Superfluid

- Liquid Helium: Bose liquid and superfluidity
- Landau’s theory: two fluid model
- Bose-Einstein Condensation and superfluid
- ODLRO, spontaneous symmetry breaking, macroscopic wavefunction
- Gross-Pitaevskii (GP) equation
- Feynman’s picture

References:
1) “Theory of quantum liquids”, David Pines & Philippe Nozieres
2) “Statistical mechanics”, R. P. Feynman
Why helium?

• Bose liquid
  – $^4$He: inert gas, no hydrogen bond, weak interaction, light atom
  – Strong zero-point oscillation: difficult to form a solid
  – Remain a liquid down to the lowest temperature (London 1938)
  – $^4$He: bosons, unique Bose liquid in the laboratory
  – Quantum liquid: quantum-mechanical description is essential for the understanding of its properties.

• Superfluid
  – Gas-liquid transition at 5.2 K: first order, latent heat
  – A new liquid phase below 2.19 K: second order transition, no latent heat
  – Two quantum liquid phases: Helium I and Helium II
Phase transition in liquid $^4$He

- $\lambda$-transition
- no latent heat
- second order
- empirical law

$$C_v = \begin{cases} 
a_+ + b \ln |T - T_c|, & T > T_c, \\
 a_- + b \ln |T - T_c|, & T < T_c. 
\end{cases}$$

- $T^3$ specific heat
- bosonic modes with linear dispersion
- phonons
Phase diagram
Superfluid phenomena

- **Superfluidity in He II**
  - Superfluid: vanishing viscosity below the $\lambda$-point
  - Thermomechanical effect
  - Fountain effect

![Diagram](image)

**Fig. 11.3** The thermomechanical effect.

- two He II containers, superleak
- constant density, temperature at each sides
- superfluid flow $\rightarrow \Delta P = \rho_s \Delta T$
- pressure difference $\rightarrow$ fountain effect

**Mechanocaloric effect**

- Pressure difference
- Mass flow $A \rightarrow B$
- $B$ cool down (zero entropy flow)
Fountain effect

(a) Electric Heater
    Capillary

(b) He II Fountain
    Capillary
    Radiant Heat
    Powder
    Cotton
Landau’s two-fluid model

Two components of fluid

- Superfluid: perfect background fluid, zero entropy and viscosity
- Normal fluid: some types of excitations, phonon gas

Density and velocity of two components

\[ \rho_n, \; \rho_s, \; v_n, \; v_s \]

Specific heat of a phonon gas

\[ C_V \propto T^3 \]

Explanation of thermomechanical effect

- Superfluid flows through a superleak
- The phonons are inherited because of the collision with the walls
Equations of motion

\[ \rho(T) = \rho_n(T) + \rho_s(T) \]

For normal fluid

\[ \rho_n \frac{\partial \vec{v}_n}{\partial t} + \rho_n \vec{v}_n \times \nabla \vec{v}_n = -\frac{\rho_n}{\rho} \nabla p - \rho_s s \nabla T + \eta \nabla^2 \vec{v}_n \]

For superfluid

\[ \rho_s \frac{\partial \vec{v}_s}{\partial t} + \rho_s \vec{v}_s \times \nabla \vec{v}_s = -\frac{\rho_s}{\rho} \nabla p + \rho_s s \nabla T \]
Andronikashvili’s experiment

Determine the fractional densities of the superfluid and normal fluid components by measuring the period and decrement of a torsional pendulum immersed in He II.

Moment of inertia

\[ I = I_{\text{disk}} + I_{\text{fluid}} \]

Measure resonant frequency

\[ I \frac{d^2 \theta}{dt^2} = -k \theta \]
Second sound

- Density wave of the phonon gas: $\rho_n / \rho$
- A temperature wave rather than pressure pulses.

Remark: second sound exists in solid state too considering the anharmonic effect.

$$c_s = c / \sqrt{3}$$

Fig. 11.6 Velocity of second sound.
Phonons and rotons

For a background fluid with velocity $v_s$:

$$ E = E(p) + \vec{p} \cdot (\vec{v}_s - \vec{v}_n) $$

$$ \left\langle N_p \right\rangle = \frac{1}{e^{[E(p) + \vec{p} \cdot (\vec{v}_s - \vec{v}_n)]/k_B T} - 1} $$

$$ \left\langle \vec{p} \right\rangle = \sum_{\vec{p}} \vec{p} \left\langle N_p \right\rangle = -\sum_{\vec{p}} \vec{p} (\vec{p} \cdot \vec{v}_n) \frac{\partial}{\partial E_p} \frac{1}{e^{\beta E_p} - 1} $$

$$ \Rightarrow \rho_n = -\int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{p^2}{3} \frac{\partial}{\partial E_p} \frac{1}{e^{\beta E_p} - 1} $$

**Fig. 11.7** Excitation curve for phonons and rotons.
For \( \nu \leq E(p) / p \) can not make spontaneous excitations, which would decay superflow, and flow is superfluid.

For \( \nu > E(p) / p \) and \( E < 0 \), can then make excitations spontaneously, and superfluidity ceases.

\[ E = E(p) + \bar{p} \cdot \bar{v} \]

\( \nu_c = 60 \text{ m/sec} \)
BEC in a nutshell

Bose distribution

\[ n(\vec{k}) = \frac{1}{e^{\beta(E_k - \mu)} - 1}, \]

Chemical potential at zero temperature

\[ n = \frac{N}{V} = \frac{1}{(2\pi)^d} \int n(\vec{k}) \ d^d \vec{k}, \]

Since \( E_k > \mu \), if \( \mu \neq 0 \), then

\[ T \to 0, \quad e^{\beta(E_k - \mu)} \to \infty, \quad n(\vec{k}) \to 0 \]
\[ \Rightarrow \quad n \to 0 \]

The only possible solution is that

\[ \mu = E_{k=0} = 0. \]
Bose distribution vs. Boltzmann distribution

\[ n(E) = \frac{1}{e^{\beta(E-\mu)} - 1} \approx \frac{k_B T}{E - \mu} \]

\[ \frac{n(E_1)}{n(E_0)} = \frac{E_0 - \mu}{E_1 - \mu} \]

\[ \mu \rightarrow 0, \quad \frac{n(E_1)}{n(E_0)} \rightarrow \frac{E_0}{E_1} = 0 \]

\[ n(\vec{k}) = n_0 \delta(\vec{k}) \]

\[ n(E) = e^{-\beta(E-\mu)} \]

\[ \frac{n(E_1)}{n(E_0)} = e^{\beta(E_0 - E_1)} \]

\[ E_1 \rightarrow E_0, \quad \frac{n(E_1)}{n(E_0)} \rightarrow 1 \]

Bose Einstein condensation

\[ n(\vec{k}) = \begin{cases} 
\frac{1}{e^{\beta E_k} - 1}, & \vec{k} \neq 0 \\
n_0(T), & \vec{k} = 0 
\end{cases} \]
BEC ≠ superfluidity

- A free boson condensate is not a superfluid
  - The absence of phonons
  - quadratic energy dispersion
  - BEC density ≠ superfluid density
- A boson fluid with phonon-like excitation spectrum is a superfluid.
- BEC is not a sufficient condition for superfluidity.
  - Considering vortices and KT transition at 2D, BEC is not a necessary condition for superfluidity too.
What’s the order parameter?

**Order parameter for condensate**

\[ \Psi(\vec{r}) = |\psi| \left\langle e^{i\phi(\vec{r})} \right\rangle \neq 0 \]

wave function of mode into which particles condense

**Off-diagonal long range order** (Penrose & Onsager)

More rigorous definition by eigenfunction of largest eigenvalue of density matrix

\[ \rho(\vec{r}, \vec{r}') = \left\langle \psi(\vec{r})\psi^*(\vec{r}') \right\rangle \rightarrow \Psi(\vec{r})\Psi(\vec{r}')^* \]

**Related to BEC**

Single particle distribution given by the eigenvalues of density matrix

C.N. Yang, Rev. Mod. Phys. 34, 694 (1962)
Physical consequences of phase coherence

Superfluid velocity:

\[ \bar{v}_s (\vec{r}) = \frac{\hbar}{m} \nabla \phi \]

Chemical potential:

\[ \mu = -\hbar \frac{\partial \phi}{\partial t} \]

Equation of motion:

\[ m \frac{\partial \bar{v}_s}{\partial t} + \nabla \mu = 0 \]
Spontaneous symmetry breaking

- Global U(1) symmetry
- Phase mode
  - Gapless Goldstone mode
  - Only one branch of phonons in superfluid phase
Summary

- Superfluidity in Helium
- Landau’s two-fluid model
  - Normal fluid and superfluid components
  - Phonons and rotons
- Bose Einstein condensation and superfluidity
- Off-diagonal long range order
  - Correspond to large eigenvalue of density matrix
  - Relation to BEC
- Spontaneous symmetry breaking