Topological order and quantum entanglements – a new direction in many-body physics

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Pu-Tuo, May 21, 2011
Traditional many-body physics is based on two organization framework:

**Symmetry breaking** (order parameters, long range correlation)
**Landau Fermi liquid theory** (energy-level-filling picture)

It allows us to understand phases, phase transitions, metals, semiconductors, magnets, superconductors, etc.

But then we discovered fractional quantum Hall (FQH) states and high Tc superconductors, where the traditional many-body theory fails.
New era in many-body physics

- The Fermi surface in high Tc sample does not even form a loop
- Different FQH phase all have the same symmetry
The Fermi surface in high Tc sample does not even form a loop.

Different FQH phase all have the same symmetry. Despite no symmetry breaking nor order parameter, FQH states contain a new kind of order – Topological order – described by new “topological non-local order parameters”. (topological ground state degeneracy and their Berry’s phases)

Wen 89; Wen-Niu 90
New era in many-body physics

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- Different FQH phase all have the same symmetry. Despite no symmetry breaking nor order parameter, FQH states contain a new kind of order – Topological order – described by new “topological non-local order parameters”. (topological ground state degeneracy and their Berry’s phases). Wen 89; Wen-Niu 90.
- High Tc SC and FQH effect opens up a new world for us.

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Topological order and quantum entanglements – a new direction in many-body physics.
The rich world of topological orders (FQH states)

- Fractional charge (Laughlin 1983) and Fractional statistics (Arovas-Schrieffer-Wilczek 1984) in FQH states.
- Integer $K$-matrix as new “order parameter” to classify all Abelian topological orders in FQH states (Blok-Wen 1990; Read 1990).
- Gapless edge conformal field theory (CFT) encode bulk topological order $\rightarrow$ exp. measurement of topological order. (Wen 1990)
  (Holographic principle 't Hooft 1993; Susskind 1994 in condensed matter)
- Theoretical discovery of Non-Abelian (NAB) FQH states: $\nu = 1/2$ Pfaffian state (Moore-Read 1991) and $SU_k(N)$ state (emergence of $SU(N)$ Chern-Simons gauge theory) (Wen 1991)
The rich world of topological orders (FQH states)

Experimental identification of NAB FQH state:

- Discovery of $\nu = 5/2$ FQH state Willett et al 1987
- Haldane-Rezayi spin-singlet $d$-wave paired state for $\nu = 5/2$ Haldane-Rezayi 1988
- Proposed that the experimental $\nu = 5/2$ state is a $p$-wave paired FQH state Greiter-Wen-Wilczek 1991
- The $p$-wave paired state is the NAB Pfaffian state Read-Green 2000
- The $d$-wave paired state is Abelian FQH state
- Ultrahigh mobility and a clear $\nu = 5/2$ state Pan-Pfeiffer et al 2008
- Observed $e/4$ charge in $\nu = 5/2$ state Dolev-Heiblum et al 2008
- Observed NAB edge CFT in $\nu = 5/2$ state Radu-Kastner, et al 2008

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The rich world of topo. orders (High Tc and spin liquid)

Topological order is not limited to FQH states

- Gapped spin liquids (with non-trivial topological order):
  - Chiral spin state (emergence of $U(1)$ Chern-Simons gauge theory from quantum spin model). Wen-Wilczek-Zee 1989
  - $Z_2$ spin liquid (emergence of $Z_2$ gauge theory)
    Read-Sachdev 1991; Wen 1991
  - Exact Kitaev model for emergence of $Z_2$ gauge theory. Kitaev 1997
  - Exact string-net model for emergence of any non-Abelian gauge theory, fermions (in any dimensions), and NAB anyons (in 2D).
    Levin-Wen 2005
Apply to gapless states (quantum order):

- Gapless non-Fermi liquid (no quasiparticle, violate Luttinger th.):
  - $U(1)$-Fermi liquid (emergent $U(1)$ gauge field with Fermi surface) in high Tc
    - Baskaran-Anderson 1988; Nagaosa-P.Lee 1990, $\nu = 1/2$ FQH
    - Read-P.Lee-Halperin 1993, $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$
    - Shimizu-Kanoda et al. 2003;
    - Y.Lee P.Lee 2005, and hyper-kagome Na$_4$Ir$_3$O$_8$
    - Zhou-P.Lee-Ng-Zhang 2008;
The rich world of “topo.” orders (High Tc/spin liquid)

- $U(1)$-Dirac liquids/algebraic spin liquids (emergent $U(1)$ gauge field with Fermi points) in under-doped high Tc $^{1}$ Affleck-Marston 1988; Rantner-Wen 2001, and in Herbertsmithsite. Helton-Y.Lee etal 2004; Ran-Hermele-P.Lee-Wen 2005

- $Z_2$-Dirac liquid (emergent $Z_2$ gauge field with Fermi points) Senthil-Fisher 2000

- “Goldstone Theorem” for emergent gapless/massless $U(1)$ gauge field: No local perturbations can give emergent gapless/massless $U(1)$ gauge boson a finite gap/mass. Hastings-Wen 2005
What is really new in topological orders?

• Why topological phases of matter have so many exotic and fascinating properties, such as fractional charge, fractional (non-Abelian) statistics, robust gapless edge excitations, emergent and robust gauge bosons, etc. Topological phases = quantum liquids (appear to have no orders).

• Topological entanglement entropies of topological phases

Kitaev-Preskill 2006; Levin-Wen 2006 → long range entanglements.

The study of topological phases is actually a study of patterns of long-range entanglements (patterns of quantum fluctuations).

Old orders: Pattern of symmetry breaking
New topological orders: Pattern of long-range entanglements

• We want to study new physics beyond symmetry breaking and/or beyond Landau Fermi liquid.
Now we know the new physics is really long-range entanglements. → Guide the direction and strategy of our research efforts
Quantum entanglements through examples

Topological order = pattern of quantum entanglements

\[ |\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)} \]
\[ |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{entangled (quantum)} \]
\[ |\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{more entangled} \]
\[ = (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle \rightarrow \text{unentangled} \]

\[ \otimes \]
\[ = |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \rightarrow \text{unentangled} \]
\[ = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \rightarrow \text{short-range entangled (SRE) entangled} \]

The old orders (crystals, ferromagnets ...) → short-range entangled.
New topological orders (FQH state ...) → long-range entangled.

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- $|↑⟩ \otimes |↓⟩ + |↓⟩ \otimes |↑⟩ \rightarrow \text{entangled (quantum)}$

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- $(|↑⟩ + |↓⟩) \otimes (|↑⟩ + |↓⟩) \rightarrow \text{unentangled}$

- $|↓⟩ \otimes |↑⟩ \otimes |↓⟩ \otimes |↑⟩ \otimes |↓⟩ \rightarrow \text{unentangled}$

- $(|↓↑⟩ - |↑↓⟩) \otimes (|↓↑⟩ - |↑↓⟩) \rightarrow \text{short-range entangled (SRE)}$

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- $|\uparrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \rightarrow$ unentangled
- $(|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) \rightarrow$ short-range entangled (SRE) entangled

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  $= (|↑⟩ + |↓⟩) ⊗ (|↑⟩ + |↓⟩) = |x⟩ ⊗ |x⟩ →$ unentangled
Quantum entanglements through examples

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• $\cdots \rightarrow \text{unentangled}$
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• .....

$= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ $\rightarrow$ unentangled

• .....

$= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots$ $\rightarrow$

short-range entangled (SRE) entangled

The old orders (crystals, ferromagnets ...)

$\rightarrow$ short-range entangled.

New topological orders (FQH state ...)

$\rightarrow$ long-range entangled.
Patterns of many-body entanglements

Old crystal order:
\[ |\varphi_{\text{crystl}}\rangle = |\varphi_{0}\rangle \times |\varphi_{1}\rangle \times |\varphi_{0}\rangle \times \ldots \]  
→ unentangled state (classical)

Particle condensation (superfluid):
\[ |\varphi_{\text{SF}}\rangle = \sum_{\text{all conf.}} |\varphi_{0}\rangle \times |\varphi_{1}\rangle \times |\varphi_{0}\rangle \times \ldots \]  
\[ = |\varphi_{0}\rangle \times |\varphi_{1}\rangle \times |\varphi_{0}\rangle \times \ldots \]  
→ unentangled state (classical)

- Superfluid, as an exemplary quantum state of matter, is actually very classical and unquantum from entanglement point of view.
- Only after the discovery of FQH effect, did condensed matter physics enter into a real quantum world (with entanglements).

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Patterns of many-body entanglements

- Old crystal order: $|\Phi_{\text{crystl}}\rangle = |\cdots\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
  
  $= \text{direct-product state} \rightarrow \text{unentangled state (classical)}$
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  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} |\ldots\rangle$
  
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\end{array}\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + \ldots) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + \ldots) \ldots \]
  = direct-product state \(\rightarrow\) unentangled state (classical)

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Patterns of many-body entanglements (topo. orders)

To make topological order, we need to sum over different product states, but we should not sum over everything.

- Sum over some of the particle configurations by first join the particles into strings:

\[
\Phi_{\text{loop}} = \sum_{\text{all loop conf.}} \ |\Phi_{\text{loop}}\rangle
\]

which is not a direct-product state and not a local perturbation of direct-product state → a new type of topological order (a new pattern entanglements)

- FQH states

\[
\prod_{ij} (z_i - z_j)^m e^{-\frac{1}{4} \sum_i |z_i|^2}
\]

→ Boson condensation with a phase factor.
Without any symmetry $\rightarrow$ SRE state and LRE states

- All SRE states belong to one phase.
- LRE states can belong to different phases $\rightarrow$ topological orders.

*Examples: FQH states, $Z_2$ spin liquid, ...*
With some symmetries → many many new phases

For systems with a given symmetry, their ground states can be:

- SRE states with different broken symmetries → Landau’s symmetry breaking orders.
- SRE states with the same symmetry can have different orders → symmetry protected topological orders (symmetry protected trivial orders). Gu & Wen 09, Pollmann & Berg, Turner & Oshikawa 09

Examples: Haldane phase and $S = 0$ trivial phase. Band and topological insulators.

- LRE states with different broken symmetries → symmetry breaking topological orders
- LRE states that do not break the symmetry → symmetry enriched topological orders
Topological order and quantum entanglements – a new direction in many-body physics

Topological non-linear sigma model
Symmetry protected topological order (e.g., topo. insulator)
Cohomology of symmetric group
Boundary states

Wave function renormalization
Local unitary transformation
Spin liquid
Vertex algebra

String-net condensation
Tensor network
Pattern of zeros
Non-Abelian statistics

Emergent photons & electrons
Emergent gravity
Tensor category
Topological quantum field theory

ADS/CFT
Modular transformation
Classification of 3-manifolds
Topological quantum comp.

Topological quantum order = Long range entanglement