Mixed-state fidelity and quantum criticality at finite temperature

Paolo Zanardi,1,2 H. T. Quan,3 Xiaoguang Wang,4 and C. P. Sun3

1Institute for Scientific Interchange, Villa Gualino, Viale Settimo Severo 65, I-10133 Torino, Italy
2Department of Physics and Astronomy, University of Southern California Los Angeles, California 90089-0484, USA
3Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, China
4Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China

(Received 30 November 2006; revised manuscript received 8 January 2007; published 16 March 2007)

We extend to finite temperature the fidelity approach to quantum phase transitions (QPTs). This is done by resorting to the notion of mixed-state fidelity that allows one to compare two density matrices corresponding to two different thermal states. By exploiting the same concept we also propose a finite-temperature generalization of the Loschmidt echo. Explicit analytical expressions of these quantities are given for a class of quasifree fermionic Hamiltonians. A numerical analysis is performed as well showing that the associated QPTs show their signatures in a finite range of temperatures.

DOI: 10.1103/PhysRevA.75.032109 PACS number(s): 03.65.Ud, 05.70.Jk, 05.45.Mt

I. INTRODUCTION

Even, small, and smooth changes in parameters governing the dynamics of a physical system can, in some circumstances, result in a dramatic change of physical properties of the system itself. The traditional approach to these so-called critical phenomena is based on the notions of order parameter, correlation functions, symmetry breaking, and a general formulation in the framework of the Landau-Ginzburg picture and renormalization group [1]. A phase transition can be triggered, e.g., by a change of temperature of the system or, at zero temperature, by a change of some of the coupling constants (e.g., external fields) defining the system’s Hamiltonian. In the first case one says that the transition is driven by thermal fluctuations, and the transition is referred to as classical, whereas in the second quantum fluctuations are held responsible for the transition and this latter is referred to as a quantum phase transition (QPT) [2].

In the last few years much interest has grown about the possibility of studying QPTs by means of ideas and tools borrowed from the new born field of quantum information science [3]. In this novel approach the key concept involved is quantum entanglement (or genuinely quantum correlations) and the idea is that quantum criticality can be suitably characterized in terms of the behavior of different entanglement measures [4]. Though the picture is still rapidly moving it is by now clear that, whereas there are no doubts that entanglement is indeed a valuable conceptual tool to analyze QPTs, one has, case by case, identified what is the entanglement measure, e.g., block entanglement or concurrence most suited to extract the relevant information.

More recently an approach to QPTs based on another quantum information notion, i.e., quantum fidelity, has been put forward [5,6]. The idea behind this novel approach is quite simple: the dramatic change of the structure of the ground state occurring at the critical points can be fruitfully studied by analyzing their degree of distinguishability. Since this latter quantity is related to the overlap, i.e., scalar product, between two different ground states, the fidelity approach is basically nothing but a metric one, the key ingredient being provided by the state-space distance between states corresponding to two slightly different sets of coupling constants. The expectation is that at the critical points a small change of the control parameters should result in an enhanced modification of the state structure and this in turn should be detected by a greater state-space distance, i.e., statistical distinguishability, between the associated quantum states. In spite of its apparent naivety, this metric-based approach turns out to be able to provide an effective way to obtain qualitative as well as quantitative information about the zero-temperature phase diagram of a large class of nontrivial quantum systems, i.e., quasifree fermionic systems [6] and matrix-product states [7]. The conceptual appeal of the fidelity approach for detecting boundaries between different phases lies in its universal geometrical as well as information-theoretic nature; in principle no a priori knowledge of the symmetry-breaking mechanism and of the associated order parameters is required.

In this paper we are going to extend the fidelity approach to finite temperature. This generalization can be achieved by exploiting the mixed-state fidelity introduced by Uhlmann [8] and related to the statistical distance between two density operators (Bures distance). This finite-temperature extension of the fidelity approach is a nontrivial technical step necessary to analyze the signatures of QPTs at nonzero temperatures and, more in general, to investigate the potential usefulness of the fidelity notion in the study of classical, i.e., temperature-driven phase transitions [9]. In the following we will focus on the first task. Moreover, we will provide an explicit finite-temperature extension of another concept that has been recently used in the context of QPTs and inspired the whole fidelity program: the Loschmidt echo [10,11].

The paper is organized as follows: In Sec. II, we analytically calculate the mixed-state fidelity of thermal states and a numerical analysis is used to demonstrate the signatures of QPTs at finite temperature. In Sec. III, we calculate the Loschmidt echo through a similar procedure. Section IV contains the conclusion.

II. MIXED-STATE FIDELITY OF THERMAL STATES

Let us consider the set of (mixed) quantum states \(S(\mathcal{H}) \defeq \{ \rho \in \mathcal{L}(\mathcal{H}) | \rho \geq 0, \text{tr}\rho = 1 \} \). The mixed-state fidelity is given by [8]

\[ F(\rho, \sigma) = \frac{1}{\sqrt{\text{tr}\rho\sigma}} \text{tr} \rho^{1/2} \sigma^{1/2} \]

for states \( \rho, \sigma \) in the set of states, where the Bures distance in a class of quasifree fermionic Hamiltonians.

A numerical analysis is performed as well showing that the associated QPTs show their signatures in a finite range of temperatures.
This quantity measures the degree of distinguishability between the two quantum states \( \rho_0 \) and \( \rho_1 \). The fidelity is related to the statistical Bures distance: \( D(\rho_0, \rho_1) = \sqrt{2(1-F)} \). We will use Eq. (1) to compare two different thermal states,

\[
\rho_{\alpha} = Z_{\alpha}^{-1} \exp(-\beta_{\alpha} H_{\alpha}),
\]

\[
Z_{\alpha} := \text{tr} \exp(-\beta_{\alpha} H_{\alpha}), \quad (\alpha = 0, 1).
\]

We define

\[
F_{H_0, H_1}(\beta_0, \beta_1) = \frac{\text{tr}(Z_0^{-1} e^{-\beta_0 H_0} Z_1^{-1} e^{-\beta_0 H_1})}{\sqrt{\text{tr}(Z_0^{1/2} \text{tr} \rho_0 Z_0^{1/2}) \sqrt{\text{tr}(Z_1^{1/2} \text{tr} \rho_1 Z_1^{1/2})}}}. \tag{6}
\]

We note in passing that this relation seems to suggest the possibility of making a direct connection between the metric or statistical notion of fidelity and purely thermodynamical quantities as well as the viability of the fidelity approach even for classical systems \([9]\). We would like also to observe that Eq. (6) also gives the (pure-state) fidelity between the pure quantum states that one associates to the thermal states \( \rho_0 \) and \( \rho_1 \) through the kind of classical-quantum correspondence discussed in \([12]\) [see Eq. (1) there].

To further exemplify this commuting case let us consider diagonal fermionic Hamiltonians \( H_{\alpha} = \sum_k \epsilon_k c_k^\dagger c_k \). One has

\[
\rho_{\alpha}(\beta) = Z_{\alpha}^{-1} \prod_k e^{-\beta_{\alpha} \epsilon_k c_k^\dagger c_k} = \prod_k \left( 1 + e^{-\beta_{\alpha} \epsilon_k} \right)^{-1} \left[ 1 + (e^{-\beta_{\alpha} \epsilon_k}) \right].
\]

From this one finds

\[
F_{H_0, H_1}(\beta_0, \beta_1) = \prod_k \frac{1 + e^{-(\beta_0 \epsilon_k + \beta_1 \epsilon_k) / 2}}{(1 + e^{-\beta_0 \epsilon_k}) (1 + e^{-\beta_1 \epsilon_k})}. \tag{7}
\]

In the zero-temperature limit \( \beta \to \infty \), from the above equation, it is easy to check that \( \forall k, \epsilon_k^0 \epsilon_k^1 > 0 \Rightarrow F_{H_0, H_1}(\infty, \infty) = 1 \), whereas if \( \exists k, \epsilon_k^0 \epsilon_k^1 < 0 \Rightarrow F_{H_0, H_1}(\infty, \infty) = 0 \). This situation is the one that encounters in the XY-model analysis of \([5]\) in the critical line with anisotropic parameter \( \gamma = 0 \). There indeed when the magnetic field \( \lambda \) (which plays the role of a chemical potential in the fermionic picture) is changed in the range \([-1, 1]\) one has exactly that one of the single-particle eigenvalues changes sign and this results in a vanishing fidelity \([5]\).
Moreover, 
\[ \exp(-\beta_{\nu}H_{\nu}^{0}) = g_{\nu}^{0}(\beta_{\nu}) \oplus 1_{k-k}, \]
where 
\[ g_{\nu}^{0}(\beta) = \exp(-\beta\Lambda_{\nu}^{z}J_{\nu}^{z}) = \cosh(\beta\Lambda_{\nu}^{z}) - J_{\nu}^{z}\sinh(\beta\Lambda_{\nu}^{z}) \]
is a 2 \times 2 operator in the even sector, and 
\[ Z_{\nu}^{k} = 2 + 2\cosh(\beta\Lambda_{\nu}^{z}). \]
The fidelity for the two thermal states is then given by 
\[ \mathcal{F}_{H_{\nu}H_{\nu}^{1}}(\beta_{\nu},\beta_{\nu}^{1}) = \frac{2 + \text{tr}[Z_{\nu}^{k}]^{1/2}g_{\nu}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu}^{1})^{1/2}}{\sqrt{Z_{\nu}^{k}}}. \]  
(16)

So, apart from the trivial terms in the odd sector in order to compute the fidelity, one has to consider the products 
\[ g_{\nu}^{1}(\beta_{\nu}/2)g_{\nu}^{1}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu}/2). \]  
As this is a product of 2 \times 2 matrices, one has 
\[ \text{tr}[g_{\nu}^{1}(\beta_{\nu})^{1/2}g_{\nu}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu}^{1})^{1/2}] = \sqrt{\text{Tr}[g_{\nu}^{1}(\beta_{\nu})g_{\nu}(\beta_{\nu}) + 2\text{det}[g_{\nu}^{1}(\beta_{\nu})g_{\nu}(\beta_{\nu})]}. \]

Note that here \[ \text{det}[g_{\nu}^{1}(\beta_{\nu})g_{\nu}(\beta_{\nu})] = 1, \text{ then substituting the above equation to Eq. (16) leads to a simple expression of the fidelity} \]
\[ \mathcal{F}_{H_{\nu}H_{\nu}^{1}}(\beta_{\nu},\beta_{\nu}^{1}) = \frac{2 + \text{tr}[g_{\nu}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu}^{1})]^{1/2}}{\sqrt{Z_{\nu}^{k}}}, \]  
(17)

Now, we are left to compute the trace of 
\[ g_{\nu}^{1}(\beta_{\nu}/2)g_{\nu}^{1}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu}/2). \]
The matrix product 
\[ g_{\nu}^{1}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu}) \]
can be written as 
\[ \text{Tr}[g_{\nu}^{1}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu})] = \text{Tr}(e^{-\beta_{\nu}\Lambda_{\nu}^{z}\sigma_{z}}e^{i\alpha_{\nu}\sigma_{x}}e^{-\beta_{\nu}^{1}\Lambda_{\nu}^{z}\sigma_{z}}e^{-i\alpha_{\nu}\sigma_{x}}) \]
\[ = \text{Tr}[(\cosh(\beta_{\nu}\Lambda_{\nu}^{z}) - \sinh(\beta_{\nu}\Lambda_{\nu}^{z})\sigma_{x})] \]
\[ \times e^{i\alpha_{\nu}\sigma_{x}}e^{-\beta_{\nu}^{1}\Lambda_{\nu}^{z}\sigma_{z}}e^{-i\alpha_{\nu}\sigma_{x}}) \]
\[ = 2\cosh(\beta_{\nu}\Lambda_{\nu}^{z})\cosh(\beta_{\nu}\Lambda_{\nu}^{z}) - \sinh(\beta_{\nu}\Lambda_{\nu}^{z}) \]
\[ \times e^{-i\alpha_{\nu}\sigma_{x}}e^{-\beta_{\nu}^{1}\Lambda_{\nu}^{z}\sigma_{z}}e^{-i\alpha_{\nu}\sigma_{x}}), \]  
(18)

where \[ \alpha_{\nu} = \frac{\beta_{\nu}^{1}-\beta_{\nu}}{2}. \]
The following trace can be evaluated as 
\[ \text{Tr}(e^{-i\alpha_{\nu}\sigma_{x}}e^{-\beta_{\nu}^{1}\Lambda_{\nu}^{z}\sigma_{z}}e^{i\alpha_{\nu}\sigma_{x}}) \]
\[ = \text{Tr}(e^{-2i\alpha_{\nu}\sigma_{x}})(\sigma_{z} \cosh(\beta_{\nu}\Lambda_{\nu}^{z}) - \sinh(\beta_{\nu}\Lambda_{\nu}^{z})) \]
\[ = \cosh(\beta_{\nu}\Lambda_{\nu}^{z})\text{Tr}(e^{-2i\alpha_{\nu}\sigma_{x}}) - 2\sinh(\beta_{\nu}\Lambda_{\nu}^{z}) \]
\[ = -2\sinh(\beta_{\nu}\Lambda_{\nu}^{z})
\]
(19)

Substituting Eq. (19) into Eq. (18) leads to

\[ \text{Tr}[g_{\nu}^{0}(\beta_{\nu})g_{\nu}^{1}(\beta_{\nu}^{1})] = 2[\cosh(\beta_{\nu}\Lambda_{\nu}^{0})\cosh(\beta_{\nu}\Lambda_{\nu}^{1}) + \sinh(\beta_{\nu}\Lambda_{\nu}^{0})\sinh(\beta_{\nu}\Lambda_{\nu}^{1})\cosh(\beta_{\nu}^{1} - \beta_{\nu})]. \]

Bringing all the terms together one eventually finds 
\[ \mathcal{F}_{H_{\nu}H_{\nu}^{1}}(\beta_{\nu},\beta_{\nu}^{1}) = \prod_{k} \left[ \left[ 1 + \cosh(\beta_{\nu}\Lambda_{\nu}^{0}) \right] \left[ 1 + \cosh(\beta_{\nu}\Lambda_{\nu}^{1}) \right] \right]^{-1/2} \]
\[ \times \left[ 1 + \frac{1}{\sqrt{2}} \left[ 1 + \cosh(\beta_{\nu}\Lambda_{\nu}^{0})\cosh(\beta_{\nu}\Lambda_{\nu}^{1}) \right] \left[ 1 + \cosh(\beta_{\nu}\Lambda_{\nu}^{0})\sinh(\beta_{\nu}\Lambda_{\nu}^{1})\cosh(\beta_{\nu}^{1} - \beta_{\nu}) \right] \right]^{1/2}. \]
(20)

It is easy to check that 
\[ \mathcal{F}_{H_{\nu}H_{\nu}^{1}}(\beta,\beta^{1}) \xrightarrow{\beta \to \infty} \prod_{k} \left| \cos(\beta_{\nu}^{1} - \beta_{\nu}^{1}) \right| \],
(21)

i.e., one recovers the zero-temperature result [5].

C. Numerical analysis

We use XY model as an example to demonstrate our main idea. Here, two Hamiltonians are \( H_{0} = H(\gamma, \lambda) \) and \( H_{1} = H(\gamma + \delta \gamma, \lambda + \delta \lambda) \) where

\[ H(\gamma, \lambda) = \sum_{i} \left( \frac{1}{2} \gamma \sigma_{x}^{i} \sigma_{x}^{i+1} + \frac{1}{2} - \gamma \sigma_{y}^{i} \sigma_{y}^{i+1} + \lambda \sigma_{z}^{i} \right), \]

where \( \gamma \) defines the anisotropy and \( \lambda \) represents external magnetic field along the z axis. Obviously, when \( \gamma = 1 \), the XY Hamiltonian (22) reduces to the transverse Ising Hamiltonian. \( \delta \gamma \) and \( \delta \lambda \) represent small perturbation to the Hamiltonian. \( \sigma_{x}, \sigma_{y}, \sigma_{z} \in \{ x, y, z \} \) are usual Pauli operators in the 3-dimensional lattice point. It is well known that this model, by means of a Jordan-Wigner transformation, can be mapped onto a quasi-free fermionic Hamiltonian of the type (8) (i.e., \( e_{k} = \cos \frac{2\pi}{N} k - \lambda, \Delta_{k} = \gamma \sin \frac{2\pi}{N} k \)). Through a straightforward calculation, it can be obtained that \( \Lambda_{k}^{0} = \Lambda_{k}^{0}(\gamma, \lambda) \) and \( \Lambda_{k}^{1} = \Lambda_{k}^{1}(\gamma + \delta \gamma, \lambda + \delta \lambda) \) where

\[ \Lambda_{k}^{0} = \sqrt{\left( \cos \frac{2\pi}{N} k - \lambda \right)^{2} + \gamma^{2} \sin^{2} \frac{2\pi}{N} k}, \]

and \( \Delta_{k} = \cos \left[ \lambda \sin \frac{2\pi}{N} k / \Lambda_{k}^{0} \right] \Delta_{k}^{0} \) \((\alpha = 0, 1, \lambda_{0} = \lambda, \lambda_{1} = \lambda + \delta \lambda)\). We plot the mixed-state fidelity according to the analytical expression obtained above Eq. (20). We choose the number of spins to be 200, and the perturbation to be 10^{-2}. In Fig. 1, we consider four different temperatures; \( \beta = 1 J^{-1}, \beta = 10 J^{-1}, \beta = 20 J^{-1}, \) and \( \beta = 100 J^{-1}, \) respectively. At low temperature, e.g., \( \beta = 100 J^{-1}, \) the sharp decay of fidelity in the critical region clearly displays the QPT, which happens at zero temperature. What is more, the pattern of fidelity in low temperature is very similar to the ground-state fidelity [5] of the XY model. This similarity is natural because, as we
mentioned above, the mixed-state fidelity approaches ground-state fidelity when the temperature decreases to zero. When the temperature increases higher, e.g., $\beta = 10 \text{ J}^{-1}$ or $\beta = 20 \text{ J}^{-1}$, the decay of fidelity, though less sharp due to the thermal excitation, still early shows the signature of QPT. However, when the temperature becomes large enough, e.g., $\beta = 100 \text{ J}^{-1}$, the dramatic decay of ground-state fidelity at the critical point is totally washed out by the thermal excitation and the signature of QPT at the critical point eventually disappears. In Fig. 2, we plot the mixed-state fidelity of thermal state of the transverse Ising model. This is actually a cross section of Fig. 1 at $\gamma = 1$. The decay of fidelity clearly indicates the critical point of the transverse Ising system.

It is important to stress that these numerical findings, besides illustrating the possibility of detecting and studying the finite-temperature signatures of QPTs, show that the ground-state fidelity approach to criticality is endowed with some robustness against “perturbations” of the ground state. In this case, we in fact have seen that mixing the ground state with excited eigenstates does not destroy the peculiar behavior of fidelity in the neighborhood of the QPTs, i.e., the fidelity drop.

III. LOSCHMIDT ECHO

In Refs. [10,11], the concept of Loschmidt echo has been used to investigate quantum criticality. The Loschmidt echo measures the sensitivity of a system to a small perturbation in its Hamiltonian. Specifically, when the system is in a pure state $|\Psi(0)\rangle$, Loschmidt echo is given by the inner product of the pure state $|\Psi(0)\rangle$ under two evolutions with a slight difference in Hamiltonians $H_0 = H(\gamma, \lambda)$ and $H_1 = H(\gamma + \delta \gamma, \lambda + \delta \lambda)$,

$$\mathcal{L}_{H_0, H_1}(t) = |\langle \Psi(0)| \exp[iH_0 t] \exp[-iH_1 t] |\Psi(0)\rangle|^2. \quad (24)$$

When the parameters $\gamma(\gamma + \delta \gamma)$ and $\lambda(\lambda + \delta \lambda)$ are far from the critical point, i.e., $H_0$ and $H_1$ are in the same phase, the system is not sensitive to the perturbation. Loschmidt echo approximately remains at unit $\mathcal{L}_{H_0, H_1}(t) = 1$ even for a long time. However, when $\gamma(\gamma + \delta \gamma)$ and $\lambda(\lambda + \delta \lambda)$ are near the critical point, $H_0$ and $H_1$ maybe in two different phases and thus have totally different symmetry. Then the system is very sensitive to this tiny perturbation and Loschmidt echo $\mathcal{L}_{H_0, H_1}(t)$ change abruptly to zero. The resulting picture is that the asymptotic value of the Loschmidt echo (for infinitely long times) is directly related to the closeness of the system to the critical points, i.e., the closer the systems to the QPT the smaller the asymptotic value of the Loschmidt echo. On the other hand, even the short-time behavior brings about information about the QPTs. Indeed, for short times the decay of the Loschmidt echo appears to be gaussian $\sim \exp(-\alpha t^2)$, where the rate $\alpha$ has a diverging derivative (as a function of, e.g., magnetic field) at the critical point [11].

Now we show how to generalize the notion of Loschmidt echo in a natural way to the thermal state case. This natural extension to the realm of mixed-state can be obtained by using Eq. (1),

$$\mathcal{L}_{\rho_0, \rho_1}(t) := F(\rho_0, U_1(t) U_0(t) \rho_1 U_0^\dagger(t) U_1(t)), \quad \rho_a := \exp(-iH_0 t) \quad (\alpha = 0, 1). \quad (25)$$

In particular, one can choose $\rho_0 = \exp(-\beta H_0)$, then Eq. (25) simplifies and one can define
\[ \mathcal{L}_{H_0,H_1}(\beta,t) = F[Z_0^{-1} e^{-\beta H_0} Z_0^{-1} U_1(t) e^{-\beta H_0} U_1(t)] = Z_0^{-1} F[e^{-\beta H_0} U_1(t) e^{-\beta H_0} U_1(t)] = \mathcal{F}_{H_0}(U_1(t)) e^{-\beta \mathcal{H}_{H_1}}(\beta,\beta). \] 

Then, from the above equation and Eq. (17), one finds

\[ \mathcal{L}_{H_0,H_1}(\beta,t) = \prod_k (Z_k^0)^{-1} (2 + \text{Tr}[\hat{\mathcal{E}}_k^0(\beta) \hat{\mathcal{E}}_k^1(\beta)] + 2) \] 

with

\[ \text{Tr}[\hat{\mathcal{E}}_k^0(\beta) \hat{\mathcal{E}}_k^1(\beta)] = \text{Tr}(e^{i(\hat{H}_0^0/2)\sigma_x e^{-\beta \Lambda_0^0 \sigma_x} e^{-i(\hat{H}_0^0/2)\sigma_x}} e^{i(\hat{H}_0^0/2)\sigma_x e^{-\beta \Lambda_0^0 \sigma_x} e^{-i(\hat{H}_0^0/2)\sigma_x}}) \times e^{i(\hat{H}_0^1/2)\sigma_x e^{-\beta \Lambda_0^1 \sigma_x} e^{-i(\hat{H}_0^1/2)\sigma_x}} e^{i(\hat{H}_0^1/2)\sigma_x e^{-\beta \Lambda_0^1 \sigma_x} e^{-i(\hat{H}_0^1/2)\sigma_x}} = \text{Tr}(e^{-\beta \Lambda_0^0 \sigma_x e^{-\beta \Lambda_0^1 \sigma_x} e^{-i(\hat{H}_0^0/2)\sigma_x}}). \]

Notice now that the unitary operator \( U_1(t) \) can be written as \( U_1(t) = e^{i(\hat{H}_0^1/2)\sigma_x e^{-i \Lambda_0^1 \sigma_x} e^{-i(\hat{H}_0^1/2)\sigma_x}}. \) Then,

\[ \mathcal{E}_k^0(\beta) = e^{i(\hat{H}_0^0/2)\sigma_x e^{-\beta \Lambda_0^0 \sigma_x} e^{-i(\hat{H}_0^0/2)\sigma_x}}, \]

and

\[ \mathcal{E}_k^1(\beta) = U_1(t) e^{i(\hat{H}_0^1/2)\sigma_x e^{-\beta \Lambda_0^1 \sigma_x} e^{-i(\hat{H}_0^1/2)\sigma_x}} U_1(t). \]

Substituting the above equation to Eq. (30) leads to

\[ \text{Tr}[\hat{\mathcal{E}}_k^0(\beta) \hat{\mathcal{E}}_k^1(\beta)] = 2 \cosh^2(\beta \Lambda_k^0) + 2 \sinh^2(\beta \Lambda_k^0) \times [\cos^2(\Lambda_k^1) + \sin^2(\Lambda_k^1) \cos(4\alpha_k)] = 2 \cos^2(\Lambda_k^1) + 2 \sin^2(\Lambda_k^1) \cos(4\alpha_k). \]

IV. CONCLUSIONS

Similarly to Sec. II, we now study mixed-state Loschmidt echo given by Eq. (33). In Fig. 3, we plot the Loschmidt echo at an instant \( t=10 \text{ s} \). The parameters are the same as that in Figs. 1 and 2. Similar to the ground-state Loschmidt echo in Ref. [10], the decay of mixed-state Loschmidt echo of a thermal state at a finite temperature well indicates the critical point. With the increase of the temperature, the decay of the mixed-state Loschmidt echo at critical point becomes less sharp until finally disappears. Figure 4 is a cross section of Fig. 3 at \( \gamma=1 \), i.e., a mixed-state Loschmidt echo of transverse Ising model. Similarly to the mixed-state Fidelity of transverse Ising model in Fig. 2, the critical point is clearly indicated by the Loschmidt echo. With temperature increase, the decay of Loschmidt echo at critical point becomes less evident, and finally disappears for sufficiently high temperature.

In this paper we have shown how to extend to finite temperature the fidelity approach to quantum phase transitions advocated in Refs. [5,6]. This generalization relies on the notion of mixed state fidelity applied to (Gibbs) thermal states. Mixed-state fidelity is strictly related to the Bures metric measuring the statistical distance between two density.
operators, therefore this approach has a geometrical as well as an operational meaning and two of them are deeply intertwined. We provided an explicit analytical expression for both the fidelity and Loschmidt echo for an important class of quasifree fermionic Hamiltonians including, e.g., the $XY$ model. A numerical analysis of these quantities has been performed; it clearly shows how the influence of the zero-temperature critical points extend over a finite range of temperatures. Besides representing a nontrivial generalization of the former zero-temperature results, the findings reported in this paper suggest directions for further investigations, the former zero-temperature results, the findings reported in the current study is focused on quasifree quadratic fermionic Hamiltonians. Both the quantum phase transition in the $XY$ model and the BCS transition in the BCS model can be described by this Hamiltonian. There is also another kind of phase transition, e.g., the BEC transition in dilute gases, which should be described by a quasifree quadratic bosonic Hamiltonian [13]. We anticipate that similar results could be obtained through an analogous calculations (see, e.g., the Dicke model example in Ref. [5]). A detailed discussion on quasifree quadratic bosonic Hamiltonian will be given in future research.

ACKNOWLEDGMENTS

The authors thank Y. Q. Li for the discussions. X.W. was supported by NSFC under Grant No. 10405019, Specialized Research Fund for the Doctoral Program of Higher Education (SRFDP) under Grant No. 20050335087, and the project sponsored by SRF for ROCS, SEM. C.P.S was supported by the NSFC with Grants No. 90203018, No. 10474104 and No. 60433050, and the National Fundamental Research Program of China with Grants No. 2001CB309310 and No. 2005CB724508.