Lecture 10: Rolling Motion and Angular Momentum

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Outline

• Rolling motion of a rigid object: center-of-mass motion + rotation around

• Generic rotational motion in the vector language
  - Kinematics/Energy/Torque

• Angular momentum of
  - A particle
  - A rotating rigid object

• Conservation of angular momentum
One light source at the center of a rolling cylinder and another at one point on the rim illustrate the different paths these two points take. The center moves in a straight line (green line), whereas the point on the rim moves in the path called a cycloid (red curve).
Pure Rolling Motion

condition for pure rolling motion

Frictional force?

\[ v_{CM} = \frac{ds}{dt} = R \frac{d\theta}{dt} = R\omega \]

\[ a_{CM} = \frac{dv_{CM}}{dt} = R \frac{d\omega}{dt} = R\alpha \]
Translation + Rotation

(a) Pure translation

CM

\[ v = v_{CM} \]

P

\[ v = 0 \]

P'

\[ v = v_{CM} \]

(b) Pure rotation

CM

\[ v = 0 \]

P

\[ v = R\omega \]

P'

\[ v = R\omega \]

\[ v = 2v_{CM} \]

(c) Combination of translation and rotation

CM

\[ v = v_{CM} \]

P

\[ v = 0 \]
At any instant, the part of the rim that is at point $P$ is at rest relative to the surface because slipping does not occur.

In other words, all points rotate about $P$. But, how to calculate the moment of inertia about $P$?
Parallel-Axis Theorem

\[ \vec{r}' = \vec{r} - \vec{r}_{CM} \]
Parallel-Axis Theorem

- Define \( \vec{r}' = \vec{r} - \vec{r}_{CM} \)

\[
K = \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)
\]

\[
= \frac{1}{2} \sum_i m_i [\vec{\omega} \times (\vec{r}_i' + \vec{r}_{CM})] \cdot [\vec{\omega} \times (\vec{r}_i' + \vec{r}_{CM})]
\]

\[
= \frac{1}{2} \sum_i m_i [\vec{\omega} \times \vec{r}_i'] \cdot [\vec{\omega} \times \vec{r}_i'] \quad (1)
\]

\[
+ \sum_i m_i [\vec{\omega} \times \vec{r}_i'] \cdot [\vec{\omega} \times \vec{r}_{CM}] \quad (2)
\]

\[
+ \frac{1}{2} \sum_i m_i [\vec{\omega} \times \vec{r}_{CM}] \cdot [\vec{\omega} \times \vec{r}_{CM}] \quad (3)
\]
Parallel-Axis Theorem

• Term (1): The rotational energy about the parallel axis through the center of mass

\[ K' = \frac{1}{2} \sum_i m_i [\vec{\omega} \times \vec{r}_i'] : [\vec{\omega} \times \vec{r}_i'] \]

• Term (2):

\[ \sum_i m_i [\vec{\omega} \times \vec{r}_i'] : [\vec{\omega} \times \vec{r}_{CM}] = 0 \]

\[ \sum_i m_i \vec{r}_i' = \sum_i m_i \vec{r}_i - \left( \sum_i m_i \right) \vec{r}_{CM} = 0 \]

• Term (3): as if the object shrink to the center of mass

\[ K_{CM} = \frac{1}{2} \left( \sum_i m_i \right) [\vec{\omega} \times \vec{r}_{CM}] : [\vec{\omega} \times \vec{r}_{CM}] \]
Parallel-Axis Theorem

Suppose the moment of inertia about an axis through the center of mass of an object is $I_{CM}$. The parallel-axis theorem states that the moment of inertia about any axis parallel to and a distance $D$ away from this axis is

$$I = I_{CM} + MD^2$$
The total kinetic energy of the rolling cylinder

\[ K = \frac{1}{2} I_P \omega^2 \]

\[ I_P = I_{CM} + MR^2 \]

\[ K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} MR^2 \omega^2 \]

\[ v_{CM} = R\omega \]

- Rotational kinetic energy about the center of mass
- Translational kinetic energy of the center of mass
Example: Rolling Sphere

- For the solid sphere shown below, calculate the linear speed of the center of mass at the bottom of the incline and the magnitude of the linear acceleration of the center of mass.

  - Accelerated rolling motion is possible only if a frictional force is present between the sphere and the incline to produce a net torque about the center of mass.
  - Despite the presence of friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant.
Moment of Inertia

\[ K = \frac{1}{2} I_{CM} \left( \frac{\nu_{CM}}{R} \right)^2 + \frac{1}{2} M \nu_{CM}^2 \]

\[ K = \frac{1}{2} \left( \frac{I_{CM}}{R^2} + M \right) \nu_{CM}^2 \]

Now, try it yourself.

Solid sphere

\[ I_{CM} = \frac{2}{5} MR^2 \]
First we calculate the moment of inertia of a spherical shell about a diagonal axis.

\[ I_z = \int dm \left( x^2 + y^2 \right) \]

By the rotational symmetry,

\[ I_x = I_y = I_z = \frac{1}{3} (I_x + I_y + I_z) \]

\[ = \frac{2}{3} \int dm \left( x^2 + y^2 + z^2 \right) = \frac{2}{3} \int dm R^2 = \frac{2}{3} M R^2 \]
Appendix: Solid Sphere

- By dimension analysis, the moment of inertia of a solid sphere (density $\rho$) is
  \[
  I(R) = cMR^2 \propto c\rho \frac{4\pi}{3} R^5
  \]
  Since a spherical shell can be written as the difference of two solid spheres
  \[
  I(R+\Delta R) - I(R) = \frac{2}{3}(\rho 4\pi R^2 \Delta R) R^2
  \]
  \[
  \Delta R \to 0 \quad dI = \frac{8\pi\rho}{3} R^4 dR
  \]
  \[
  I = \frac{8\pi\rho}{15} R^5 = \frac{2}{5} MR^2
  \]
Conservation of Energy

\[ \frac{1}{2} \left( \frac{I_{CM}}{R^2} + M \right) v_{CM}^2 = M g h \]

\[ v_{CM} = \left( \frac{2gh}{1 + \frac{2}{5} MR^2} \right)^{1/2} = \left( \frac{10}{7} gh \right)^{1/2} \]

Kinematic equation

\[ v_{CM}^2 = 2a_{CM} x \]

\[ a_{CM} = \frac{5}{7} g \sin \theta \]
• All homogeneous solid spheres experience the same speed and acceleration on a given incline, independent of
  - mass, and
  - the radius of the sphere
• Try hollow sphere, solid cylinder, or hoop, you will find:
  - Only the dimensionless factor would differ.
  - The factors depend only on the moment of inertia about the center of mass for the specific body.
• In all cases, the acceleration of the center of mass is less than the value the acceleration would have if the incline were frictionless and no rolling occurred.
The frictional force does not change the total kinetic energy, but it contributes to $F$ and decreases the acceleration of the center of mass.
• The torque $\tau$ involves the two vectors $\mathbf{r}$ and $\mathbf{F}$, and its direction is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$. We can establish a mathematical relationship between $\tau$, $\mathbf{r}$, and $\mathbf{F}$, using a new mathematical operation called the vector product, or cross product.

$$\tau \equiv \mathbf{r} \times \mathbf{F}$$

Will discuss more next week.
Angular Momentum: Motivation

Newton's second law

\[ \sum \mathbf{F} = \frac{d\mathbf{p}}{dt} \]

\[ \sum \tau = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \]

\[ \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p} \]

\[ \sum \tau = \frac{d\mathbf{L}}{dt} \]

(rotational analog)

\[ \mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \]

Angular momentum

\[ \mathbf{v} \times (m\mathbf{v}) = 0 \]
Angular Momentum: Definition

- The instantaneous angular momentum \( L \) of the particle relative to the origin \( O \) is defined as the cross product of the particle’s instantaneous position vector \( \mathbf{r} \) and its instantaneous linear momentum \( p \).

\[
L = \mathbf{r} \times p
\]

Magnitude:

\[
L = mvr \sin \phi
\]

SI Unit:

\[
\text{kg} \cdot \text{m}^2/\text{s}
\]

\( L \) is zero when \( \mathbf{r} \) is parallel or antiparallel to \( p \).
The net torque acting on a particle is equal to the time rate of change of the particle’s angular momentum.

\[ \sum \tau = \frac{dL}{dt} \]

Both the magnitude and the direction of \( L \) depend on the choice of origin.

The expression is valid for any origin fixed in an inertial frame, as long as \( L \) and \( \tau \) are measured about the same origin.
A System of Particles

- The time rate of change of the total angular momentum of a system about some origin in an inertial frame equals the net external torque acting on the system about that origin.

\[
\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i
\]

\[
\sum \tau_{\text{ext}} = \sum_i \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt} \sum_i \mathbf{L}_i = \frac{d\mathbf{L}}{dt}
\]

The moment arm \( d \) from \( O \) to the line of action of the internal forces is equal for both particles.

Recall \( \sum \mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}}{dt} \)
A Rotating Rigid Object

\[ L_z = \sum_i m_i r_i^2 \omega = \left( \sum_i m_i r_i^2 \right) \omega \]

\[ L_z = I \omega \]

\[ \frac{dL_z}{dt} = I \frac{d\omega}{dt} = I \alpha \]

\[ \sum \tau_{\text{ext}} = \frac{dL_z}{dt} = I \alpha \]
The Scalar Version

- Now extend to a rigid object of arbitrary shape rotating about a fixed axis.

\[ dF_t = (dm) a_t \]

\[ d\tau = r \, dF_t = (r \, dm) a_t \]

\[ d\tau = (r \, dm) r\alpha = (r^2 \, dm) \alpha \]

\[ \sum \tau = \int (r^2 \, dm) \alpha = \alpha \int r^2 \, dm = I\alpha \]

\( I \): The moment of inertia
\[ \vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i = \sum_i \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i) \]

\[ = \sum_i \left[ m_i r_i^2 \vec{\omega} - m_i \vec{r}_i (\vec{r}_i \cdot \vec{\omega}) \right] \]

Identity: \( \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) \)

Let \( \vec{r}_i = x_i \hat{x} + y_i \hat{y} + z_i \hat{z} \)

\[ \vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z} \]

\[ L_x = \sum_i m_i (y_i^2 + z_i^2) \omega_x - \sum_i m_i x_i y_i \omega_y - \sum_i m_i x_i z_i \omega_z \]

\[ L_y = -\sum_i m_i y_i x_i \omega_x + \sum_i m_i (z_i^2 + x_i^2) \omega_y - \sum_i m_i y_i z_i \omega_z \]

\[ L_z = -\sum_i m_i z_i x_i \omega_x - \sum_i m_i z_i y_i \omega_y + \sum_i m_i (x_i^2 + y_i^2) \omega_z \]
Moment of Inertia is a Matrix

\[
L_x = \sum_i m_i (y_i^2 + z_i^2) \omega_x - \sum_i m_i x_i y_i \omega_y - \sum_i m_i x_i z_i \omega_z
\]

\[
L_y = -\sum_i m_i y_i x_i \omega_x + \sum_i m_i (z_i^2 + x_i^2) \omega_y - \sum_i m_i y_i z_i \omega_z
\]

\[
L_z = -\sum_i m_i z_i x_i \omega_x - \sum_i m_i z_i y_i \omega_y + \sum_i m_i (x_i^2 + y_i^2) \omega_z
\]

\[
\begin{bmatrix}
L_x \\
L_y \\
L_z
\end{bmatrix} =
\begin{bmatrix}
I_{xx} & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} & I_{yz} \\
I_{zx} & I_{zy} & I_{zz}
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
\]

Now you should be ready to advance by yourself.
Symmetry Tells Us

\[ \bar{I} = \begin{pmatrix}
I_0 & 0 & 0 \\
0 & I_0 & 0 \\
0 & 0 & I_0 \\
\end{pmatrix} \]

\[ \bar{I} = \begin{pmatrix}
I_x & 0 & 0 \\
0 & I_0 & 0 \\
0 & 0 & I_0 \\
\end{pmatrix} \]
A rigid rod of mass $M$ and length $l$ is pivoted without friction at its center. Two particles of masses $m_1$ and $m_2$ are connected to its ends. The combination rotates in a vertical plane with an angular speed $\omega$. 

(a) Find an expression for the magnitude of the angular momentum of the system.

(b) Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle $\theta$ with the horizontal.
\[ I = \frac{1}{12} M \ell^2 + m_1 \left( \frac{\ell}{2} \right)^2 + m_2 \left( \frac{\ell}{2} \right)^2 \]
\[ = \frac{\ell^2}{4} \left( \frac{M}{3} + m_1 + m_2 \right) \]

\[ L = I \omega = \frac{\ell^2}{4} \left( \frac{M}{3} + m_1 + m_2 \right) \omega \]

\[ \sum \tau_{\text{ext}} = \tau_1 + \tau_2 = \frac{1}{2} (m_1 - m_2) g \ell \cos \theta \]

\[ \alpha = \frac{\sum \tau_{\text{ext}}}{I} = \frac{2 (m_1 - m_2) g \cos \theta}{\ell (M/3 + m_1 + m_2)} \]

\[ \tau_2 = -m_2 g \frac{\ell}{2} \cos \theta \quad \text{(}\tau_2 \text{ into page)} \]

\[ \tau_1 = m_1 g \frac{\ell}{2} \cos \theta \quad \text{(}\tau_1 \text{ out of page)} \]
• The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

\[ \sum \tau_{\text{ext}} = \frac{dL}{dt} = 0 \quad \Rightarrow \quad L = \text{constant} \]

If the system is an object rotating about a fixed axis, such as the \( z \) axis,

\[ I_i \omega_i = I_f \omega_f = \text{constant} \]
Figure Skating

Remember that momentum is conserved in the absence of external forces. Likewise, **angular momentum** is conserved in the absence of external torques.

The skater begins spinning with her arms extended. But when she pulls in her arms, her rotational inertia goes down. Her angular momentum remains constant—so her angular velocity increases!

Anything wrong?
Kepler’s Second Law

- When a force is directed toward or away from a fixed point and is function of \( r \) only, it is called a central force.

\[
\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F\hat{r} = 0
\]

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times M_p\mathbf{v} = M_p\mathbf{r} \times \mathbf{v} = \text{constant}
\]
Kepler’s Second Law

- The radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

\[ dA = \frac{1}{2} \left| \mathbf{r} \times d\mathbf{r} \right| = \frac{1}{2} \left| \mathbf{r} \times \mathbf{v} \right| dt = \frac{L}{2M_p} \, dt \]

\[ \frac{dA}{dt} = \frac{L}{2M_p} = \text{constant} \]
Isolated Systems

- The (mechanical) energy, linear momentum, and angular momentum of an isolated system all remain constant.

\[
\begin{align*}
K_i + U_i &= K_f + U_f \\
p_i &= p_f \\
L_i &= L_f
\end{align*}
\]

For an isolated system
Example: Ball and Stick

\[ m_d v_{di} = m_d v_{df} + m_s v_s \]
\[ -r m_d v_{di} = -r m_d v_{df} - I\omega \]
\[ \frac{1}{2} m_d v_{di}^2 = \frac{1}{2} m_d v_{df}^2 + \frac{1}{2} m_s v_s^2 + \frac{1}{2} I\omega^2 \]