Lecture 11: Simple Harmonic Motion

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Outline

- The conditions for static equilibrium
- Motion near the stable equilibrium
- Simple harmonic motion
- The block-spring system in quantitative treatment
- Energy of the simple harmonic oscillator
For Point-like Objects

- The necessary condition for equilibrium is that the net force acting on an object be zero.
For Extended Objects

• A second condition is that the net torque about any axis be zero.

• The two conditions are the statements of
  - Translational equilibrium
  - Rotational equilibrium

Not in equilibrium!
Static Equilibrium

• An object is in equilibrium means that both the linear acceleration and the angular acceleration are zero.

• If the object is at rest and so has no linear speed or angular speed, the object is in static equilibrium.
The torque $\mathbf{t}$ involves the two vectors $\mathbf{r}$ and $\mathbf{F}$, and its direction is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$. We can establish a mathematical relationship between $\mathbf{t}$, $\mathbf{r}$, and $\mathbf{F}$, using a new mathematical operation called the vector product, or cross product.

$$\mathbf{\tau} \equiv \mathbf{r} \times \mathbf{F}$$

Will discuss more next week.
Torque about Any Point?

\[ \sum \tau_{O'} = (r_1 - r') \times F_1 + (r_2 - r') \times F_2 + (r_3 - r') \times F_3 + \cdots \]

\[ = r_1 \times F_1 + r_2 \times F_2 + r_3 \times F_3 + \cdots - r' \times (F_1 + F_2 + F_3 + \cdots) \]

\[ \sum \tau_O = r_1 \times F_1 + r_2 \times F_2 + r_3 \times F_3 + \cdots \]

\[ \sum F = F_1 + F_2 + F_3 + \cdots = 0. \]

If an object is in translational equilibrium and the net torque is zero about one point, then the net torque must be zero about any other point.
To compute the torque due to the gravitational force on an object of mass $M$, we need only consider the force $Mg$ acting at the center of gravity of the object.

The center of gravity is located at the center of mass as long as the object is in a uniform gravitational field.
Example: A Weighted Hand

\[ F_d - mg\ell = 0 \]

\[ F = 583 \text{ N} \]

- \( mg = 50.0 \text{ N} \)
- \( d = 3.00 \text{ cm} \)
- \( \ell = 35.0 \text{ cm} \)
Example: Standing on a Beam

Diagram showing a person standing on a beam with a force of 200 N and a tension force of 600 N. The beam is tilted at an angle of 53.0° and extends 8.00 m from the wall.
Example: Standing on a Beam

\[ \sum F_x = R \cos \theta - T \cos 53.0^\circ = 0 \]
\[ \sum F_y = R \sin \theta + T \sin 53.0^\circ - 600 \text{ N} - 200 \text{ N} = 0 \]

\[ \sum \tau = (T \sin 53.0^\circ)(8.00 \text{ m}) - (600 \text{ N})(2.00 \text{ m}) - (200 \text{ N})(4.00 \text{ m}) = 0 \]

\[ R \cos \theta = 188 \text{ N} \]
\[ R \sin \theta = 550 \text{ N} \]

\[ T = 313 \text{ N} \]
Example: The Leaning Ladder

Find the minimum angle at which the ladder does not slip.

\[ \sum F_x = f - P = 0 \]
\[ \sum F_y = n - mg = 0 \]

Take the torques about an axis through the origin O at the bottom of the ladder

\[ \sum \tau_O = P\ell \sin \theta - mg \frac{\ell}{2} \cos \theta = 0 \]

Because the ladder is stationary, the three forces acting on it must all pass through some common point.
More Example: Up the Curb

When the wheel is just about to be raised from the street, the reaction force exerted by the ground on the wheel at point $Q$ goes to zero.

$$mgd - F(2r - h) = 0$$
In general, positions of stable equilibrium correspond to points for which $U(x)$ is a \textbf{minimum}.

$$\frac{dU}{dx} = 0 \quad \frac{d^2U}{dx^2} > 0$$
Positions of **stable equilibrium** correspond to points for which $U(x)$ is a **minimum**.

Equilibrium ($F = 0$)

$$\frac{dU}{dx} = 0$$

Stable

$$\frac{d^2U}{dx^2} > 0$$
We can Taylor expand the potential energy near the equilibrium

\[ U(x) = U(x_0) + \frac{dU}{dx}igg|_{x=x_0} (x - x_0) + \frac{1}{2} \frac{d^2U}{dx^2}igg|_{x=x_0} (x - x_0)^2 + \cdots \]

Quite generically, we expect linear restoring force like in the block-spring system, if we neglect the higher order terms (which are usually small if we stay close enough to the equilibrium).

This universal family of motion is known as the simple harmonic motion.
Simple Harmonic Motion

- An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed.

\[ F_s = -kx = ma \]

\[ a = -\frac{k}{m} x \]
Quantitative Analysis

Define

\[ \omega = \sqrt{\frac{k}{m}} \]

\[ \frac{d^2x}{dt^2} = -\frac{k}{m} x = -\omega^2 x \]

Solution:

\[ x = A \cos(\omega t + \phi) \]

Please check that the differential equation is satisfied.
The Meaning of the Solution

\[ x = A \cos(\omega t + \phi) \]

- **A**: Amplitude of the oscillation
- **T = \(2\pi / \omega\)**: Period of the oscillation
- **\(\phi\)**: Phase constant, or phase angle

Understand how to draw the \(x-t\) curve, and how to read information from the \(x-t\) curve.
Period and Frequency

- The **period** $T$ of the motion is the time it takes for the particle to go through one full cycle.

\[ T = \frac{2\pi}{\omega} \]

- The **frequency** represents the number of oscillations that the particle makes per unit time.

\[ f = \frac{1}{T} = \frac{\omega}{2\pi} \]

The units of $f$ are cycles per second = $s^{-1}$, or **hertz** (Hz).

Angular frequency: \[ \omega = 2\pi f = \frac{2\pi}{T} \]
Not in Phase

$\pi/2$ out of phase with the displacement

$\pi$ out of phase with the displacement
Important Observations

- The acceleration of the particle is proportional to the displacement but is in the opposite direction. This is the necessary and sufficient condition for simple harmonic motion, as opposed to all other kinds of vibration.

- The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time but are not in phase.

- The frequency and the period of the motion are independent of the amplitude.
Energy of the Harmonic Oscillator

\[ K = \frac{1}{2} \, m v^2 = \frac{1}{2} \, m \omega^2 A^2 \sin^2(\omega t + \phi) \]

\[ U = \frac{1}{2} \, k x^2 = \frac{1}{2} \, k A^2 \cos^2(\omega t + \phi) \]

\[ E = K + U = \frac{1}{2} \, k A^2 \left[ \sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) \right] \]

\[ = \frac{1}{2} \, k A^2 \quad \omega = \sqrt{\frac{k}{m}} \]

The total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.
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<th>v</th>
<th>a</th>
<th>K</th>
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<td>$\frac{1}{2} kA^2$</td>
</tr>
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Energy Transform

\[ U = \frac{1}{2} kx^2 \]

\[ K = \frac{1}{2} mv^2 \]