Bipartite Entanglement in the Two-Mode Quantum Kicked Top

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(Received 24 July 2006)

We show that the bipartite entanglement in the two-mode quantum kicked top can reveal the underlying chaotic and regular structures in phase space: namely, the entanglement displays a rapid rise after a very short time for an initial spin coherent state centred in a chaotic region of the phase space, whereas the entanglement displays a periodic modulation for the coherent state centred at an elliptic fixed point. The quantum–classical correspondence is investigated by studying the mean and maximal linear entropy.

PACS: 05.45.Mt, 03.65.Ud, 42.50.Dv

Classical and quantum chaos are important physical phenomena in various fields of physics. Different signatures of quantum chaos have been identified, such as the spectral properties of the generating Hamiltonian,\textsuperscript{[1]} phase space scarring,\textsuperscript{[2]} hypersensitivity to perturbation,\textsuperscript{[3]} and fidelity decay,\textsuperscript{[4]} which indicate an underlying chaotic presence in the corresponding quantum dynamics. Recently, the entanglement was identified as another signature of quantum chaos.\textsuperscript{[5–14]} Since entanglement is at the heart of quantum mechanics and is a crucial resource for quantum information processing,\textsuperscript{[15, 16]} the entanglement inherent in quantum chaotic systems could provide a valuable approach to studying quantum chaos.

Quantum entanglement results from the superposition principle in quantum mechanics and the tensor product structure in Hilbert space. The most well known example is the singlet state for two spin halves, which displays maximal entanglement. It is pure quantum mechanical effect and has potential applications in the study of quantum chaos and quantum phase transitions.

In this Letter, we consider the quantum kicked top (QKT) model,\textsuperscript{[17–22]} i.e. a representative spin model, which exhibits chaos in the classical limit. The Hilbert space for the QKT model is finite and the Poincaré section of the phase space compact, allowing analysis of quantum and classical dynamics, and facilitating the study of the role of entanglement in the system. An advantage of the dynamics of the QKT model is that it obeys a spin algebra symmetry. Since the QKT model is a single spin system, we cannot discuss entanglement in the system. Here we use the Schwinger representation to represent the spin, namely, use two bosonic modes to realize the spin.

In the Schwinger representation, the angular mo-

mentum operators are defined as
\begin{equation}
J_x = \frac{1}{2}(a_1^+ a_2 + a_2^+ a_1),
\end{equation}
\begin{equation}
J_y = \frac{1}{2i}(a_1^+ a_2 - a_2^+ a_1),
\end{equation}
\begin{equation}
J_z = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2),
\end{equation}
where $J_\alpha (\alpha \in \{x, y, z\})$ are spin operators and states are restricted to irreps $j$, for which $J^2 = j(j + 1)$, and $a_1^+$ and $a_2^+$ denote the creation operators in the two bosonic modes, respectively. For later use, we define the integer $N = 2j$.

The QKT model is described by the Hamiltonian\textsuperscript{[17–20]}
\begin{equation}
H = \frac{\kappa}{2j\tau}J_z^2 + p J_y \sum_{n=-\infty}^{\infty} \delta(t - n\tau),
\end{equation}
where $\tau$ is the duration between periodic kicks, $p$ is the strength of each kick (which is manifested as a turn by angle $p$), and $\kappa$ is the strength of the twist. The Hamiltonian is an alternative sequence of twists ($J_z^2$-term) and turns ($J_y$-term). The two-mode QKT can be viewed as a composite system with two interacting modes, and the entanglement naturally exists. An entanglement measure between two modes can be given by the linear entropy\textsuperscript{[23]}
\begin{equation}
E_l = 1 - \text{Tr}_1(\rho_1^2),
\end{equation}
where $\rho_1$ is the reduced density matrix for the first mode by tracing out the second mode. Finally, it is important to note that the QKT model in principle can be realized in Bose–Einstein condensates.\textsuperscript{[23]} In the two-component BEC, each BEC can be approximated by a single bosonic mode. The twist operations

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\textsuperscript{*} Supported by the National Natural Science Foundation of China under Grant No 10405019, the Specialized Research Fund for the Doctoral Programme of Higher Education (SRFDP) of China under Grant No 20050335087, and the Project-sponsored by SRF for ROCS, SEM.

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originate from atom–atom interactions and the turn operations stem from atom tunneling.

A standard dynamical description of the QKT model is via the Floquet operator
\[ F = \exp \left( -i \frac{\kappa}{2j^r} J_z^2 \right) \exp(-ip J_y), \]
where the energy is rescaled so that \( \tau = 1 \) and \( p = \pi/2 \) are henceforth assumed. An arbitrary state \( |\Psi(0)\rangle \) evolves to \( |\Psi(n)\rangle = F^n |\Psi(0)\rangle \).

The classical limit of the QKT model is obtained by expression \( X = \langle J_x/j \rangle \) and similar for \( Y \) and \( Z \) and factorizing all moments such as \( \langle J_x J_y / j^2 \rangle = XY \) to products of first-order moments. Then the classical equations of motion, obtained from the Heisenberg operator equations of motion and applying the factorization rule above, are given by \( X' = Z \cos(\kappa X) + Y \sin(\kappa X), Y' = -Z \sin(\kappa X) + Y \cos(\kappa X), Z' = -X \).

The stroboscopic evolution described by this equation can be represented in a phase space given by a sphere \( S^2 \) of unit radius. The classical, normalized angular momentum variables \( (X, Y, Z) \) can be parametrized in polar coordinates as \( (X, Y, Z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), where \( \theta \) and \( \phi \) are the polar and azimuthal angles, respectively. Thus the map is essentially two-dimensional.

For completeness, we give the stroboscopic dynamics of the classical map, as shown in Fig. 1. In the plot, we choose the chaoticity parameter \( \kappa = 3 \), which yields a mixture of regular and chaotic areas of significant size. Elliptic fixed points surrounded by the chaotic sea are evident. Two such elliptic fixed points have coordinates \( (\theta, \phi) = (2.25, -2.5) \) and \( (\theta, \phi) = (2.25, 0.63) \). As we will see, this phase space structure of the classical kicked top determines the behaviour of quantum entanglement in the QKT model.

The connection between the quantum and classical dynamics of QKT is performed by choosing the initial state to be the spin coherent state (SCS)\(^\text{[24]}\) whose centre is at a classical phase space point. The SCS is defined as \( \{ \theta_0, \phi_0 \} = R(\theta_0, \phi_0 | j, j) ; -\pi \leq \phi_0 \leq \pi, 0 \leq \theta_0 \leq \pi \} \) with \( \text{[24]} \)
\[ R(\theta_0, \phi_0) = \exp \{ i \theta_0 [J_x \sin \phi_0 - J_y \cos \phi_0] \}. \]

The mean of \( J / j \) is
\[ \langle \theta, \phi | J / j | \theta, \phi \rangle = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \]

Now, we start from studying the dynamics of entanglement for initial states with a mean value in four different regions of the phase space, specifically a fixed point, a chaotic region, and the border between the integrable and the chaotic region. This line includes all four regions we are exploring. An elliptic fixed point arises at \( \phi = 0.63 \), the border occurs at \( \phi = 0.3 \), and a point well in the chaotic sea is located at \( \phi = -1.0 \). The states with means at each of these points in the phase space are chosen to be SCSs.

![Fig. 1. Stroboscopic phase space dynamics of the classical kicked top for \( \kappa = 3 \). Three hundreds of stroboscopic trajectories are plotted, each for a duration of 233 kicks.](image1.png)

![Fig. 2. Dynamical evolution of the linear entropy for the initial SCS with \( \theta = 2.25 \) and different \( \phi \). The parameters \( N = 300, \kappa = 3 \).](image2.png)

We display the numerical results of dynamical evolution of the linear entropy in Fig. 2. Initially, the linear entropy is nonzero, implying that the initial state is an entangled state. The entanglement for the state is enhanced for the initial state centred in the chaotic region after a very short time. As the dynamics evolves, the entropy exhibits a rapid rise for a state centred in the chaotic region. The curve with \( \phi = 0.3 \) displays the intermediate behaviour.

In Fig. 3, we plot the long-time behaviour of the linear entropy for \( \phi = 0.63 \) and \( n = 5000 \). We observe strong oscillations of entanglement. Indeed, we can see that there is a good quantum–classical correspondence, namely, the underlying classical chaos has a strong effects on the quantum behaviour of entangle-
ment. Conversely, we can know the underlying classical chaos by studying the quantum entanglement.

![Graph](image1)

**Fig. 3.** Long-time behaviour of the linear entropy for \( \theta = 2.25 \) and \( \phi = -1 \). The parameters \( N = 300, \kappa = 3 \).

In this study, we also explore both the mean entanglement and maximal entanglement over all time to study the quantum-classical correspondence. In practice, for numerical purposes we consider a finite time domain (the average is over \( M = 200 \) steps here). We define mean and maximal entanglement as follows:

\[
E_{\text{mean}} = \frac{1}{M} \sum_{k=1}^{M} E_i(k), \quad E_{\text{max}} = \max_{k=1, \ldots, M} \{E_i(k)\}.
\]

These two quantities can be considered as entangling powers of our unitary operations.

![Graph](image2)

**Fig. 4.** Maximal linear entropy and the mean linear entropy against \( \phi \). The parameters \( N = 50, \theta = 2.25, \kappa = 3 \). The average is over 200 steps.

We fix the polar angle \( \theta = 2.25 \) of the SCS and vary the azimuthal angle. The centre of the SCS wave packet thus commences in the chaotic region and passes through two regular islands. Figure 4 displays the mean and max linear entropy a function of the azimuthal angle. When the azimuthal angle goes from \(-\pi\) to the first regular region, the mean linear entropy decreases until it reaches a minimum which approximately corresponds to the fixed point. Subsequently the mean entropy increases to larger region corresponding to the chaotic one. Then, the mean entropy decreases again due to the approaching to another fixed point. It is very clear that the two dips of the entropy corresponding to the two classical fixed points. The maximal linear entropy displays similar behaviour, besides the more oscillations in the chaotic region. Hence we find a good classical-quantum correspondence.

In the previous context, we did not pay more attention to the effects of dimension on the entanglement. Here we consider how the maximal linear entropy vary with \( N \). The numerical results are given in Fig. 5. With the increase of \( N \), initially the maximal linear entropy first displays a rapid rise, and then an oscillation. The oscillations are due to the parity effects (the parity of \( N \)). When \( N \) is large enough, parity effects become not obvious, the oscillation vanishes, and the maximal entanglement reaches a steady value close to one.

![Graph](image3)

**Fig. 5.** Maximal linear entropy versus \( N \). The parameters \( \theta = 2.25, \phi = 0.63, \kappa = 3 \). The maximization is over 100 steps.

In conclusion, we have studied the bipartite two-mode entanglement in the QKT model via the Schwinger representation. The dynamics of the linear entropy are found to be strongly dependent on the initial SCS. The entanglement displays a rapid rise after a very short time for an initial spin coherent state centred in a chaotic region of the phase space, whereas the entanglement displays a periodic modulation for the coherent state centred in an elliptic fixed point.

The mean and maximal linear entropy over time have been used to study the quantum-classical correspondence. We observe a good quantum-classical correspondence. We also studied the effects of dimension on entanglement. It is desirable to study entanglement in other two-mode systems which exhibits quantum chaos, and investigate the universality of the entanglement features observed here.
References

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