Lecture 3-7: General formalism at finite temperature

Reference: Negele & Orland (N&O) Chapter 2

- Lecture 6
- Irreducible diagrams and integral equations

To derive exact integral equations relating connected Green’s functions and irreducible vertex functions.

- Generating function for connected Green’s functions

- Couple the field operators to an external source,

\[ S = \sum_{\alpha} \int d\tau [(J^{\ast}_{\alpha})(\tau)a_{\alpha}(\tau) + a_{\alpha}^{\dagger}(\tau)J_{\alpha}(\tau)] \]

- The generating function may be written as

\[ G(J^{\ast}_{\alpha}(\tau), J_{\alpha}(\tau)) \equiv \frac{1}{Z} \int \mathcal{D}[\psi^{\ast}_{\alpha}(\tau)\psi_{\alpha}(\tau)] e^{-\beta \int_{0}^{\beta} d\tau \left[ \sum_{\alpha} \psi_{\alpha}(\tau)(\partial_{\tau} - \mu)\psi_{\alpha}(\tau) + H[\psi^{\ast}_{\alpha}(\tau), \psi_{\alpha}(\tau)] \right]} \times e^{\int_{0}^{\beta} d\tau \sum_{\alpha} [J^{\ast}_{\alpha}(\tau)\psi_{\alpha}(\tau) + \psi^{\ast}_{\alpha}(\tau)J_{\alpha}(\tau)]} \]

Differentiation with respect to the sources yields

\[ \frac{\delta G(J^{\ast}_{\alpha}(\tau), J_{\alpha}(\tau))}{\delta J^{\ast}_{\alpha_1}(\tau_1)} = -\langle \psi_{\alpha_1}(\tau_1) e^{\int_{0}^{\beta} d\tau \sum_{\alpha} [J^{\ast}_{\alpha}(\tau)\psi_{\alpha}(\tau) + \psi^{\ast}_{\alpha}(\tau)J_{\alpha}(\tau)]} \rangle \]

and

\[ \frac{\delta G(J^{\ast}_{\alpha}(\tau), J_{\alpha}(\tau))}{\delta J_{\alpha_1}(\tau_1)} = -\zeta \langle \psi^{\ast}_{\alpha_1}(\tau_1) e^{\int_{0}^{\beta} d\tau \sum_{\alpha} [J^{\ast}_{\alpha}(\tau)\psi_{\alpha}(\tau) + \psi^{\ast}_{\alpha}(\tau)J_{\alpha}(\tau)]} \rangle \]

So that the \( n \)-particle imaginary-time Green’s function may be written as
The generating function for connected Green’s functions may be obtained using replica technique once again. Note that all the connected diagrams will be proportional to \( p \).

\[
W(J^*_\alpha(\tau), J_\alpha(\tau)) = \lim_{p \to 0} \frac{\partial}{\partial p} (\mathcal{G}(J^*_\alpha(\tau), J_\alpha(\tau)))^p
\]

So that

\[
W(J^*_\alpha(\tau), J_\alpha(\tau)) = \ln \mathcal{G}(J^*_\alpha(\tau), J_\alpha(\tau))
\]

and

\[
\mathcal{G}^{(n)}_c(\alpha_1, \tau_1; \ldots \alpha_n, \tau_n | \alpha'_1, \tau'_1; \ldots \alpha'_n, \tau'_n)
= \zeta^n \frac{\delta^{2n} W(J^*_\alpha(\tau), J_\alpha(\tau))}{\delta J^*_\alpha_1(\tau_1) \ldots \delta J^*_\alpha_n(\tau_n) \delta J^*_\alpha'_1(\tau'_1) \ldots \delta J^*_\alpha'_n(\tau'_n)} \bigg|_{J^*_\alpha=J_\alpha=0}
\]

Example: one-particle Green’s function \((n=1)\)

\[
\mathcal{G}^{(1)}_c(1|1') = \zeta \frac{\delta^2}{\delta J^*_1 \delta J^*_{1'}} \left[ \ln (e^{-(J^*\psi + \psi^* J)}) \right]_{J^*_\alpha=J_\alpha=0}
= -\frac{\delta}{\delta J^*_1} \left[ (e^{-(J^*\psi + \psi^* J)})^{-1} \langle \psi^*_1, e^{-(J^*\psi + \psi^* J)} \rangle \right]_{J^*_\alpha=J_\alpha=0}
= \left[ (e^{-(J^*\psi + \psi^* J)})^{-1} \langle \psi^*_1 \psi^*_{1'}, e^{-(J^*\psi + \psi^* J)} \rangle - (e^{-(J^*\psi + \psi^* J)})^{-2} \langle \psi^*_1, e^{-(J^*\psi + \psi^* J)} \rangle \langle \psi^*_{1'}, e^{-(J^*\psi + \psi^* J)} \rangle \right]_{J^*_\alpha=J_\alpha=0}
\]

For the one-body connected Green’s function in the absence of symmetry breaking, we have \( \langle \psi \rangle = \langle \psi^* \rangle = 0 \), thus

\[
\mathcal{G}^{(1)}_c(1|1') = \langle \psi^*_1 \psi^*_{1'} \rangle = \mathcal{G}^{(1)}(1, 1')
\]

Note that \( \langle \psi \rangle \neq 0 \) will happen in the case of Bose condensation.
Example: two-particle Green’s function

\[ G_c^{(2)}(1, 2|1', 2') = \frac{\delta^4}{\delta J_1^* \delta J_2^* \delta J_2' \delta J_1'} \left[ \ln \langle e^{-(J^* \psi^* \psi J)} \rangle \right]_{J=J^*=0} \]

\[ = \frac{\delta^2}{\delta J_1^* \delta J_2^*} \left[ \langle e^{-(J^* \psi^* \psi J)} \rangle^{-1} \langle \psi_2^*, \psi_1^* e^{-(J^* \psi^* \psi J)} \rangle \right. \\
- \left. \langle e^{-(J^* \psi^* \psi J)} \rangle^{-2} \langle \psi_2^*, e^{-(J^* \psi^* \psi J)} \rangle \langle \psi_1^*, e^{-(J^* \psi^* \psi J)} \rangle \right]_{J=J^*=0} \]  

(2.15)

Assuming no symmetry breaking, we have

\[ G_c^{(2)}(1, 2|1', 2') = \langle \psi_1 \psi_2 \psi_2^*, \psi_1^* \rangle - \langle \psi_2 \psi_2^* \rangle \langle \psi_1 \psi_1^* \rangle - \zeta \langle \psi_1 \psi_2^* \rangle \langle \psi_2 \psi_1^* \rangle \]

(2.15)

\[ = G^{(2)}(1, 2|1', 2') - \left[ G^{(1)}(1|1') G^{(1)}(2|2') + \zeta G^{(1)}(1|2') G^{(1)}(2|1') \right] \]

The above can be represented graphically,

\[ G^{(2)}_c = \]

\[ = G^{(2)}_c - \left[ G^{(0)}_c G^{(0)}_c + \zeta G^{(0)}_c G^{(0)}_c \right] \]

\[ = \]

\[ = G^{(2)}_c - G^{(0)}_c G^{(0)}_c + \text{exch.} \]

where exch is the abbreviation for exchange indicating the sum of all possible permutations \( P \) of the external points with associated factor \( \zeta^P \).

Example: higher order connected Green’s function,

\[ G^{(3)}_c = \]

\[ = G^{(3)}_c - \left[ G^{(2)}_c G^{(1)}_c + G^{(1)}_c G^{(2)}_c \right] \]

\[ = G^{(3)}_c - G^{(2)}_c G^{(1)}_c - G^{(1)}_c G^{(2)}_c \]

The effective potential and vertex function

In the presence of external sources, say, when \( J \neq 0 \), the operators \( \{a_c(r), a_c^*(r)\} \) acquire non-zero expectation values. Let’s define the average field
\[ \phi_\alpha = \langle a_\alpha \rangle_{J^\ast, J} = \langle \psi_\alpha \rangle_{J^\ast, J} \]
\[ = \int D[\psi_\alpha^* \psi_\alpha] \psi_\alpha e^{-\int d\tau \left[ \sum_{\alpha} \psi_\alpha (\partial_\tau - \mu) \psi_\alpha + H \psi_\alpha^* \psi_\alpha + \sum_{\alpha} [J^\ast_\alpha \psi_\alpha + \psi_\alpha^* J_\alpha] \right]} \]
\[ = -\frac{\delta}{\delta J^\ast_\alpha(\tau)} W[J^\ast_\alpha(\tau), J_\alpha(\tau)] \]

and its complex conjugate field
\[ \phi^\ast_\alpha(\tau) = \langle a^\dagger_\alpha(\tau) \rangle_{J^\ast, J} \]
\[ = -\frac{\delta}{\delta J_\alpha(\tau)} W[J^\ast_\alpha(\tau), J_\alpha(\tau)] \]

- Legendre transformation of the fields \( \phi^\ast, \phi^\dagger \)?

- Consider the familiar example of a spin system in a magnetic field, the \( \mathcal{H}(s) \), the free energy as function of an external magnetic field \( \vec{H} \) is given by
\[ \text{Tr} e^{-\beta(\mathcal{H}(s) - \vec{H} \cdot \sum_i \vec{s}_i)} = e^{-\beta F(H)} \]
from which it follows that the magnetization is given by
\[ M = -\frac{\partial F(H)}{\partial H} \]

An equation of state which depends on the magnetization instead of the external magnetic field is obtained by inverting \( M(H) \) to obtain \( H(M) \) and defining the Legendre transformation,
\[ G(M) = F(H(M)) + MH(M) \]

Thus
\[ \frac{\partial G}{\partial M} = \frac{\partial F}{\partial H} \frac{\partial H}{\partial M} + H + M \frac{\partial H}{\partial M} = H \]

Note that both \( F(H) \) and \( G(M) \) contain the same physical information.
The effective potential (or action) is defined as the Legendre transformation,

$$\Gamma[\phi^*_\alpha(\tau), \phi_\alpha(\tau)] = -W[J^*_\alpha(\tau), J_\alpha(\tau)] - \sum_\gamma \int_0^\beta dr' [\phi^*_\gamma(\tau') J_\gamma(\tau') + J^*_\gamma(\tau') \phi_\gamma(\tau')]$$

which satisfies the reciprocity relation

$$\frac{\partial}{\partial \phi^*_\alpha(\tau)} \Gamma[\phi^*_\alpha(\tau), \phi_\alpha(\tau)] = \sum_\gamma \int_0^\beta dr' \left[ -\frac{\partial W}{\partial J^*_\gamma(\tau')} \frac{\partial J_\gamma(\tau')}{\partial \phi^*_\alpha(\tau)} - \frac{\partial W}{\partial J_\gamma(\tau')} \frac{\partial J^*_\gamma(\tau')}{\partial \phi^*_\alpha(\tau)} \right]$$

$$-\delta_{\alpha \gamma} \delta(\tau - \tau') J_\gamma(\tau') - \phi^*_\gamma(\tau') \frac{\partial J_\gamma(\tau')}{\partial \phi^*_\alpha(\tau)} \phi_\gamma(\tau')$$

$$= -J_\alpha(\tau)$$

(2.165)

and

$$\frac{\partial}{\partial \phi_\alpha(\tau)} \Gamma[\phi^*_\alpha(\tau), \phi_\alpha(\tau)] = -\phi^*_\alpha(\tau)$$

When the sources are set equal to zero,

$$\frac{\delta \Gamma(\tilde{\phi}_\alpha(\tau), \tilde{\phi}_\alpha(\tau))}{\delta \tilde{\phi}_\alpha(\tau)} = 0$$

The nonvanishing solutions to the above equation will be obtained when the symmetry breaking happens.

The effective potential is a generating function for vertex functions

$$\Gamma_{m\phi^*, n\phi}(\alpha_1 \tau_1, \ldots \alpha_m \tau_m | \alpha'_1 \tau'_1, \ldots \alpha'_n \tau'_n)$$

(2.1)

$$= \frac{\delta^{m+n}}{\delta \phi^*_{\alpha_1}(\tau_1) \cdots \delta \phi^*_{\alpha_m}(\tau_m) \delta \phi_{\alpha'_1}(\tau'_1) \cdots \delta \phi_{\alpha'_n}(\tau'_n)} \Gamma[\phi^*_\alpha(\tau), \phi_\alpha(\tau)] \bigg|_{J^*_\alpha = J_\alpha = 0}$$

The vertex function is one-particle irreducible and thus cannot be disconnected by removing a single internal propagator.

The connected Green’s function can be constructed from vertex functions using only tree diagrams, i.e., diagrams containing no closed propagator loops. This property is extremely useful in renormalization of field theories.
The self-energy and Dyson’s equation

Writing the matrix form of one-particle (connected) Green’s function $g^{(1)}_{c(1,2)}$ as $g$, one has

$$g = g_0 - g_0 \Sigma g$$

$$= g_0 - g_0 \Sigma g_0 + g_0 \Sigma g_0 \Sigma g_0 \ldots$$

The self-energy $\Sigma$ can be obtained graphically with $g$ and the non-interacting corresponding $g_0$,

By construction, we can separate $g_0$ from $g$ to obtain an iterative equation,

Thus the self-energy is **one-particle irreducible**, namely, it can not be separated into two or more disconnected diagrams by cutting a single internal propagator.

The Dyson equation then can be written as

$$g^{-1} = [g_0]^{-1} + \Sigma$$

or explicitly,

$$g^{(1)}_{c(1,2)}(\alpha_1 \tau_1 | \alpha_4 \tau_4) = g^{(1)}_{0,c}(\alpha_1, \tau_1 | \alpha_4, \tau_4)$$

$$- \sum_{\alpha_2 \alpha_3} \int_0^\beta d\tau_2 d\tau_3 g^{(1)}_{0,c}(\alpha_1, \tau_1 | \alpha_2 \tau_2) \Sigma(\alpha_2 \tau_2, \alpha_3 \tau_3) g^{(1)}_{c}(\alpha_3, \tau_3 | \alpha_4, \tau_4)$$

Note that there is a sign difference between some literatures due to the definition.
The physical interpretation of $\Sigma$ as a self-energy is evident from the following,

$$[G^{(1)}_c]^{-1}(\alpha_1 \tau_1 | \alpha_2 \tau_2) = \left( \delta_{\alpha_1 \alpha_2} \left( \frac{\partial}{\partial \tau_1} - \mu \right) + \langle \alpha_1 | H_0 | \alpha_2 \rangle \right) \delta(\tau_1 - \tau_2) + \Sigma(\alpha_1 \tau_1 | \alpha_2 \tau_2)$$

An economical method to derive Dyson equation can be given by taking successive derivatives to the generating function and using the chain rule for functional derivatives,

$$\frac{\delta F(J^*, J)}{\delta \phi_{\alpha_1}(\tau_1)} = \sum_{\alpha_2} \int_0^\beta d\tau_2 \left[ \frac{\delta F}{\delta J_{\alpha_2}^*(\tau_2)} \frac{\delta J_{\alpha_2}(\tau_2)}{\delta \phi_{\alpha_1}(\tau_1)} + \frac{\delta F}{\delta J_{\alpha_2}(\tau_2)} \frac{\delta \phi_{\alpha_1}(\tau_1)}{\delta \phi_{\alpha_2}(\tau_2)} \right]$$

$$= \sum_{\alpha_2} \int_0^\beta d\tau_2 \left[ - \zeta \frac{\delta F}{\delta J_{\alpha_2}^*(\tau_2)} \frac{\delta^2 \Gamma}{\delta \phi_{\alpha_1}(\tau_1) \delta \phi_{\alpha_2}(\tau_2)} \right.$$

$$- \frac{\delta F}{\delta J_{\alpha_2}(\tau_2)} \frac{\delta^2 \Gamma}{\delta \phi_{\alpha_1}(\tau_1) \delta \phi_{\alpha_2}(\tau_2)}$$

and similarly,

$$\frac{\delta F(J^*, J)}{\delta \phi_{\alpha_1}^*(\tau_1)} = \sum_{\alpha_2} \int_0^\beta d\tau_2 \left[ \frac{\delta F}{\delta J_{\alpha_2}^*(\tau_2)} \frac{\delta^2 \Gamma}{\delta \phi_{\alpha_1}(\tau_1) \delta \phi_{\alpha_2}(\tau_2)} \right.$$

$$- \frac{\delta F}{\delta J_{\alpha_2}(\tau_2)} \frac{\delta^2 \Gamma}{\delta \phi_{\alpha_1}(\tau_1) \delta \phi_{\alpha_2}(\tau_2)} \right].$$

So that

$$\delta_{\alpha_2 \alpha_1} \delta(\tau_3, \tau_1) = \frac{\delta \phi_{\alpha_3}(\tau_3)}{\delta \phi_{\alpha_1}(\tau_1)} = \frac{\delta}{\delta \phi_{\alpha_1}(\tau_1)} \left[ - \frac{\delta W}{\delta J_{\alpha_3}^*(\tau_3)} \right]$$

$$= \sum_{\alpha_2} \int_0^\beta d\tau_2 \left[ \zeta \frac{\delta^2 W}{\delta J_{\alpha_2}^*(\tau_2) \delta J_{\alpha_3}(\tau_3)} \frac{\delta^2 \Gamma}{\delta \phi_{\alpha_1}(\tau_1) \delta \phi_{\alpha_2}(\tau_2)} \right.$$

$$+ \frac{\delta^2 W}{\delta J_{\alpha_2}(\tau_2) \delta J_{\alpha_3}^*(\tau_3)} \frac{\delta^2 \Gamma}{\delta \phi_{\alpha_1}(\tau_1) \delta \phi_{\alpha_2}(\tau_2)} \right].$$

The above can be written in terms of more compact notation,

$$\delta(31) = \int d2 \left[ \frac{\delta^2 W}{\delta J^*(2) \delta J^*(3)} \frac{\delta^2 \Gamma}{\delta \phi^*(1) \delta \phi(2)} + \frac{\delta^2 W}{\delta J(2) \delta J^*(3)} \frac{\delta^2 \Gamma}{\delta \phi(1) \delta \phi^*(2)} \right].$$
The remaining three similar derivatives yields equations,

$$\delta(31) = \frac{\delta}{\delta \phi^*(1)} \phi^*(3) = \frac{\delta}{\delta \phi^*(1)} \left[ -\zeta \frac{\delta W}{\delta J(3)} \right]$$

$$= \int d^2 \left[ \frac{\delta^2 W}{\delta J^*(2) \delta J(3)} \frac{\delta^2 \Gamma}{\delta \phi^*(1) \delta \phi(2)} + \zeta \frac{\delta^2 W}{\delta J(2) \delta J(3)} \frac{\delta^2 \Gamma}{\delta \phi^*(1) \delta \phi^*(2)} \right]$$

and

$$0 = \frac{\delta}{\delta \phi^*(1)} \phi^*(3) = \frac{\delta}{\delta \phi^*(1)} \left[ -\frac{\delta W}{\delta J(3)} \right]$$

$$= \int d^2 \left[ \frac{\delta^2 W}{\delta J^*(2) \delta J(3)} \frac{\delta^2 \Gamma}{\delta \phi^*(1) \delta \phi(2)} + \frac{\delta^2 W}{\delta J(2) \delta J(3)} \frac{\delta^2 \Gamma}{\delta \phi^*(1) \delta \phi^*(2)} \right]$$

The above four equations can be expressed in matrix form

$$\int d^2 \begin{pmatrix} \frac{\delta^2 W}{\delta J^*(2) \delta J(2)} & \frac{\delta^2 W}{\delta J^*(3) \delta J(2)} \\ \frac{\delta^2 W}{\delta J(3) \delta J^*(2)} & \frac{\delta^2 W}{\delta J(3) \delta J^*(3)} \end{pmatrix} \begin{pmatrix} \frac{\delta \Gamma}{\delta \phi^*(2) \delta \phi(1)} & \frac{\delta \Gamma}{\delta \phi^*(2) \delta \phi^*(1)} \\ \frac{\delta \Gamma}{\delta \phi(2) \delta \phi(1)} & \frac{\delta \Gamma}{\delta \phi(2) \delta \phi^*(1)} \end{pmatrix} = \delta(31) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

With the help of matrix form, we have that

$$\begin{pmatrix} \Gamma_{\phi^* \phi} & \Gamma_{\phi^* \phi^*} \\ \Gamma_{\phi \phi} & \Gamma_{\phi \phi^*} \end{pmatrix} = \zeta \begin{pmatrix} \langle \psi \psi^* \rangle & \langle \psi \psi \rangle \\ \langle \psi^* \psi \rangle & \langle \psi^* \psi^* \rangle \end{pmatrix}^{-1}$$

Consider a system without symmetry breaking, the above are simplified,

$$\int d^2 \mathcal{G}_c^{(1)}(1,2) \Gamma_{\phi^* \phi}(2,3) = \int d^2 \mathcal{G}_c^{(1)}(1,2) \mathcal{G}_c^{(1)}(2,3) = \delta(1,3)$$

and

$$\Gamma_{\phi^* \phi}(1,2) = [\mathcal{G}_c^{(1)}]^{-1}(1,2)$$

For a non-interacting system, the Green’s function yields

$$\sum_{\alpha_2} \left( \delta_{\alpha_1 \alpha_2} \left( \frac{\partial}{\partial \tau_1} - \mu \right) + \langle \alpha_1 | H_0 | \alpha_2 \rangle \right) \mathcal{G}^{(1)}_{\alpha_2, \tau_1} (\alpha_2, \tau_1 | \alpha_3, \tau_3) = \delta_{\alpha_1 \alpha_3} \delta(\tau_1 - \tau_3)$$
Thus, for a non-interacting system

\[ \Gamma^{(0)}_{\phi^*\phi}(\alpha_1, \tau_1 | \alpha_2, \tau_2) = [\mathcal{G}^{(1)}_{0,c}]^{-1}(\alpha_1 \tau_1 | \alpha_2 \tau_2) \]

\[ = \left( \delta_{\alpha_1 \alpha_2} \left( \frac{\partial}{\partial \tau_1} - \mu \right) + \langle \alpha_1 | H_0 | \alpha_2 \rangle \right) \delta(\tau_1 - \tau_2) \]

For an interacting system, the one-particle vertex function can be divided into two parts, the non-interacting part and the self-energy,

\[ \Gamma^{(1)}_{\phi^*\phi}(1,2) \equiv \Gamma^{(0)}_{\phi^*\phi}(1,2) + \Sigma(1,2) \]

which yields

\[ \mathcal{G}^{-1} = [\mathcal{G}_0]^{-1} + \Sigma \]

Higher-order vertex functions (Home reading)

**Homework:** Problem 2.12
• Feynman diagram rule for self-energy

Rule 1:

Draw all distinct, unlabeled, one-particle irreducible, amputated diagrams composed of \( n \) interaction vertices \( \cdots \langle \cdot \rangle \) with the label \( \{\alpha, \beta\} \) assigned to one outgoing arrow of an interaction vertex, the label \( \{\alpha', \beta'\} \) assigned to one ingoing arrow of an interaction vertex, and all other arrows of the interaction vertices connected by directed lines \( \downarrow \). Two diagrams are distinct if, holding the points \( \{\alpha, \beta\} \) and \( \{\alpha', \beta'\} \) fixed, the lines and propagators cannot be deformed to coincide completely including the direction of arrows on propagators. The contribution for each distinct unlabeled diagram is evaluated as follows:

Rule 2:

Assign an internal time label \( \tau_i \) to each interaction vertex which is not assigned to one of the external time values \( \beta \) or \( \beta' \). For every directed line, assign a single-particle index \( \gamma \) and include the factor

\[
\tau_{\gamma} = g_\gamma (\tau - \tau') = e^{-i(\gamma-\mu)(\tau-\tau')} [(1 + \gamma n_\gamma) \theta(\tau - \tau' - \eta) + \gamma n_\gamma \theta(\tau' - \tau + \eta)]
\]

where \( \tau, \tau' \) denote either internal or external times.

Rule 3:

For each interaction vertex include the factor

\[
\tau \quad \gamma \quad \lambda \quad U \quad \delta \quad (\alpha \lambda | U | \gamma \delta)
\]

Note that if the external points \( \{\alpha, \beta\} \) and \( \{\alpha', \beta'\} \) are associated with the same interaction vertex, since the interaction is instantaneous the factor \( \delta (\beta - \beta') \) must also be included.

Rule 4:

Sum over all internal single-particle indices and integrate all internal times \( \tau_i \) over the interval \([0, \beta]\).

Rule 5:

Multiply the result by the factor \( (-1)^{n-1} \zeta^{n_L} \) where \( n_L \) is the number of closed propagator loops and the extra minus sign accounts for the fact that (2.179c) specifies \(-\Sigma\).