Lecture 1-2: Second quantization and coherent states

Reference: Negele & Orland (N&O) Chapter 1

- Lecture 1
  - Quantum mechanics of a single particle (Home reading, N&O 1.1)
  - System of identical particles (Home reading, N&O 1.2)
    - Hilbert space of an N-particle system
      \[ \mathcal{H}_N = \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H} \]
    - Boson and Fermion: total symmetric and antisymmetric states
  - Occupation number representation
    - The occupation number representation of bosons
      - Mapping a single-mode many-particle system to a harmonic oscillator
        \[ H = \hbar \omega \left( \hat{a}^+ \hat{a} + \frac{1}{2} \right), \quad [\hat{a}, \hat{a}^+] = 1 \]
        \[ \hat{n} = \hat{a}^+ \hat{a}, \quad \hat{n} |n\rangle = n |n\rangle, \]
        \[ \hat{a}^+ |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle \]
      - Quantum harmonic oscillators vs. boson occupation
        | identical bosons |
        | use harmonic oscillator eigenstates to label single mode bosonic state |
        | eigenvalue ↔ boson occupation number |
        | ground state ↔ empty state |
        | n- th excited state ↔ n bosons |
        | \( \hat{a}^+ \) ↔ create a boson |
        | \( a \) ↔ annihilate a boson |
  - Fock space: the Hilbert space in occupation number representation
    \[ \mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \ldots = \bigoplus_{n=0}^{\infty} \mathcal{B}_n \]
The occupation number representation of fermions

The single-mode system

\[
[\hat{a}, \hat{a}^+] = \hat{a} \hat{a}^+ + \hat{a}^+ \hat{a} = 1, \quad [\hat{a}, \hat{a}^+] = [\hat{a}^+, \hat{a}^+] = 0 \Rightarrow \hat{a} \hat{a} = \hat{a}^+ \hat{a}^+ = 0
\]

\[
\hat{n} = \hat{a}^+ \hat{a}, \quad \hat{n} |n\rangle = n |n\rangle \quad \Rightarrow \quad \hat{n}^2 = \hat{n} \quad \Rightarrow \quad n = 0, 1
\]

\[
\hat{a}^+ |0\rangle = |1\rangle, \quad \hat{a} |1\rangle = |0\rangle, \quad \hat{a}^+ |1\rangle = \hat{a} |0\rangle = 0
\]

Fock space

\[
\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \ldots = \oplus_{n=0}^{\infty} \mathcal{F}_n
\]

The creation and annihilation operators

Notice: a typo in (1.73)

\[
[a_\lambda, a_\mu]_{-\zeta} = a_\lambda a_\mu - \zeta a_\mu a_\lambda = 0
\]

\[
[a_\lambda, a_\mu^+]_{-\zeta} = a_\lambda a_\mu^+ - \zeta a_\mu^+ a_\lambda = \delta_{\lambda \mu}
\]

\(\zeta = +1\) for boson, \(\zeta = -1\) for fermion

Coherent states: another set of basis for the Fock space

Eigenstates for annihilation operators

Why annihilation operators? The creation operator can NOT have an eigenstate.

Since the minimum number of particles is increased by one.

\[
a_\alpha |\phi\rangle = \phi_\alpha |\phi\rangle
\]

A significant difference between bosons and fermions

\[
[a_\lambda, a_\mu]_{-\zeta} = a_\lambda a_\mu - \zeta a_\mu a_\lambda = 0
\]

\[
[a_\alpha, a_\beta]_{-\zeta} = 0
\]

We need “anticommuting” numbers for fermions!
Boson coherent states

\[ |\phi\rangle = \sum_{n_{a_1}, n_{a_2}, \ldots n_{a_p}} \phi^{n_{a_1} n_{a_2} \ldots n_{a_p}} |n_{a_1} n_{a_2} \ldots n_{a_p} \rangle \]

\[ \Rightarrow \]

\[ \phi^{a_i} \phi^{n_{a_1} n_{a_2} \ldots (n_{a_i}-1)} = \sqrt{n_{a_i}} \phi^{n_{a_1} n_{a_2} \ldots n_{a_i}} \]

\[ \Rightarrow \]

\[ \phi^{n_{a_1} n_{a_2} \ldots n_{a_i}} = \frac{\phi^{n_{a_1}} \phi^{n_{a_2}} \ldots \phi^{n_{a_i}} \ldots}{\sqrt{n_{a_1}}! \sqrt{n_{a_2}}! \ldots \sqrt{n_{a_i}}! \ldots} \]

Using the fact that

\[ |n_{a_1} n_{a_2} \ldots n_{a_p} \rangle = \frac{\left( a_{a_1}^\dagger \right)^{n_{a_1}} \left( a_{a_2}^\dagger \right)^{n_{a_2}} \ldots \left( a_{a_p}^\dagger \right)^{n_{a_p}}}{\sqrt{n_{a_1}}! \sqrt{n_{a_2}}! \ldots \sqrt{n_{a_p}}! \ldots} |0\rangle \]

we finally obtain

\[ |\phi\rangle = \sum_{n_{a_1}, n_{a_2}, \ldots n_{a_p}} \frac{\left( \phi^{a_1} a_{a_1}^\dagger \right)^{n_{a_1}} \left( \phi^{a_2} a_{a_2}^\dagger \right)^{n_{a_2}} \ldots \left( \phi^{a_p} a_{a_p}^\dagger \right)^{n_{a_p}}}{n_{a_1}! n_{a_2}! \ldots n_{a_p}!} |0\rangle \]

\[ = e^{\sum_{a} \phi^{a} a_{a}^\dagger} |0\rangle \]

\[ \text{◆ The action of a creation operator} \]

\[ a_{a}^\dagger |\phi\rangle = a_{a}^\dagger e^{\sum_{a'} \phi^{a'} a_{a'}^\dagger} |0\rangle \]

\[ = \frac{\partial}{\partial \phi^{a}} |\phi\rangle \]

\[ \text{◆ The overlap of two coherent states} \]

\[ \langle \phi | \phi' \rangle = e^{\sum_{a} \phi^{a} \phi^{a'}} \]

\[ \text{◆ The overcompleteness in the Fock space} \]

\[ \int \prod_{a} \frac{d\phi_{a}^* d\phi_{a}}{2i\pi} e^{-\sum_{a} \phi_{a}^* \phi_{a}} |\phi\rangle \langle \phi| = 1 \]

where the measure is defined by

\[ \frac{d\phi_{a}^* d\phi_{a}}{2i\pi} = \frac{d(\text{Re} \phi_{a}) d(\text{Im} \phi_{a})}{\pi} \]
The trace of an operator

\[ \text{Tr } A = \sum_n \langle n | A | n \rangle \]

\[ = \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} \sum_n \langle n | \phi \rangle \langle \phi | A | n \rangle \]

\[ = \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} \langle \phi | A | \sum_n n | n \rangle \langle n | \phi \rangle \]

\[ = \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} \langle \phi | A | \phi \rangle . \]

Holomorphic representation

Any state of the Fock space can be represented as

\[ |\psi\rangle = \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} \psi(\phi^*) \cdot |\phi\rangle \]

where by definition

\[ \psi(\phi^*) = \langle \phi | \psi \rangle \]

is the coherent state representation in which \( \psi \) is an analytical function of \( \phi^* \).

How the annihilation/creation operator act in the coherent state representation

\[ \langle \phi | a_\alpha^\dagger | f \rangle = \frac{\partial}{\partial \phi_\alpha^*} f(\phi^*) \]

\[ \langle \phi | a_\alpha | f \rangle = \phi_\alpha^* f(\phi^*) \]

Symbolically

\[ a_\alpha = \frac{\partial}{\partial \phi_\alpha^*} \]

\[ a_\alpha^\dagger = \phi_\alpha^* \]

Matrix elements of normal-order operators

\[ \langle \phi | A(a_\alpha^\dagger, a_\alpha) | \phi' \rangle = A(\phi_\alpha^*, \phi_\alpha') e^{\sum_\alpha \phi_\alpha^* \phi_\alpha'} \]