular $n \times n$ matrices) the invariants are simply homogeneous polynomials in the generators,

$$ C_\lambda = \sum_{\alpha} a^\alpha \psi^\alpha \psi^\alpha \cdots \psi^\alpha \tag{9} $$

whose coefficients $a^\alpha \psi^\alpha \psi^\alpha \cdots \psi^\alpha \psi^\alpha$ are prescribed functions of the structure constants of the group. The simplest, $C_2$, is a quadratic function of the generators, like $J^2 = \frac{1}{2} J_+ J_- + \frac{1}{2} J_- J_+ + J_z^2$; $C_3$ is third order in the generators, etc. The invariants of $SU_3$ are $C_2$ and $C_3$.

In spite of some of the explicit prescriptions it should be noted that the invariants of a group are not unique. If $C$ and $C'$ are invariants for a rank-$l$ group, $C = C + C'$ and $C = -C'$ (for instance) is another pair, which is just as adequate for labelling the multiplets as the first pair. If the operators of the group are unitary, we can take advantage of this freedom to assert the following:

**Theorem 6.** The invariant operators of a unitary semisimple Lie group can always be chosen Hermitian.

This is readily seen, for if $C$ is an invariant, $CU = UC$ for every $U$ in the group. The adjoint of this equation is $U^T C^T = C^T U^T$, but $U^T = U^{-1}$ for unitary operators, so $C^T$ commutes with all inverses, which by the group property is the same as commuting with the whole group. Thus if $C$ is an invariant, $C^T$ is as well, so instead of $C$ we can use, e.g., $(C + C^T)$, which is Hermitian. Since we are only interested in unitary groups, we assume all invariants to be Hermitian.

In spite of their nonuniqueness, an important prop-

---

property of any set of \( l \) independent operators is their completeness, in the following sense:

**Theorem 7.** Any operator which commutes with all the operators of a semisimple Lie group is necessarily a function of the invariant operators of the group.\(^{15}\)

The invariant operators, in other words, are the largest set which commutes with the group. This is a powerful and useful theorem. For instance, if a Hamiltonian operator possesses a certain symmetry, it commutes with the corresponding symmetry group, and Theorem 7 is often used to show that \( H \) is then a function of the invariants of the group. The spherically symmetric Hamiltonian of a spinless particle in a central force field commutes with the rotation group and so the part of it which operates on the angle coordinates of the particle (the parameters of the rotation group) must be a function of \( I^2 \), the invariant of the group; it is in fact \( I^2 \) itself. More significantly, the invariant of the isotropic spin group is \( T^2 \). Consequently we conclude that a charge-independent Hamiltonian for the nucleon-nucleon system (i.e., one which commutes with the isotropic spin group) can depend on \( I \)-spin operators only as a function of \( T^2 = \frac{1}{4}(\tau_1^2 + \tau_2^2) = \frac{1}{4}(6+2\tau_1 \cdot \tau_2). \) But because the squares of the \( \tau \) operators are unity, any function is simply a linear function, and the only charge-independent combination is the scalar \( \tau_1 \cdot \tau_2 \).

We conclude by listing the invariant operators for a few familiar symmetry groups. In doing so we remark that the simplest invariant operators of an Abelian group are just the generators themselves, since by definition they all commute with the group, and there are just \( l \) of them.

<table>
<thead>
<tr>
<th>Group</th>
<th>Generators</th>
<th>Rank</th>
<th>Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translation</td>
<td>( p )</td>
<td>3</td>
<td>( p )</td>
</tr>
<tr>
<td>Inversion</td>
<td>( J )</td>
<td>1</td>
<td>( J^2 )</td>
</tr>
<tr>
<td>Rotation</td>
<td>( J )</td>
<td>1</td>
<td>( J^2 )</td>
</tr>
<tr>
<td>Rotation–Inversion</td>
<td>( J \cdot p )</td>
<td>2</td>
<td>( J^2, p )</td>
</tr>
<tr>
<td>Isotropic Spin</td>
<td>( T )</td>
<td>1</td>
<td>( T^2 )</td>
</tr>
<tr>
<td>Rotation–Translation</td>
<td>( J \cdot p )</td>
<td>3</td>
<td>( J^2, p, J \cdot p )</td>
</tr>
</tbody>
</table>

Everything "checks" except the last group, which has rank 3, but only two invariants. It is at liberty to do so, however, since we saw earlier that its Abelian subgroup (the translations) is just the kind which takes the group out of the semisimple class, and Racah's theorem (5) is only stated for semisimple groups.

**IV. SYMMETRY GROUPS AND PHYSICS**

With this machinery in hand, we turn to the central problem in the theory of symmetry groups: how much can be learned about a physical system from its symmetry properties alone? The fullest answer can be given for the case in which the corresponding symmetry group of the Hamiltonian is a unitary semisimple Lie group, with \( n \) generators and \( l \) invariant operators (\( l \leq n \)), all chosen Hermitian. In this happy situation (it includes the symmetry groups which have so far had physical application), the apparatus assembled in the previous sections provides an unambiguous answer:

1. The system will have \( n \) conserved quantum numbers, defined by the generators of the symmetry group, \( l \) of which commute with each other. (Example for the case that the symmetry is charge independence, and the symmetry group is the isotopic spin group: \( T_1, T_2, T_3 \).)

2. It will also possess \( l \) more conserved quantum numbers, defined by the invariant operators of the symmetry group, all of which commute with each other (and with the generators). These \( l \) quantum numbers uniquely specify the multiplets of the group, including their dimensionalities. (Example: \( T^2 \); dimension of a multiplet: \( 2T^2+1 \).)

3. All the members of a multiplet of the symmetry group will have the same mass (energy eigenvalue). They will also have the same spin and parity if \( J^2 \) and \( P \) commute with the group. (Example: \( \pi^+ \), \( \pi^0 \), \( \pi^- \).)

4. In fact, the degenerate energy eigenstates of the system are exactly the multiplets of the full symmetry group.

\(^{14}\) By independent we mean that no two distinct multiplets of the group have the same eigenvalues; independent invariants suffice to label the multiplets of the group uniquely. For example, in SU\(_2\), \( C_0 \) and \( C_1 \), though they commute with the entire group, are not independent invariants.

\(^{15}\) The meaning of this is the following. If two operators \( A \) and \( B \) commute, they have a common set of eigenvectors \( |i \rangle \) and so can be written in terms of the projections onto these vectors and their respective eigenvalues, \( a_i \) and \( b_i \):

\[
A = \sum a_i |i \rangle \langle i |, \\
B = \sum b_i |i \rangle \langle i |.
\]

Let the degeneracy of \( a_i \) (the number of times it appears in the sum) be \( a_i \) and that of \( b_i \) be \( b_i \). Then if \( b_i \leq a_i \) for each \( i \), \( b_i \) can be thought of as a function of \( a_i \), \( b_i = f(a_i) \), and in this sense \( B \) is a function of \( A \), \( B = f(A) \). This is because specifying \( a_i \) determines a set of \( a_i \) vectors, and they in turn determine a value for \( b_i \), since \( b_i \leq a_i \) guarantees that these \( a_i \) vectors (and perhaps more) all have the same \( B \) eigenvalue; \( b_i \) is determined by \( a_i \), so \( b_i = f(a_i) \). (It is clear that the converse is not true unless \( a_i = b_i \) for all \( i \).)

We already know that any operator \( B \) which commutes with the entire group is at least degenerate on the multiplets of the group. But the invariants are the operators which are \emph{exactly} degenerate on the multiplets. Consequently specifying their eigenvalues determines a multiplet and so determines the eigenvalue of \( B \) on the multiplet, meaning that \( B \) is a function of the generators.
metry group; consequently the \( l \) eigenvalues of the invariant operators of the group provide a complete set of “good quantum numbers” for labeling the energy eigenvalues (bound states or resonances) of the system. [Example: The energy eigenvalues (masses) of a charge-independent system are labeled by \( T \).]

(5) All reactions which cause a transition from one multiplet of the system to another are forbidden, i.e., scattering and reactions occur only within a given multiplet. The phase shifts are the same for all states of a multiplet, and so can be labeled by the good quantum numbers of the multiplet. (Example: The \( T = \frac{1}{2} \) and \( T = \frac{3}{2} \) multiplets of the \( \pi N \) system scatter independently; their phase shifts are labeled by \( T \) and are independent of \( T \) within a multiplet.)

The remainder of this section proves and elaborates on these assertions. A way of doing this which keeps the wheat separated from the chaff is the theorem-proof format, since it conveniently distinguishes concise statements of fact from the machinery used in proving them.

To avoid the complication of handling more than one symmetry property at a time, we assume in this section that the system under consideration possesses only the symmetry \( S_\alpha \) (e.g., spherical symmetry) described by a single unitary semisimple Lie group (e.g., the rotation group). Direct products of two or more symmetry groups are discussed in Sec. V.

**Theorem A.** The statement that a system (or the interaction between two systems) possesses a certain symmetry \( S_\alpha \) is equivalent to the requirement that its Hamiltonian commute with every operator of the corresponding symmetry group \( G_\alpha \).

This in turn is equivalent to the condition

\[
[H, L_\alpha] = 0
\]

for every generator \( L_\alpha \) of the group. Consequently the existence of the symmetry \( S_\alpha \) establishes the (Hermitian) generators of \( G_\alpha \) as conserved (but in general noncommuting) physical observables.

\( G_\alpha \) is said to be a “symmetry group of the Hamiltonian” for such a system.

**Proof:** If a unitary transformation \( U \) (such as a rotation of axes) is performed on some coordinate of the system, all the wave functions of the Hilbert space are transformed to \( \psi' = U\psi \), and the Hamiltonian is replaced by \( H' = UHU^{-1} \). If the system is invariant under the transformation, its Hamiltonian is unchanged: \( H = H' = UHU^{-1} = HU^2 = UH \).

If \( U(\alpha)H = HU(\alpha) \) for all \( \alpha \), differentiation with respect to \( \alpha \) and reference to Eq. (4) (the definition of the generators) shows that \( L_\alpha H = H L_\alpha \). Conversely if \( H \) commutes with all the generators, it commutes with any \( U \) of the group, which is just a function of the generators.

**Discussion:** The reason \( H \) is singled out for special consideration is that it is the generator of the one-parameter “time-translation” group, the operators \( V(t) = \exp(-itH) \). The “rotated” time-development operator is \( V'(t) = U(\alpha)V(t)U^{-1}(\alpha) \), so the commutation of the group \( \{ U(\alpha) \} \) with \( H \) means that the time development of the system is exactly the same in all states which differ only by a “rotation” \( U(\alpha) \). If the Hamiltonian formalism is inapplicable, the remarks of footnote 5 apply.

As an example, the rotation group is a symmetry group of the Hamiltonian for any system not subject to external torques, i.e., a system possessing spherical symmetry. That this should be equivalent to \( [H, J] = 0 \) (the \( J \)'s being the generators of the group) is physically reasonable, since invariance under rotation about each of the three axes separately guarantees invariance under an arbitrary rotation.

In such a case we immediately have three conservation laws for the system, the conservation of \( J_1 \), \( J_2 \), and \( J_3 \). The fact that they fail to commute means that only one can be diagonalized at a time (or \( l \) in the general case of a symmetry group of rank \( l \)). The simultaneous conservation laws determined by the generators of \( G_\alpha \) define “linear quantum numbers” of the system (i.e., linear in the number of “particles” in the system), such as \( J_3, T_3 \) (charge) and \( S \) (strangeness); this will be clearer after direct product groups have been discussed.

If \( G_\alpha \) has two or more generators, the system will also possess \( l \) nonlinear commuting observables (which are also nonlinear functions of the generators). Even more interesting than the generators, they are the observables which “explain” the degeneracies in the energy spectrum of the system, and provide exactly the quantum numbers necessary to label energy levels and phase shifts by their symmetry properties. These observables are the \( l \) invariant operators \( \{ C_{l}, C_{l+1}, \ldots, C_{l+1} \} \) of the symmetry group—the generalizations of \( J^2 \) for the rotation group whose simultaneously degenerate eigenstates are the multiplets. Their physical properties are summarized in the following theorem.

**Theorem B.** If a system possesses a symmetry \( S_\alpha \), its Hamiltonian commutes with each of the \( l \) Hermitian invariant operators \( \{ C_{l}, C_{l+1}, \ldots, C_{l+1} \} \) of the symmetry group \( G_\alpha \),

\[
[C_{l}, H] = 0, \tag{11}
\]

so that each \( C_{l} \) is a conserved physical observable. Since the \( C_{l} \) commute with each other this establishes \( l \) simultaneous conservation laws for the system.

The \( l \) simultaneous eigenvalues \( \{ c_{l}, c_{l+1}, \ldots, c_{l+1} \} \) of the invariant operators are thus “good quantum numbers” of the system, conserved in all processes. Their great usefulness is due to the fact that they also uniquely label the multiplets of \( G_\alpha \); one important conclusion is that all transitions from one multiplet of \( G_\alpha \) to another are absolutely forbidden if the system possesses the symmetry \( S_\alpha \).

**Proof:** All the generators of \( G_\alpha \) commute with \( H \) if \( G_\alpha \) is a symmetry group of the Hamiltonian, so any
function of them, in particular $C_\lambda$ will as well. The lack of intermultiplet transitions follows from the conservation of the $\{c_\lambda\}$ (conservation of $P$ in the rotation-group case), since at least one $c_\lambda$ would have to change in such a transition. More explicitly, if multiplets $S$ and $S'$ are different, let $c_\lambda$ and $c_\lambda'\neq c_\lambda$ be the eigenvalues of a $C_\lambda$ on the two multiplets. Then

$$0 = \langle c_\lambda' \mid C_\lambda H - HC_\lambda \mid c_\lambda \rangle = (c_\lambda' - c_\lambda) \langle c_\lambda' \mid H \mid c_\lambda \rangle,$$

so that

$$\langle c_\lambda' \mid H \mid c_\lambda \rangle = 0,$$

(12)
i.e., the matrix elements of $H$ (and of the $S$ matrix, which is a function of $H$) vanish between any states from different multiplets.

**Discussion.** The eigenvalues $(c_0, c_1, \ldots, c_{14})$ are clearly destined to play a key role. From a mathematical point of view, as we saw in Sec. III, they completely characterize the essential symmetries of the group $G_\alpha$, and so classify the multiplets of $G_\alpha$ according to these symmetries. Their importance in physics now becomes clear from the fact that they are also automatically good quantum numbers of any system which has the symmetry $S$, and that it is their conservation which explains the lack of intermultiplet transitions.

All that remains is to understand the relationship of the multiplets themselves to the bound and scattering energy eigenstates of the system. It takes slightly different forms for positive and negative energy states, but both are special cases of a very general result known as:

**Schur's Lemma.** Any operator which commutes with every member of a group $G_\alpha$ has each vector in a multiplet of $G_\alpha$ as eigenvector, and is completely degenerate on each multiplet.

**Discussion.** The lemma was proved in Sec. II.C and describes an essential property of the invariant operators, which we have used repeatedly. Since both the Hamiltonian and the $S$ matrix of any system possessing the symmetry $S$ commute with the entire group $G_\alpha$, we can draw the following conclusions.

**Theorem C. If a system possesses a symmetry $S$, all its scattering states belonging to the same multiplet of $G_\alpha$ have the same phase shifts (or more generally the same $S$-matrix elements in the many-channel case).**

**Discussion.** The equality of the $S$-matrix elements follows directly from Schur's lemma, and that of the phase shifts from the fact that the phase shift in any channel is simply the logarithm of the diagonal $S$-matrix element in that channel.

As an example, a system composed of a nucleon (an isotopic doublet) and a sigma (a triplet) has wave functions which can be classified by the invariant operator $T^3$ (assuming charge independence) as belonging to the $T=\frac{1}{2}$ or $T=\frac{3}{2}$ multiplet. If they are scattering states, the two multiplets will scatter entirely independently of each other, with no transitions between them. The individual states within the multiplets are themselves labeled by $T_3$, a generator of the group; since it also commutes with the $S$ matrix, it is conserved (charge conservation), so that each $T_3$ eigenstate scatters independently of every other. The content of the present theorem is that all four charge states with $T=\frac{3}{2}$ have the same phase shift, as do the two $T=\frac{1}{2}$ states. Consequently the phase shifts, as well as the multiplets, can be labeled by the eigenvalues of $T^2$ [or the $l$ eigenvalues $(c_0, c_1, \ldots, c_{14})$ if the symmetry group has rank $l$]. For the bound states we have:

**Theorem D. If a system possesses a symmetry $S_\alpha$, all its bound (or resonant) states belonging to the same multiplet of $G_\alpha$ have the same energy eigenvalue, or mass.**

Conversely, the set of degenerate energy eigenstates corresponding to a single energy level of the system is either one multiplet or ("accidental degeneracy") a sum of multiplets of $G_\alpha$; it can never be part of a multiplet.

Excluding the case of accidental degeneracy, there is a one-to-one correspondence of multiplets to energy levels, so that the set of eigenvalues $(c_0, c_1, \ldots, c_{14})$ which labels the multiplets is exactly the set of quantum numbers necessary to identify the energy levels of the system; changing any of these quantum numbers necessarily changes the energy eigenvalue.

**Proof.** The degeneracy of the Hamiltonian on any multiplet of a symmetry group of the Hamiltonian was proved in Sec. II.C and is a special case of Schur's lemma. As for the converse, if $\Psi_\Phi$ is an energy eigenstate, $U\Psi_\Phi$ is another with the same energy ($U$ is any member of the symmetry group $G_\alpha$), for $H(U\Psi_\Phi) = UH\Psi_\Phi = U(E\Psi_\Phi)$. Consequently the vector space $V_B$ spanned by all the eigenvectors $\Psi_\Phi$ with the same energy eigenvalue is transformed into itself by any $U$ in $G_\alpha$. That is, it is an invariant subspace of $G_\alpha$ and so cannot be a part of a multiplet (which by definition contains no smaller invariant subspaces). In fact, according to the Peter–Weyl theorem (Theorem 3), knowing $G_\alpha$ to be semisimple guarantees that $V_B$ is either a single multiplet of $G_\alpha$ or at worst a sum of multiplets.

**Discussion.** This important theorem immediately explains the occurrence of degenerate energy levels as being a consequence of whatever special symmetries the system possesses—and at the same time provides the quantum numbers necessary to classify the energy levels according to these symmetries. For example, any bound system which is invariant under rotations and inversion has energy levels labeled by $(j\pi) (1^-, \frac{3}{2}^+, \ldots)$, with $(2j+1)$-fold degeneracies.

It also explains why "unnecessary" degeneracies such as, e.g., two levels with different $T$ at the same mass, are inherently suspicious. This is not implied by charge independence alone. Although it could be an accident, it could also (especially if there are more than two
V. SEVERAL SYMMETRIES: DIRECT PRODUCT GROUPS AND THE FULL SYMMETRY GROUP

Throughout Sec. IV we considered the existence of only a single type of symmetry, \( S_n \), described by a single group \( G_a \). In general a system may possess several distinct symmetries simultaneously, such as rotational symmetry, inversion symmetry, charge independence, baryon conservation, etc. Hence the question arises: What symmetries can a system possess simultaneously and what is the group, if any, which describes them all at once?

The simplest approach is to begin with the smallest symmetry group of the Hamiltonian, and ask how it can be extended to form a larger group, which will demand more symmetry of the system. Since the groups under consideration are Lie groups, the answer is straightforward: a Lie group is enlarged by adding more generators (i.e., more “coordinates” which the group can transform).

There are two distinct ways in which this can happen: either the added generators all commute with the invariant operators of the original group or they do not. If they do not, they will in general have matrix elements between different multiplets of the original group, i.e., the original multiplets will not be invariant under the operators of the larger group, and two or more of them will have to coalesce to form the multiplets of the new group, thus increasing the degeneracies of the energy levels. As described above, this is what happens if the isospin group is enlarged to form \( SU_3 \). \( T \) does not commute with the added generators and so is not an invariant operator of \( SU_3 \); states of different \( T \) can, and do, appear in the same \( SU_3 \) multiplet.

Alternatively, the original invariants may commute with the added generators; in this case they remain invariants of the new group, so their multiplets and associated energy levels remain distinct. The most important example of this situation is the case in which the added operators commute with the entire original group \( G_a \), rather than just with its invariants. In this case one readily sees that the extended set will be a group only if the added operators themselves form a group, \( G_b \). In the case that \( G_b \) is a Lie group, its generators will commute with those of \( G_a \), so the two groups define symmetries which are compatible with each other. (For example, rotations are compatible with

Inversion because \([ J, P ] = 0 \), but they are not compatible with translations because \([ J, p ] \neq 0 \); a linear-momentum eigenstate cannot, in general, also be an angular-momentum eigenstate.)

In this case one can construct a larger group which summarizes the total (but distinct) symmetry properties of \( G_a \) and \( G_b \) simultaneously. It is called the direct product group, \( G_a \otimes G_b \), and its elements consist of all possible products \( U(\alpha) V(\beta) \) of elements from the two factors\(^8\); it contains \( G_a \) in the form \( U(\alpha) V(0) \), where \( V(0) \) is the identity in \( G_b \) and \( G_b \) as subgroups. Each element of \( G_a \otimes G_b \) is specified by the double set of parameters \((\alpha, \beta)\), and its generators are

\[
\frac{\partial}{\partial \alpha_i} U(\alpha) V(\beta) \bigg|_{\alpha = 0, \beta = 0} = \frac{\partial}{\partial \alpha_i} U(\alpha) V(0) \bigg|_{\alpha = 0}
\]

and similarly for \( \beta_j \), which is clearly just the total set of generators of \( G_a \) and \( G_b \) taken together. Similarly, since the two sets commute with each other, the full set of mutually commuting generators for the direct product group consists of the corresponding sets for \( G_a \) and \( G_b \) taken together; hence the important conclusion, the rank of \( G_a \otimes G_b \) is the sum of the ranks of its factors. Now \( G_a \) has \( l_a \) invariants and \( G_b \) has \( l_b \) invariants. Each set commutes with the entire group, there are \((l_a + l_b)\) of them, and it will be clear shortly that they uniquely label the multiplets of the product group, so we conclude: The invariant operators of \( G_a \), together with those of \( G_b \), provide a complete set of invariants for \( G_a \otimes G_b \).

The wave functions \( \psi(\alpha, \beta) \) on which \( G_a \otimes G_b \) operates can be expanded in terms of the corresponding functions \( \psi(\alpha) \) and \( \phi(\beta) \) for the groups \( G_a \) and \( G_b \),

\[
\psi(\alpha, \beta) = \sum \sum a_{\alpha \beta} \psi(\alpha) \phi(\beta)
\]

so the product group operates in the space of product wave functions. For this reason it is natural to introduce the concept of a product of two multiplets. Given an \( m_a \)-dimensional multiplet \( \mathfrak{M}_a(\alpha) \) of \( G_a \) with an orthogonal basis \( (\psi_1, \psi_2, \ldots, \psi_{m_a}) \), and an \( m_b \)-dimensional multiplet \( \mathfrak{M}_b(\beta) \) of \( G_b \) (we consider products of multiplets of the same group in the next section) with an orthogonal basis \( (\phi_1, \phi_2, \ldots, \phi_{m_b}) \), we form the set of \( m_a m_b \) product vectors \( \psi \phi \). They are all orthogonal to each other, so the set of all their linear combinations is an \( m_a m_b \)-dimensional vector space, which is called the product \( \mathfrak{M}_a \otimes \mathfrak{M}_b \) of the two multiplets. Since \( \mathfrak{M}_a \) is invariant under \( G_a \) and \( \mathfrak{M}_b \) under \( G_b \), \( \mathfrak{M}_a \otimes \mathfrak{M}_b \) is invariant under \( G_a \otimes G_b \) — i.e., it is an invariant subspace of the direct product group. In fact, it is actually a multiplet of that group. This fact, as well as all other questions about the structure of the multiplets of the

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8 In fact, the direct product of any two commuting groups can be formed this way, whether they are Lie groups or not, a fact we shall use below.
direct product group, is settled by

**Theorem 8.** The product $\mathfrak{so}_3(\alpha) \otimes \mathfrak{sl}_2(\beta)$ of any two multiplets of commuting groups $G_a$ and $G_b$ is a multiplet of $G_a \otimes G_b$, of dimension $m_{ab}$. All multiplets of the direct product group can be constructed in this way, and they are labeled by all the invariants of $G_a$ and $G_b$ taken together.

This is nothing but a formal statement of a familiar situation. For instance, the strong interactions are charge-independent, i.e., they commute with the isospin group. They also commute with the rotation group, of course, and since these two commute with each other, we may describe both symmetry properties at once by the statement that the direct product of the two groups commutes with the strong interactions. This means (if we neglect any further symmetries) that the physical states will be grouped into mass-degenerate multiplets of this (rank 2) product group, which are identified by its two invariable operators $T^0$ and $J^z$. For instance, two of its six-dimensional multiplets are those identified as $(T=1, j=\frac{1}{2})$ and $(T=\frac{1}{2}, j=1)$, i.e., the product of an isospin triplet with a spin doublet, or vice versa. An example of the first is the $\Sigma$ hyperon (three charge states, two spin states) and of the second is the $K^\pm$ (885-Mev) meson (two charge states, three spin states). Increasing the isospin group to (isospin) \(\otimes\) (rotations) has not made different $T$ multiplets degenerate because of the commutativity of the two groups.

An important special case is that in which either $G_a$ or $G_b$ is an Abelian group (i.e., one whose members all commute with each other, like the one-dimensional rotation group), for it is a well-known theorem that all the multiplets of an Abelian group are singlets. Then since the product of a singlet with, say, a triplet is itself a triplet, we conclude that whenever $G_a$ is Abelian and commutes with $G_b$, the multiplets of $G_a \otimes G_b$ are identical in size with those of $G_b$. They will in general not be identical with the multiplets of $G_a$, for in addition to the invariant operators of $G_b$, they will also be labeled by the invariant operator of $G_a$. Increasing the symmetry group from $G_a$ to $G_b \otimes G_a$ has in this case not increased the multiplets in size, but merely added a quantum number to their label.

The inversion group is a prime example of this, for it consists of only $P$ and $I$, and so is certainly Abelian. If the strong interactions, e.g., were only rotation-invariant, nuclear energy levels would be labeled by $j$ and would be $(2j+1)$-fold degenerate, but the states would not be parity eigenstates. (The deuteron, as an example, would be a $j=1$ state composed of $S$, $P$, and $D$ waves.) Insisting on inversion invariance as well adds the parity of its degenerate states to the label on each energy level, and so provides two levels (of opposite parity) where before there was one, but each is still only $(2j+1)$-fold degenerate.

In exactly the same way, the group of gauge transformations corresponding, e.g., to baryon conservation, though not a Lie group, is Abelian, so that adding baryon conservation to the symmetry group of any system adds a quantum number to each state and energy level, but does not change the sizes of the multiplets of the group.

In summary, the energy levels and phase shifts of any system are labeled by its good quantum numbers, which are the eigenvalues of the invariant operators of all its commuting symmetry groups. This is most concisely stated by asserting the invariance of the system under its full symmetry group, which is simply the direct product of all these individual groups. Its rank is the sum of the ranks of the factors, and its invariant operators define exactly all the good quantum numbers which label the energy levels of the system, as well as their corresponding sets of degenerate eigenstates. The latter are just the multiplets of the full symmetry group.

At present it is well-established that the full symmetry group of the strong interactions is at least as large as the direct product: \((\text{rotations}) \otimes (\text{inversion}) \otimes (\text{baryon gauge transformations}) \otimes (\text{strangeness conservation}) \otimes (\text{isotopic spin})\)—all of which commute with each other—corresponding to the conservation of $J^z$, parity, baryon number, strangeness and $T^0$.

### VI. COMPOSITE SYSTEMS: PRODUCTS OF MULTIPLETS

In Sec. V, the simultaneous consideration of two distinct symmetry groups leads us naturally to contemplate products $\mathfrak{so}_3(\alpha) \otimes \mathfrak{sl}_2(\beta)$ of their multiplets.

Products of multiplets of the same group are of even greater importance in physics, but in an entirely different context, namely that of building “composite” systems out of “elementary” ones. As the simplest example, the He atom is composed of two $s_1$ electrons whose ground-state wave function is $2^{-1}(\alpha \beta_2 - \beta \alpha_2) \phi (r_1, r_2)$, i.e., a sum of products of states from two different doublets of the same (rotation) group, which itself is a singlet of this group. In the same way, the ground state of the deuteron, composed of a spin-$\frac{1}{2}$ neutron and a spin-$\frac{1}{2}$ proton, is a product of two rotational doublets which itself is a rotational triplet. At the same time it is an isospin multiplet (singlet in this case) composed of isospin doublets—and all other nuclei are as well, hence the importance of the isospin group in understanding nuclear levels.

To fix ideas, we again turn to the rotation group as an example and recall that, e.g., $\exp (i \alpha f_2)$ is the
operator which changes the azimuthal angle \( \varphi \) of any spatial function by \( \alpha \). In particular, \( f(\varphi)g(\varphi) \) is such a function, and

\[
\exp(i\alpha J_z)f(\varphi)g(\varphi) = f(\varphi + \alpha)g(\varphi + \alpha),
\]

i.e., we can write

\[
\exp(i\alpha J_z)[f(\varphi)g(\varphi)] = [\exp(i\alpha J_z)f(\varphi)][\exp(i\alpha J_z)g(\varphi)].
\]

This is clearly true for a rotation about an arbitrary axis as well; in particular,

\[
\exp(i\hat{\theta} \cdot \hat{J})[Y_{l \pm 1}(\theta, \varphi) Y_{l \pm 1}(\theta, \varphi)]
\]

\[
= [\exp(i\hat{\theta} \cdot \hat{J})Y_{l + 1}(\theta, \varphi)][\exp(i\hat{\theta} \cdot \hat{J})Y_{l + 1}(\theta, \varphi)].
\]

But each of these functions is a member of a multiplet of the rotation group, so that a rotation operator acting on it must produce another state in the same multiplet. Consequently if we label the rotational multiplets by \( l \) and consider the product \( \mathfrak{M}_{l_1} \otimes \mathfrak{M}_{l_2} \) of two multiplets of the rotation group (as defined in the previous section), this result tells us that a rotation operator acting on any element from the product produces another element in the product—i.e., \( \mathfrak{M}_{l_1} \otimes \mathfrak{M}_{l_2} \) is an invariant subspace of dimension \((2l_1+1)(2l_2+1)\) of the rotation group. It is different from the product of multiplets of two distinct and commuting groups in that it is generally not a single multiplet of the rotation group, i.e., it contains smaller invariant spaces. However since the rotation group is a semisimple Lie group, the Peter–Weyl theorem assures us that \( \mathfrak{M}_{l_1} \otimes \mathfrak{M}_{l_2} \) being a finite-dimensional invariant space of it, is decomposable into nonoverlapping multiplets, e.g., the familiar doublet \( \otimes \) doublet \( \rightarrow \) singlet \( \otimes \) triplet, etc. This is the vector addition of angular momentum and is stated in general as the theorem

\[
\mathfrak{M}_{l_1} \otimes \mathfrak{M}_{l_2} = \mathfrak{M}_{l_1+l_2} \oplus \mathfrak{M}_{l_1+l_2+1} \oplus \cdots \oplus \mathfrak{M}_{l_1+l_2}. \quad (13)
\]

It states that the \((2l_1+1)(2l_2+1)\)-dimensional space consisting of all products of members of \( \mathfrak{M}_{l_1} \) and \( \mathfrak{M}_{l_2} \) can be decomposed into the multiplets appearing on the right. Exactly the same type of thing is true for the multiplets of any semisimple Lie group, but the exact form of the decomposition must be worked out separately (from the commutation relations of the generators) for each group. The fact that each multiplet on the right of Eq. (13) appears only once is a peculiarity of the rotation group. In \( SU_3 \), e.g., \( 8 \otimes 8 = 27 \oplus 10 \oplus 10 \oplus 8 \oplus 8 \oplus 1 \), where the multiplets are identified, for simplicity, only by their dimensions rather than by the invariants of the group.

To be more explicit, the symbolic expansion (13) of the multiplets also implies the existence of the Clebsch–Gordan series (for half-integral as well as integral angular momentum),

\[
Y_{l \pm 1}(\theta, \varphi) Y_{l \pm 1}(\theta, \varphi)
\]

\[
= \sum_{l=-L}^{L} \sum_{M=-L}^{L} C_{m_1 m_2}^{m_1 m_2} Y_{LM}(\theta, \varphi) \quad (14)
\]

in which the coefficients weighting the terms on the right are the Clebsch–Gordan coefficients. They vanish unless \( M = m_1 + m_2 \), so that the sum is only over \( L \), with one term contributing from each multiplet. Again, such a series will exist for each group, the Clebsch–Gordan coefficients being determined, basically, by the structure constants.

In practice, one is usually more interested in products of multiplets of two different “versions” of the same group—e.g., the addition of spin and orbital angular momentum, or the addition of the angular momenta of two different particles. This is only a slight generalization of the above ideas. Consider, for definiteness, two “versions” of the rotation group corresponding to two spinless particles: the set of operators \( i\hat{a}_1 \cdot \hat{J}_1 \) and the set \( i\hat{a}_2 \cdot \hat{J}_2 \), where \( \hat{J}_1 \) are the angular momentum operators acting on the spatial coordinates of particle 1, and \( \hat{J}_2 \) are those of particle 2; \( \hat{a}_1 \) and \( \hat{a}_2 \) are the respective angles of rotation of these coordinates. If we call the first set the rotation group \( \mathfrak{G}_1 \) and the second set the rotation group \( \mathfrak{G}_2 \), then corresponding to the product \( \mathfrak{M}_{l_1} \otimes \mathfrak{M}_{l_2} \) it would seem to make sense to consider the direct product group \( \mathfrak{G}_1 \otimes \mathfrak{G}_2 \), consisting of the operators \( i(\hat{a}_1 \cdot \hat{J}_1 + \hat{a}_2 \cdot \hat{J}_2) \) (since \( \hat{J}_1 \) and \( \hat{J}_2 \) commute). However, this is clearly too large a group for the usual physical applications. It is a six-parameter Lie group (and so is certainly not the rotation group) which has the disconcerting property of rotating the coordinates of the two particles independently, i.e., by the different amounts \( \hat{a}_1 \) and \( \hat{a}_2 \). Any interaction between the two (spinless) particles will depend on \( |r_1 - r_2| \), which is not invariant under such independent rotations, so that \( \mathfrak{G}_1 \otimes \mathfrak{G}_2 \) is too large to be the symmetry group of the Hamiltonian for interacting particles.

\[
|r_1 - r_2| \quad \text{is invariant, though, under simultaneous (equal) rotations of } r_1 \text{ and } r_2 \text{ together. Consequently its symmetry group, which we might call } \mathfrak{G}_2(1, 2), \text{ is that subgroup of } \mathfrak{G}_2(1) \otimes \mathfrak{G}_2(2) \text{ composed of operators of the form } i(\hat{a}_1 \cdot \hat{J}_1 + \hat{a}_2 \cdot \hat{J}_2), \text{ with } \hat{a}_1 = \hat{a}_2. \text{ This is a three-parameter Lie group whose three generators } \hat{J}_1 = \hat{J}_2 \text{ are the “total angular-momentum operators” of the system. Using } [\hat{J}_1, \hat{J}_2] = 0, \text{ it is readily seen that they satisfy the same rotation group commutation relations as either } \hat{J}_1 \text{ or } \hat{J}_2, \text{ which confirms that they are angular momentum operators; consequently } \mathfrak{G}_2(1, 2) \text{ is indeed a rotation group, composed of rotations of the system as a whole.}

In these terms the simplest description of the “addition” of the angular momenta of the two particles is obtained by considering \( \mathfrak{M}_{l_1}(1) \) to be a multiplet of

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23 See M. Hamermesh or G. Ya. Lyubarskii, Ref. 1.
\[ \mathfrak{M}_L(1, 2) \text{ [which it is since } \exp (i \mathbf{L} \cdot \mathbf{J}_0) = 1 \text{ on states which depend only on } \mathbf{r}_0, \text{ and } \mathfrak{M}_L(2) \text{ as well. Then the product decomposition is in terms of multiplets of this same group, as in Eq. (13),} \]
\[ \mathfrak{M}_4(1) \otimes \mathfrak{M}_4(2) = \mathfrak{M}_4 \{ 1 \} + \mathfrak{M}_4 \{ 2 \} + \cdots + \mathfrak{M}_4 \{ 1 \} + \mathfrak{M}_4 \{ 2 \}, \]

in which the multiplets \( \mathfrak{M}_L(1, 2) \) are composed of members obtained from \( \mathfrak{M}_4(1) \) and \( \mathfrak{M}_4(2) \) by inverting the Clebsch–Gordan series (14),

\[ Y_{LM}^{i_1 i_2}(\tilde{\rho}_1, \tilde{\rho}_2) = \sum_{m_1 m_2} C_{m_1 m_2}^{i_1 i_2} Y_{1 m_1}(\tilde{\rho}_1) Y_{1 m_2}(\tilde{\rho}_2), \]

(16)

In this way “composite particle” states \( Y_{LM}^{i_1 i_2}(1, 2) \) are constructed of “elementary particle” states in such a way that the composite states are eigenstates of the invariants of the group we started with, and so form multiplets (mass-degenerate states) of this group.\(^{24}\)

The great power of this technique follows from the fact that we are assured that these composite states provide the correct angular dependence of the wave functions of the composite system, no matter what the detailed interaction between the particles, provided only that we know it is rotationally invariant (i.e., central in the case of spinless particles), and that \((i_1, i_2)\) is the only “configuration” present in the true function. At worst, the function will have the form

\[ \psi_{LM}(1, 2) = \sum_{i_1 i_2} \sum_{s_1 s_2} \alpha_{i_1 i_2} f_{i_1}(s_1) f_{i_2}(s_2) Y_{LM}^{i_1 i_2}(\tilde{\rho}_1, \tilde{\rho}_2) \]

(17)
in the case of configuration mixing.

In this sense the He-atom ground state is a rotational singlet composed of two rotational doubles, i.e., the singlet part of \( \mathfrak{M}_4(1) \otimes \mathfrak{M}_4(2) = \mathfrak{M}_6(1, 2) \otimes \mathfrak{M}_4(1, 2) \). The process is extendible, of course, to form lithium from \( \mathfrak{M}_4 \otimes \mathfrak{M}_4 \otimes \mathfrak{M}_4 \), and so on through the periodic system. In exactly the same way, nuclei are composed of \( Z \) protons and \( N \) neutrons, i.e., \( A = N + Z \) isotopic doubles, the deuterium ground state being the isotopic singlet part of \( \mathfrak{M}_4 \otimes \mathfrak{M}_4 \), and its (virtual) excited spin-singlet “state” being the triplet part of the same product, etc.

Finally, we mention the important Wigner–Eckart theorem. For the rotation group, its simplest form is obtainable from the Clebsch–Gordan expansion

\[ Y_{LM} | j m \rangle = \sum_{j' m'} \sum_{m''} C_{mm''}^{j j'} Y_{j' m''} | j m \rangle, \]

whose projection onto \( | j' m' \rangle \) is

\[ | j' m' \rangle | Y_{LM} | j m \rangle = C_{mm''}^{j j'} | j' m' \rangle | Y_{j' m''} \rangle. \]

(18)

But \( | j' m' \rangle \) and \( | Y_{j' m''} \rangle \) are both basis states of the \((2j'+1)\)-dimensional multiplet of the rotation group, with identical transformation properties under rotations. Writing \( \langle j' m' | Y_{j' m''} \rangle = \langle j' m' | 1 | j' m'' \rangle \) makes it a diagonal matrix element of an operator which commutes with the entire group, which by Schur’s lemma is independent of \( m' \). It can only depend on \( j', L \), and \( j \) and is generally written as \( (j' || Y_L | | j) \), the reduced matrix element.

The only property of \( Y_{LM} \) which was needed for this result was its rotation properties. Consequently any set of \( 2L+1 \) “objects” \( \{ T_{LM} \} \) (functions, operators, etc.), called spherical tensors, which transform under rotations the way the \( Y_{LM} \)’s do will satisfy the equation

\[ \langle j' m' | T_{LM} | j m \rangle = C_{mm''}^{j j'} (j' || T_L || j), \]

(19)

the Wigner–Eckart theorem for the rotation group.

Its significance is that the entire \( m \)-dependence of the matrix element is given by the Clebsch–Gordan coefficients. The reduced matrix element depends on the invariant operator quantum numbers \( j \) and \( j' \) which identify the multiplets concerned, as well as on any further (nonrotational) quantum numbers \( T_{LM} \) may possess.

An analogous result clearly holds for any group which permits Clebsch–Gordan expansions, with obvious modifications: (a) The reduced matrix element will be labeled by all the invariant operators of the group, and (b) there may be more than one reduced matrix element, for the appearance of each multiplet \( \mathfrak{M}_L \) only once in the decomposition of a product of multiplets into a sum [Eq. (13)] was a peculiarity of the rotation group. The “mass formula” for the \( SU_3 \) octet is a case in point, which contains two reduced matrix elements.

VII. NONGEOMETRIC SYMMETRY GROUPS: I SPIN, \( SU_3 \), AND BEYOND

The geometric symmetry groups are the rotation group, the space-translation group, the time-translation group, the space-inversion group, the time-reversal group and the Lorentz group. Wigner\(^{25}\) has recently drawn a careful distinction between these groups of operations on geometric coordinates and the physically (though not mathematically) rather different “dynamical” groups such as the isotopic spin group and \( SU_3 \), which are peculiar to certain interactions and which are most comfortably thought of as operating on discrete quantum numbers such as charge and strangeness, rather than on continuous, geometric-like coordinates. Since they occupy the center of the stage at the

\(^{24}\) The functions \( Y_{LM}^{i_1 i_2} \) are labeled by the invariants \( i_1 \) and \( i_2 \) of the components as well as by the invariant \( L \) of the result. This means that the multiplets \( \mathfrak{M}_L \) (1, 2) should really be labeled \( \mathfrak{M}^{i_1 i_2} \) (1, 2), so that there will in general be many \( \mathfrak{M}^{i_1 i_2} \)'s for a given \( L \), all with identical rotation properties, but differing in other respects. A \( P \) state constructed of particles in \( s \) and \( p \) states is quite different from a \( P \) state constructed from \( d \) and \( f \) states. In shell-model terms the two \( P \) states are described as belonging to different configurations.

A. The Isotopic-Spin Group: Charge and Its Operators

The generators of the rotation group are \( J_\theta \), \( J_\phi \), and \( J_\phi \). One can consider them, as we have throughout this discussion, as operators on the continuous coordinates \( \theta \) and \( \phi \). Alternatively, however, they are just as reasonably viewed as operators on the discrete quantum number \( m \), the three-component of angular momentum. \( J_\theta \) is the (Hermitian) operator which defines it, and \( J_\phi \) and \( J_\phi \) are the operators which "raise" and "lower" it. In this alternative light the rotation group becomes a certain set of unitary operators \( \exp (i \theta \cdot \mathbf{J}) \) which do peculiar things to \( J_\phi \) eigenstates. The physical meaning of the group, in fact, becomes rather dim from this point of view, and its principal usefulness seems to be confined to defining the multiplets and determining the decomposition of a product of them, i.e., the Clebsch–Gordan series.

It is exactly in this light, however, that we wish to view the isotopic-spin group, not primarily as a set of operators \( \exp (i \mathbf{a} \cdot \mathbf{T}) \) which perform "rotations in charge space," but rather as the group generated by certain physically obvious operations on the charges of particles, a group which serves primarily to determine the properties of composite charged systems and provide the Clebsch–Gordan coefficients necessary to construct them.

The structure of the group can be introduced in a variety of ways, such as postulating it outright, or insisting that it be a simple Lie group of rank 1 (there is only one such). A more palatable approach would be to determine as many of its multiplets as possible experimentally and use the fact that multiplets are to a certain extent characteristic of the group which determines them in order to make an intelligent guess at the group structure. Getting the multiplets right is the final test to which any promising group must be subjected, but since the \( I \)-spin group is so simple we feel it worthwhile to emphasize a somewhat more direct route from the experimental data on families of charged particles to the group which determines their multiplet structure.

It begins with the assertion that charge exists, as a measurable, quantized observable, so that there must be a Hermitian operator \( \hat{Q} \) which counts it. \( \hat{Q} \) has (normalized) eigenstates and eigenvalues which we write as

\[
\hat{Q} \left| m \right\rangle = m \left| m \right\rangle.
\]  

From the further experimental evidence that it is absolutely conserved,

\[
[\hat{Q}, \hat{H}] = 0
\]  

must be true for any admissible Hamiltonian.

Since certain particles have more than one charge state (as indicated by the existence of \( \beta \)-decay and charge-exchange reactions), there is a clear and present need in the formalism for operators which raise and lower charge (by one unit at a time, to fit the observed particles). If the particle under consideration possesses a higher charge state than \( \left| m \right\rangle \), define

\[
\hat{Q}_+ \left| m \right\rangle = \lambda_m \left| m+1 \right\rangle
\]  

where \( \lambda_m \) is a number, from which

\[
[\hat{Q}, \hat{Q}_+] \left| m \right\rangle = \left( (m+1)\lambda_m - \lambda_m m \right) \left| m+1 \right\rangle = \lambda_m \left| m+1 \right\rangle - \hat{Q}_+ \left| m \right\rangle
\]

If \( \left| m \right\rangle \) is the highest charge state, we define \( \hat{Q}_- \left| m \right\rangle = 0 \), so that Eq. (23) holds for all states and is an operator identity. Doing the same thing for \( \hat{Q}_- \), the results are

\[
[\hat{Q}, \hat{Q}_\pm] = \pm \hat{Q}_\mp
\]

which express nothing but the definition of the \( \hat{Q}_\pm \) operators. Since this is just the form of the commutation relations between the generators of a Lie group, one might reasonably suspect that the \( \hat{Q} \)'s could generate such a group. Without further restrictions, however, it would not be a very useful group, for Eq. (24) merely states that charge can be raised and lowered without limit (translation group on a lattice), and the multiplets of the group would be infinite dimensional.

What is missing is the commutator \( [\hat{Q}_+, \hat{Q}_-] \). One readily sees from Eq. (24) that \( [\hat{Q}_+, \hat{Q}_-] = 0 \), i.e., \( [\hat{Q}_+, \hat{Q}_-] \) is an operator with the same eigenstates as \( \hat{Q} \) which can be written symbolically as

\[
[\hat{Q}_+, \hat{Q}_-] = f(\hat{Q})
\]

To determine what function it is of \( \hat{Q} \) (subject to the condition that its trace vanish), more information is needed about the nature of the charged systems the formalism is to describe.

The exact information necessary is not entirely clear, for apparently different restrictions lead to the same result.

\* One reasonable condition, for instance, kindly pointed out by Dr. S. Epstein, is the requirement that the operators \( \hat{Q}_+, \hat{Q}_- \) and \( \hat{Q}_0 \) for a composite system be expressible in terms of the charge operators of its components in the linear fashion, \( \hat{Q} = \Sigma \hat{Q}(i) \) and \( \hat{Q}_0 = \Sigma \hat{Q}_0(i) \), just as \( \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \) for the angular momentum operators of a composite system. Demanding in addition that Eq. (25) be satisfied by the composite operators determines \( f(\hat{Q}) \) to be a linear function.
to be constructed. In group theory terms this means the identification of the fundamental multiplet of the group, from which all others are to be constructed by multiplication.

In the case of nuclear physics we are fortunately aware of the fact that all nuclei are composed of neutrons and protons, i.e., of charge doublets. Similar evidence on the composition, if any, of the "elementary" particles is so far lacking. What is known is that charge multiplets do exist (\(\Sigma^+, \Sigma^0, \Sigma^-, K^+, K^0, p^+, p^0, p^-; \) etc.), which empirically combine with each other via the same Clebsch–Gordan coefficients as those which describe nuclear charge states. This strongly suggests that the same symmetry group is active in both cases, and consequently many current attempts to understand the applicability of the I-spin group to the elementary particle charge multiplets also construct them out of charge doublets (e.g., in its simplest version \(\pi^{\pm}\) as the triplet part of \(\bar{N} \otimes N, \) etc.).

Our question is then: Is the insistence that the charge group possesses a two-dimensional multiplet adequate to determine the commutator of Eq. (25) and specify the group uniquely? The answer is yes, and it follows directly from the familiar properties of the two-dimensional representation of the group (the matrices of the \(Q\)'s relative to the doublet states). Using the two charge states as a basis for the space, \(Q\) is diagonal,

\[
Q = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}
\]  

(26)

with the charges of these states as eigenvalues. From Eq. (20),

\[
Q_+ = \begin{pmatrix} 0 & \lambda' \\ 0 & 0 \end{pmatrix},
\]  

(27a)

and similarly

\[
Q_- = \begin{pmatrix} 0 & 0 \\ \lambda' & 0 \end{pmatrix}.
\]  

(27b)

Consequently

\[
Q_+ Q_- = \lambda' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q_+ (Q_- m_2 1),
\]  

\[
Q_- Q_+ = \lambda' \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Q_- m_1 1. \]  

(28)

Eq. (24) is homogeneous in \(Q_{\pm}\), and so is unchanged by a renormalization of these operators. It is clearly simplest to take \(\lambda\lambda' = 1\), which makes

\[
[Q_+, Q_-] = \frac{2}{\langle m_1 - m_2 \rangle} [Q - \frac{1}{2}(m_1 + m_2) 1].
\]

(29)

But the basic Eq. (20) clearly holds only if \(m_1 - m_2 = \pm 1\), and we choose the order which makes it +1, so that

\[
[Q_+, Q_-] = 2 [Q - \frac{1}{2}(m_1 + m_2) 1].
\]

Considering \(Q_+, Q_-\), and \(Q\) as directly observable operators on charge, the definition of \(Q_{\pm}\) and the requirement that all charged systems be constructed of charge doublets imply Eqs. (24) and (29), which show these operators to be closed under commutation. This means that they (without the help of any further operators) automatically generate a Lie group (of "rotations in charge space"), the isotopic-spin group, which is clearly of rank 1. A slight change in definition makes the generators traceless on any multiplet of the group, and since there is only one such group (SU_2, the rotation group), the I-spin group must be isomorphic to it. The doublet is the fundamental multiplet of SU_2, so the multiplets obtained by products of the charge doublet with itself are precisely all the multiplets of SU_2: one singlet, one doublet, one triplet, etc.

To change to the more customary notation, note that for the antiparticles of the doublet, Eq. (29) is changed only in that \((m_1 + m_2) \rightarrow -(m_1 + m_2)\). Introducing the hypercharge, \(Y = S + B\), which is \(Y = -Y\) for the antiparticle, Eq. (29) can be written for a \(Y=1\) doublet as

\[
[Q_+, Q_-] = 2(Q - m_1 + m_2 \frac{1}{2} Y 1),
\]  

(30)

which we transform to the customary isotopic-spin notation by noting that Eq. (24) is unchanged by the addition of a constant to \(Q\):

\[
T_{\pm} = Q_{\pm}, \quad T_3 = Q - m_1 + m_2 \frac{1}{2} Y.
\]  

(31)

Then

\[
[T_3, T_{\pm}] = \pm T_{\pm}, \quad [T_3, T_-] = 2 T_3,
\]  

(32)

and the isomorphism with the rotation group is obvi-
ous by comparison with Eq. (7). (Customarily the
doublet has isomorphisms 1 and 0, so that $T_3 = Q - \frac{1}{2} Y$.)

This isomorphism answers all further questions about the
group: $T_3$ uniquely answers basis states within a
multiplet; the invariant operator which labels the multipllets themselves is $T^2 = \frac{1}{2} (T_1^2 + T_2^2 + T_3^2)$; the
multiplets have dimension $(2T+1)$; and the Clebsch–
Gordan coefficients are the same as those for the rotation
group.

Composite charge states (nuclei, "elementary" particles) are constructed from products of the doublet
with itself by the Clebsch–Gordan techniques used for the
rotation group in Sec. VI. This dictates the relation
$T = \sum T_i$ for the isotopic spin operators of the composite
particle in terms of those of its constituents, and
$Y = \sum Y_i$ for its hypercharge. The commutation relations
(32) then hold for $T$ on any multiplet, but Eqs.
(28), a special property of the fundamental representation
(i.e., the Pauli matrices), do not.

The charge independence of the interactions of the doublet is stated concisely as

$$\left[(T_i)_{\pm}, H\right] = 0,$$

which together with charge conservation means
$[T_i, H] = 0$; the entire $I$-spin group commutes with $H$ and so is a symmetry group of the Hamiltonian.

Furthermore this implies $[T_i, H] = 0$ for a system composed of these doublets, so that it is charge-independent and all of the theorems of Sec. IV immediately apply. In particular,

$$[T_{\pm}, H] = 0$$

makes it abundantly clear that the mass of such a system is indeed independent of its charge ($T_3$, position in a multiplet), although not of $T$.

B. $SU_3$, etc.

1. An Analogy with the Shell Model

The search for a pattern of family relationships be-
 tween the "elementary" particles, which would help
us to understand them in the way we understand the
Balmer series or the occurrence of magic nuclei, has
been under way for more than ten years. It has ranged
in approach from postulated relations between coupling
constants (global symmetry) to specific composite-particle models (e.g., the Sakata model), neither of
which has proved very successful. As more particles
have come to light, it has been realized that the families,
if they exist, will not be small ones, and recently
a somewhat more modest group-theoretic approach has been attempted, based on the hope that the particles
can be classified by the "broken multiplets" of some
symmetry group, even though the origin of the group and
the nature of the symmetry-breaking interaction may remain a mystery for the present.

The masses, spins, parities, strangeness, and $I$-spins
of a large number of levels are the experimental data
to be classified. The problem is clearly a spectroscopic
one, and the present group-theoretic approach bears a
remarkable resemblance to the classic tool of atomic
and nuclear spectroscopy, the shell model. The analogy
is so close, in fact, that the simplest way of intro-
ducing the terminology necessary for a discussion of
elementary particles in terms of $SU_3$ is by a brief recapitulation of the shell-model point of view in symmetry-group language.

The shell model is a two-step theory of atomic or
nuclear energy levels whose first step consists of re-
placing the actual many-particle Hamiltonian by a
self-consistent single-particle approximation,

$$H(1, 2, \cdots, N) \rightarrow H^0 = \sum_{i=1}^{N} H^0(i),$$

$H^0(i)$ being the average one-particle interaction felt
by particle $i$. If the $H^0$ problem is soluble and provides
a reasonably accurate approximation to the true eigen-
states, the second step is to put back the interaction
between the particles, $H = H^0 + H'$, and treat it, e.g.,
by low-order perturbation theory.

Suppose, for simplicity, that the spins of the particles
are neglected and $H^0(i)$ is spherically symmetric
(as is near a closed shell). Its symmetry group is
then the rotation group on $r_i$, $\theta_{3}(i)$; because of the
absence of coupling between particles, $^{39}$ the symmetry
group of $H^0$ is

$$S^0 = \theta_2(1) \otimes \theta_2(2) \otimes \cdots \otimes \theta_2(N),$$

the group of independent rotations on the coordinates of
all the particles. This is an extremely large group (corresponding to the high symmetry of $H^0$), which
contains as a small subgroup the symmetry group
$S = \theta_2(1, 2, \cdots, N)$ of the correct Hamiltonian $H =
H^0 + H'$. $\theta_2(1, 2, \cdots, N)$ is the group of simultaneous
rotations of the coordinates of all the particles, i.e.,
rotations of the entire system; its invariant operator is

$$L^2 = \sum_{i=1}^{N} L_i^2,$$

with eigenvalue $L(L+1)$.

The large multiplets of $S^0$, called supermultiplets,
are composed (Theorem D) of the degenerate eigen-
states of $H^0$. These supermultiplets are invariant under
the subgroup $S'$, and so by the Peter–Weyl Theorem
must each consist of a set of multiplets of $S'$. But
each multiplet of $S'$, labeled by $L$, corresponds (by
Theorem D) to a single energy level of $H$, hence the
famous result: if a supermultiplet of $S^0$ contains

$^{39}$ A point which has been made with incisive humor by H. J.
Lipkin.

$^{40}$ Actually a complete lack of coupling is not necessary; e.g., a
factorable interaction, $V_{ii} = V(r_i) W(r_i)$, is also invariant under
the direct product group. It is unphysical for the rotation group,
but might not be for an "internal symmetry" (charge, strange-
ness, etc.) group.
multiplets of $S'$, turning on the symmetry-breaking interaction $H'$ will cause the corresponding energy level of $H^0$ to split into $n$ distinct levels of $H$. (A representative example is shown in Fig. 1.)

As an example phrased more explicitly in shell-model terms, consider two particles outside a closed shell, and for simplicity neglect entirely interactions between these particles and the "inert" closed-shell particles (since distant levels interact least anyway). Neglecting the interaction between the outer particles as well gives the zero-order Hamiltonian, $H^0 = H^0(1) + H^0(2)$, whose eigenstates are of course products of the eigenstates of $H^0(1)$ and $H^0(2)$. That is, since $S^0 = \mathcal{S}_0(1) \otimes \mathcal{S}_0(2)$, its supermultiplets are $\mathcal{S}_0(1) \otimes \mathcal{S}_0(2)$ by Theorem 8. They are the familiar "configurations" of the shell model [labeled by $(l_1, l_2)$, the invariant operators of $S^0$], a typical one being $(p_d)$, the $3 \times 5 = 15$-dimensional set of states obtained by putting one particle in a $p$ state and one in a $d$ state. They are all degenerate in this limit, with energy $E^0 = E^0_p + E^0_d$.

An interaction $H' = V_{12}$ between the two particles has as its symmetry group the subgroups $S' = \mathcal{S}_2(1, 2)$ of $S^0$, whose multiplets are labeled by its quantum number $L$. $(2L+1)$ serves as well, and in terms of it the 15-dimensional supermultiplet of $S^0$ decomposes into three multiplets of $S'$ according to $3 \otimes S = 3 \otimes S \oplus 7$.\footnote{Finding the multiplets of $S'$, the group of the perturbation, is the familiar process of finding the correct zero-order wave functions of degenerate perturbation theory.} Knowing nothing about the symmetry-breaking interaction $H'$ except its invariance under simultaneous rotations of $r_1$ and $r_2$, we can conclude that it will split the $(p_d)$ configuration into $P$, $D$ and $F$ states. (More explicitly, $1^-$, $2^-$, and $3^-$ states, since the entire configuration has odd parity.)

If its symmetry-breaking properties (i.e., its transformation properties under the full group $S^0$) were known in detail, one could use a generalization of the Wigner–Eckart theorem for $S^0$ to find the dependence of these energies on their $S'$ quantum number $L$ (to first order in $V_{12}$),

$$E^0 + \Delta E_L (p_d).$$

(35)

The dependence of $\Delta E_L (p_d)$ on $L$ occurs entirely in $S^0$ Clebsch–Gordan or Racah coefficients;\footnote{In this context, $L$ serves to identify certain states within an $S^0$ supermultiplet. It is not an invariant operator of $S^0$, and so does not appear in the $S^0$ reduced matrix elements.} while $V_{12}$ occurs as a factor in the reduced matrix elements. In the special case that $V_{12}$ transforms like a member of a single multiplet of $S^0$, there is only one such factor. Although it determines the absolute size of the splittings, in this case it divides out of the splitting ratio, $(E_a - E_b)/(E_c - E_d)$, which depends only on $(l_1, l_2)$, the configuration. Eq. (35) is accurate, of course, only if $H'$ is small compared to $\Delta E$, the separation of this configuration from the nearest one which can mix with it, since $H'/\Delta E$ measures the contribution of the second-order energy shift.

2. Elementary Particles Again

Getting back to the case at hand, the disconcerting aspect of the elementary particle situation at the moment is the great profusion of energy levels which abound, with as yet no clearly discernible pattern relating them. The search which is under way at present is for an understanding of these states at exactly the level of our understanding of the above $1^-$, $2^-$, and $3^-$ states. The "pattern" which related them was their common origin in a single supermultiplet of a larger group, whose symmetry was broken by that "perturbing" interaction which moved them into their final, observed, positions. Meager consolation though this may be to the empirically minded observer, it is the stuff of which spectroscopies are made. The shell models, of course, have a physically convincing framework on which to build. Until a comparable physical substructure emerges to unify our picture of the elementary particle world, a rather blind faith in the benevolence of dimly guessed at groups seems to be all we have to light the way: a shell model with imagined particles being put into unknown shells.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Level splitting and multiplet structure. In the approximation that the Hamiltonian of the system is $H^0$, the energy levels are the eigenvalues of $H^0$ and the (super) multiplets of its symmetry group $S^0$ are the large sets of degenerate $H^0$ eigenstates. Turning on a symmetry-breaking interaction $H'$ reduces the symmetry group to $S'$, a subgroup of $S^0$. The number of levels into which each $H^0$ level splits is equal to the number of $S'$ multiplets contained in the corresponding $S^0$ supermultiplet. In the two superoctets shown, the splitting is assumed to fit the Gell-Mann–Okubo mass formula (Ref. 34), $M(T, V) = M_0 + M_1 Y + M_2 T(T+1) - Y^2/4$; purely for illustrative purposes the unperturbed mass is taken to be $M_0$. The dimensionsality of the multiplets of the levels are indicated in parentheses.}
\end{figure}
What, then, is the Big Group \( S^0 \), and how are we to recognize its supermultiplets in the barrage of levels confronting us? Since spin, parity, baryon number, isotopic spin, and strangeness are empirically conserved by the strong interactions, the direct product of their groups is \( S' \), the symmetry group whose quantum numbers label the observed strong-interaction energy levels. If we suppose, as is commonly done, that the operators which are added to \( S' \) in enlarging it to \( S^0 \) all commute with \( J^2, P, \) and \( B \), then the particles of each supermultiplet of \( S^0 \), although they may have different isotopic spin and/or strangeness, will at least all have the same \( J, \pi, \) and \( B \).

If this is the case, we could hope that the symmetry-breaking interaction \( H' \) might be small compared to the distance between \( H^0 \) levels, so that the pattern by which the supermultiplets could be recognized would be a spectrum of well-separated "levels," each with a "fine structure." The fine structure would be several levels with the same \( J, \pi, \) and \( B \), but with different \( T \) and/or strangeness, and the total multiplicity of such a fine-structure level group would be the dimension of one of the supermultiplets of \( S^0 \). With a few such supermultiplets as clues, one might be able to guess the identity of \( S^0 \) with some confidence.

Unfortunately, \( H' \) shows signs of being large, and of producing level shifts in some cases comparable to the rest energies of the observed particles. The supermultiplet pattern in the observed spectrum is not nearly as simple or obvious as one might have hoped, and a more ambitious technique will evidently be necessary to discern the pattern, if it exists. The one being employed at the moment is to do all one's guessing straightforwardly. The idea is to choose a promising group with reasonable multiplets, and at the same time assume that \( H' \), the symmetry-breaking interaction, transforms like a single multiplet of the group. From it, the ratios of the fine-structure level splittings can be calculated for each supermultiplet by a formula analogous to Eq. (35), which is called the "mass formula" for those assumptions. These splitting ratios then provide the pattern to which a set of levels must fall if they are to be associated as a supermultiplet of the assumed group (with the assumed splitting mechanism).

Using this technique to identify supermultiplets, it has been found possible to group many of the presently known levels into multiplets of a single, quite famous group. It is the group \( SU_3 \), a rank-2 simple Lie group, considered as a symmetry group on the charge and strangeness quantum numbers. At the moment it looks like an excellent candidate for \( S^0 \), and we close with a brief review of its mathematical properties and a glance at its physical interpretation.

We begin by asking, must the guesses at \( S^0 \) be made completely at random, or are there general physical considerations which limit the range of possibilities? A natural first step is to inquire whether all the operators added in enlarging \( S' \) to \( S^0 \) could commute with \( S' \). This can be settled at once, for if they did, we saw in Sec. V that the added operators would themselves form a group (call it \( \bar{S} \)). In this case \( S^0 \) would be \( S' \otimes \bar{S} \), and its multiplets would be the products of the multiplets of the two direct factor groups, so that all the particles of a given supermultiplet would have exactly the same \( S' \) quantum numbers. This would mean, e.g., that the nucleon could not be in the same supermultiplet with the \( \Sigma \) (wrong \( T \) and \( S \)) nor even with the cascade particle (wrong \( S \)), but could only associate with other particles having the same \( J, \pi, B, T, \) and \( S \). The existence of supermultiplets of this nature is not inconceivable, but as yet there is no evidence in their favor.

In order to produce the more conventional type of supermultiplet, \( S^0 \) must contain operators which do not commute with all the operators of \( S' \), for it is this noncommutation which mixes \( S' \) multiplets [Eq. (12)] and forces two or more of them to collaborate to form the supermultiplets of \( S^0 \). Conventionally, one has assumed even the higher symmetry part of the interaction to conserve \( J, \pi, \) and \( B \). That is, its symmetry group \( S^0 \) is assumed to contain the rotation, inversion, and baryon gauge groups as direct factors, so that its supermultiplets will be composed of particles all having the same \( J, \pi, \) and \( B \). This assumption narrows the choice of operators which must be added in enlarging \( S' \) to \( S^0 \), to those which commute with \( J, \pi, \) and \( B \), but do not commute with \( S \) and/or the \( I \)-spin group. In this sense \( J, \pi, \) and \( B \) are "inert." They play exactly the same role in \( S' \) and \( S^0 \), so we "factor them out" of both groups and keep for further consideration only the remaining factors \( \bar{S} \) (\( I \)-spin\( \otimes \)strangeness) and its higher symmetry enlargement \( \bar{S}^0 \).

Finally, \( \bar{S}^0 \) is unitary by physical assumption, and can without loss of generality be taken unimodular (all its operators have determinant \( = \pm 1 \) on any multiplet), since this merely corresponds to factoring off explicitly any Abelian direct factor group (e.g., baryon gauge group) it may contain. If \( \bar{S}^0 \) is a Lie group, its rank must be at least 2, since that is the rank of \( \bar{S} \).

This seems to be as far as we can go on general considerations alone, and there are still plenty of feasible groups in the running. Our understanding of why the particular group \( SU_3 \) seems to have been chosen

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22 This is only a first-order perturbation result, and in view of the large level-shifts mentioned, might be suspect. Its accuracy is governed, though, by the ratio of the shift of a given level \( E_{\text{shift}} \) to the distance of the nearest supermultiplet which has a level with the same quantum numbers as \( E_{\text{shift}} \). Since this distance (in general greater than the separation of adjacent levels of the \( H^0 \) spectrum) is not yet known, there is no direct evidence that a first-order calculation must be inaccurate. The baryon octet and decuplet (which have different spins), e.g., can interchange without "distorting" one another in the slightest.

23 The \( \Sigma \) hyperon, e.g., was considered in this light in Sec. V as a six-dimensional multiplet of the \( \text{spin}\bar{S} \) (\( I \)-spin) group, in which all six members of the multiplet have the same \( J \) and the same \( T \).
for glory is practically nil at the moment. However in casting about for a possible explanation, one is struck by the suggestiveness of the assertion: The smallest group \( S^0 \) which satisfies the above conditions, contains the known \( S^1 \), and would permit all strongly interacting particles to be composed of triplets, is \( SU_3 \).

The recognition of this fact clearly played a part in motivating the original introduction\(^{14}\) of \( SU_3 \) as a candidate for \( S^0 \). Its final role in the scheme of things is by no means clear, but at present the group is being most actively exploited in composite-particle models, in which the observed states are constructed of certain mysterious fundamental triplets,\(^{15,16}\) in much the way nuclei are constructed of fundamental \( I \)-spin doublets.\(^{17}\)

The argument behind the assertion that triplets imply \( SU_3 \) can be formulated as follows. \( S^0 \) is a group of unitary operators, which can always be written as exponentials of Hermitian operators. Just as our previous argument that doublets imply \( SU_2 \) for the charge group was phrased in terms of the Hermitian charge operators, it is also most convenient to discuss triplets in terms of the Hermitian charge and strangeness operators. The requirement that \( S^0 \) possess a three-dimensional representation (triplet) is very stringent, for all the Hermitian 3X3 matrices, only nine are linearly independent, so all the rest can be expressed in terms of them. (In fact, only the eight which are traceless can be used here, because of the \( d \)et = +1 condition which the \( S^0 \) operators must satisfy.) Their commutators are manifestly anti-Hermitian and so can be expanded in terms of the eight, with purely imaginary "structure constants." This means that, on the triplet, every \( S^0 \) operator can be written as the exponential of a sum of these eight basic matrices, which satisfy the commutation relations of the generators of a Lie group; whatever its properties on other multiplets, \( S^0 \) is an eight parameter Lie group of operators when acting on this three-dimensional space. All that remains is to find its rank and see which Lie group it is.

To do so, recall that \( S^0 \) must contain \( SU_2 \) (the \( I \)-spin group) as a subgroup, since \( SU_3 \) is a subgroup of \( S^0 \). This means that the \( S^0 \) triplet must be an \( I \)-spin triplet, or must decompose into the sum of either an \( I \)-spin doublet and singlet, or three singlets. The triplet is out because it has \( T = 1 \) and cannot be used to construct \( T = \frac{1}{2} \) particles. Similarly singlet \( \otimes \) singlet = singlet eliminates the three singlets, so the \( S^0 \) triplet must be composed of an \( I \)-spin doublet and singlet.\(^{18}\)

The rank of \( S^0 \) is at least 2, i.e., it has at least two commuting generators, which can simultaneously be taken diagonal. It is customary to take the two known ones as \( T_3 \) and the hypercharge \( Y = S + B \) (rather than \( T_3 \) and \( S \)); in the three-dimensional representation we have insisted the group possess, the above consideration unambiguously fix their forms (aside from normalization) to be:

\[
T_3 = \begin{pmatrix}
\frac{1}{2} \\
-\frac{1}{2}
\end{pmatrix}, \quad Y = \begin{pmatrix}
0 \\
1 \\
-2
\end{pmatrix}.
\]

(36)

The simultaneous eigenstates of the two operators are being used as a basis for the triplet, with the doublet spanning the upper left 2X2 submatrix. This gives \( T_3 \) immediately. The form of \( Y \) follows from the fact that \( [T_3, Y] = 0 \) forces \( Y \) to be degenerate on the \( I \)-spin doublet (Schur's lemma), and the \( (-2) \) is necessary to make it traceless. But these are the only two real, diagonal, traceless 3X3 matrices possible, so \( S^0 \) has rank 2.

We thus have the explicit forms of four of the eight matrices, \( T_3, T_3, T_3, \) and \( Y \). Four more linearly independent ones can be written down in any convenient form, and it is then just a matter of algebra (e.g., Ref. 2, p. 10) to check that no fewer than eight (except the original four) are closed under commutation, and that the structure constants which follow from their commutators are indeed those of \( SU_3 \). In this sense triplets imply \( SU_3 \).\(^{19}\)

The four added generators have off-diagonal elements relative to this basis, and so do not define new quantum numbers, but are instead raising-and-lowering-type operators for \( T_3 \) and/or \( Y \). If they are chosen to be exactly raising and lowering operators (i.e., non-Hermitian operators which transform one \( (T_3, Y) \) eigenstate into another, like \( T_3 \) and \( T_3 \), rather than Hermitian "mixtures" like \( T_3 \) and \( T_3 \)), they are found to change both \( T_3 \) and \( Y \) simultaneously (the "slanting arrows" of the weight-space diagrams). This is in accord with \( T_3 + \frac{1}{2} Y = Q \), which demands that \( \Delta T_3 = \) integer if \( T_3 \) alone is changed, \( \Delta Y = 0 \) (transitions) or that \( \Delta T_3 = \pm \frac{1}{2} \Delta Y \) when the two change together (\( \Delta Q/\Delta Y = +1, \Delta Q = 0 \) transitions), but forbids \( \Delta Y = \pm 1 \) alone if charge is quantized in integral units of e.

\( SU_3 \) has two inequivalent triplets (one a triplet and one an antitriplet, in terms of physical quantum numbers). Composite particles can be built of them in exact analogy to the "vector addition of isotopic spin" described in Sec. VI, by multiplying two or more triplets.


\(^{15}\) M. Gell-Mann, Phys. Letters 4, 214 (1964); G. Zweig, CERN preprint.


\(^{17}\) An alternative argument for \( SU_3 \) in terms of \( CP \) invariance, has recently been considered by A. Pais, Phys. Rev. Letters 12, 432 (1964).

\(^{18}\) It is in this sense that the present models construct all particles (except singlets) from \( I \)-spin doublets.

\(^{19}\) All of this proves nothing whatever about the physical existence of triplets. The relation between triplets and \( SU_3 \) is included merely as a suggestive fact, with the full realization that the occurrence of this group in nature need not have any connection with triplets at all. This seems to be the attitude adopted by Gell-Mann in Ref. 42, although for a "concrete" illustration he returns to the quark scheme.
plets together and decomposing the product space into other multiplets of SU2. This requires that the generators of the composite particle be the sum of the generators of the components, \( T = T_1 + T_2 \), \( Y = Y_1 + Y_2 \), etc., forming the eight components of the "total SU3-spin." Just as in the case of the rotation group, this means that the composite generators satisfy the same commutation rules as their components, and so have the same physical interpretation; it also defines the group on these larger multiplets. As might have been expected, the two triplets of SU3 are its fundamental multiplets, and so by Theorem 4 the multiplets obtained from products of these triplets include all the multiplets of SU3. In attempting to fit observed particles into the assumption that \( SU_3 = SU_2 \), one would expect all the SU3 multiplets of low dimension to occur, as is the case with SU2.41

Most unfortunately for this point of view, no sign of a triplet has yet (i.e., at a mass below a few BeV) appeared among the strongly interacting particles, either fermion or boson. The triplets, if they exist at all, as yet give little indication of being physical, observable particles. On the other hand, this gives free rein to imaginative invention and allows one to speculate, for instance, on the possible occurrence of two (or more) types of triplets (call them 1 and 2), distinguished by some non-SU3 quantum number. If they

\[ \begin{align*}
C_1 &= (\lambda_1^+ + \lambda_2^+ + \lambda_3^+ + \lambda_4^+ + \lambda_5^+ + \lambda_6^+ + \lambda_7^+ + \lambda_8^+)/9, \\
C_2 &= (\lambda_1^- + \lambda_2^- + \lambda_3^- + \lambda_4^- + \lambda_5^- + \lambda_6^- + \lambda_7^- + \lambda_8^-)/162,
\end{align*} \]

with a particular choice of normalization. The dimensionality of a multiplet is of course also determined by \( C_1 \) and \( C_2 \); expressed in terms of \( \lambda_1 \) and \( \lambda_2 \) it is \( N = (1+4(\lambda_1 + \lambda_2))(1+\lambda_1)(1+\lambda_2) \). The smaller multiplets are almost uniquely designated simply by their dimensions, as \( 1, 3 \) (\( = D^2(1,0) \)), \( 3^\overline{1} \) (\( = D^2(0,1) \)), etc. The four 15-dimensional multiplets are called \( 15, 15', 15'' \) and \( 15''' \).

If this same notation is used to indicate the decomposition of SU3 multiplets into SU2 triplets, then \( \begin{align*}
3 \otimes 3 &= 8 \oplus 3, \\
3 \otimes 3' &= 8' \oplus 3', \\
3 \otimes 3'' &= 10 \oplus 3'', \\
3 \otimes 3''' &= 10' \oplus 3''',
\end{align*} \]

the quark model, Ref. 35, explains this by the assignment of baryon number \( \frac{1}{3} \) to the fundamental triplet particles, or quarks. The simplest objects with baryon number 1 are then obtained from \( 3 \otimes 3 = 1 \oplus \overline{8} \oplus 10 \), and the simplest mesons from \( 3 \otimes 3 = 1 \oplus 8 \). A restriction which permits only \( \frac{1}{3} \) of the possible representations to occur in nature is not unlike the rotational bands \( J = 0, 2, 4, 6 \), or \( J = 1, 3, 5, 7 \); both nuclear and molecular physics, in which special symmetries eliminate \( \frac{1}{3} \) of the possible representations. Alternatively, the "restriction" could be due to dynamics, as in the model of F. Gurney, T. D. Lee, and M. Nauenberg (to be published).

were to exist, one could go on to imagine an "independent triplet" limit, in which (1) and (2), though coupled to other particles, were not coupled to each other, so that the Lagrangian for the system would be invariant under the even larger symmetry group. \( SU_3 = SU_3(1) \otimes SU_3(2) \), the group of SU3 transformations on (1) and (2) independently.

There would then have two kinds of hypersupermultiplets of \( SU_3 \). The first would be constructed of only one type of triplet, the conventional \( 3(1) \otimes 3(1) = 1(1) \oplus 8(1) \), which are just the SU3 multiplets. The other kind would be, e.g., \( 3(1) \otimes 3(2) = 9(1, 2) \), which is irreducible by Theorem 8 because of the independence of SU3(1) and SU3(2). This would imply nine mass-degenerate particles (with equal spins and parities) in the \( SU_3 \)-symmetry limit. Turning on an SU3-invariant interaction between (1) and (2) would ordinarily reduce the symmetry group to \( SU_3 = SU_3(1, 2) \), the group of simultaneous SU3 transformations on (1) and (2) generated by the "total SU3 spin," and the hypersupernot level would split into two, \( 9 = 8 \oplus 1 \). Finally, reducing the symmetry group to just the SU3(2) generated by the total isotopic spin \( T = T_1 + T_2 \) gives the observed particles, \( 9 = 2 \oplus 3 \oplus 2 \oplus 1 \oplus 1 \), presumably with the members of the eight clustered most tightly together.

Conjecture in this direction have recently been made by Schwinger and by Gell-Mann. In Schwinger's scheme (1) and (2) are distinguished by baryon number, and it is so constructed that no trilinear interactions of (1) (a boson) with (2) (a fermion) exist which are invariant under SU3(1, 2). Consequently the model skips directly from \( SU_3 \) to SU3, and the "ninth baryon" is not predicted at a vastly different mass from the mean of the other eight.

The Gell-Mann model distinguishes (1) from (2) by parity. The generators of SU3(1) and SU3(2), \( F^+ \) and \( F^- \), satisfy \( [F^+, F^-] = 0 \), but \( FF^+ = FF^- \), so neither group commutes with space-inversion. Consequently \( P \) cannot be factored out of \( SU_3 \) as was assumed earlier. For this reason Gell-Mann considers \( SU_3 \) to be what we might call \( SU_3^+ \otimes SU_3^- \mathbb{K}P \), the group obtained from \( SU_3^+ \otimes SU_3^- \) by multiplying every element of it by \( P \). Since parity interchanges \( F^+ \) and \( F^- \), it mixes such pairs of multiplets of \( SU_3^+ \otimes SU_3^- \) as \( 3^+ \oplus 3^- \) and \( 3^- \oplus 3^+ \), or \( 1^+ \otimes 8^- \) and \( 8^+ \otimes 1^- \), which means that these pairs coalesce to form the hypersupermultiplets of Gell-Mann's \( SU_3 \). This makes them twice as large as Schwinger's hypersupermultiplets. As an example, in Gell-Mann's \( SU_3 \)-symmetry limit, there are 18 mass-
Rotations and Vibrations in Deformed Nuclei

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A general survey of phenomenological, collective-model theories, especially those based upon the hydrodynamic model of rotations and vibrations of a deformed liquid drop, is presented. Details of models of even-even, odd-odd and odd-odd nuclei are given as well as a discussion of electromagnetic moments and transitions in these models. The influence of the vibrating and rotating nuclear surface has an important role in the alpha and beta decay of deformed nuclei is outlined. The discussion is extended to collective model calculations of the photonuclear process from which information can be obtained on the shape of nuclei in their ground states. Comparison of theory with the results of some recent experiments is given where this has been considered useful.

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I. INTRODUCTION

The need to consider a deformation degree of freedom, and thus the low-energy collective rotations and vibrations of atomic nuclei, occurred some time ago with the first attempts to explain the phenomenon of nuclear fission. Immediately after its discovery, and essentially at the same time, Feenberg (Fe 39) and Bohr and Wheeler (Bo 39) studied the shape and stability of a deformed and charged liquid drop and showed how the total energy changed as a function of deformation from the spherical equilibrium shape. Feenberg's original note was concerned with demonstrating the stability of nuclei against spontaneous fission, while Bohr and Wheeler investigated in detail the theory of the process. The nuclear instability arises from the fact that the total energy considered in this process is the sum of surface and Coulomb energies [ideas originally introduced by Weizsäcker (We 35) in the empirical binding energy formula] and distortions from spherical increase the surface area, and hence the surface energy, while necessarily decreasing the Coulomb energy. Long ago Lord Rayleigh considered a similar classical problem in his studies on the stability of electrically charged liquid drops (St 82), and some of the early work on the hydrodynamical model of the nucleus parallels quite closely his theoretical investigation.

While it has been usual to consider the axially symmetric deformation from the spherical equilibrium shape, a more general treatment is only slightly more involved. Following Feenberg's notation (Fe 39) we may define an ellipsoid of volume equal to that of a sphere of radius \( R_0 \) by

\[ \lambda x^2 + \mu y^2 + \eta z^2 = R_0^2, \]  
(I-1)

where the eccentricities may be taken as

\[ e_1 = (a-c)/R_0 \equiv 2(1 - \sqrt{\lambda}) + (1 - \sqrt{\mu}), \]
\[ e_2 = (b-c)/R_0 \equiv 2(1 - \sqrt{\eta}) + 2(1 - \sqrt{\lambda}), \]  
(I-2)

and the semi-axes are made definite by requiring \( a > b > c \). For a deformed liquid drop, the change in Coulomb plus surface energies is then, to lowest order

\[ \Delta E = \left(4/45\right) \left(2E_0 - E_0^0\right) (e_1^2 + e_2^2 + e_3^2), \]  
(I-3)

where \( E_0^0 \) is the spherical energy in question. For axial symmetry this reduces to the usual value (Fe 39). Treating the eccentricities as variational parameters we see that the spherical shape is the most stable. From