Enhancement of parameter-estimation precision in noisy systems by dynamical decoupling pulses

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(Received 20 December 2012; published 4 March 2013)

We present a scheme to enhance the precision of parameter estimation (PPE) in noisy systems by employing dynamical decoupling pulses. An exact analytical expression for the estimation precision of an unknown parameter is obtained by using the transfer matrix and time-dependent Kraus operators. We show that the PPE in noisy systems can be preserved in the Heisenberg limit by control of the dynamical decoupling pulses. It is found that a larger number of pulses and longer reservoir correlation time can greatly protect the PPE.

DOI: 10.1103/PhysRevA.87.032102 PACS number(s): 03.65.Ta, 06.20.Dk, 03.65.Yz

I. INTRODUCTION

The ultraprecise estimation of parameters plays an important role in quantum metrology such as quantum frequency standards, measurement of gravity acceleration, clock synchronization [1–5], and so on. In the quantum metrology field, the quantum Fisher information (QFI) [6–13] is a key concept, giving a theoretically achievable limit on the precision when estimating an unknown parameter \( \phi \). According to the quantum Cramér-Rao theorem [9,14], the mean square fluctuation of \( \phi \) becomes

\[
\Delta \phi \geq \Delta \phi_{\text{QCB}} \equiv \frac{1}{\sqrt{\mathcal{F}(\phi)}},
\]

where \( \nu \) represents the number of independent measurements and \( \mathcal{F}(\phi) \) is the QFI with respect to the unknown parameter \( \phi \),

\[
\mathcal{F}(\rho(\phi)) = \text{Tr}[\rho(\phi)L_{\phi}^2],
\]

where \( L_{\phi} \) is the so-called symmetric logarithmic derivative determined by the equation \( \frac{\partial}{\partial \phi} \rho(\phi) = \frac{1}{2} (\rho(\phi)L_{\phi} + L_{\phi}^\dagger \rho(\phi)) \). Equation (1) implies that large Fisher information means a high precision of the estimation. Thus increasing the Fisher information has important theoretical and practical value in enhancing the precision of parameter estimation (PPE) [15].

Recently, many works have demonstrated that entangled states can improve the precision of parameter estimation [15–24]. When using a maximally entangled state the precision can be improved to the Heisenberg limit (HL) (proportional to \( 1/N \)), where \( N \) is the particle number. This limit gives better resolution than the best estimation limit with separable states, called the standard quantum limit (SQL) (proportional to \( 1/\sqrt{N} \)) [25–27].

Until recently, most of the work in quantum metrology involved isolated systems undergoing unitary evolution [1]. However, under realistic physical conditions, the unavoidable interaction with the environment leads to decoherence. In a recent paper [1], Escher and co-workers proposed a general framework for the quantum metrology of noisy systems, and obtained useful analytic bounds for optical interferometry and atomic spectroscopy. It is found that in the presence of decoherence, even with the use of entanglement, the Heisenberg level estimation cannot be achieved, since the entangled states are very sensitive to the action of the environment [23,28–30].

In Ref. [4] the authors studied the QFI of a Greenberger-Horne-Zeilinger (GHZ) state under three typical types of noise source [i.e., an amplitude-damping channel (ADC), a phase-damping channel, and a depolarizing channel] by means of Kraus operators. They found that, when the decoherence strength is sufficiently large, in all these channels precision higher than the SQL level cannot be achieved. Therefore, it is important to find strategies to suppress decoherence in order to obtain an ultraprecise measurement.

In quantum information theory, there are two main classes of approaches to overcome decoherence: passive techniques, in which quantum information is encoded within a decoherence-free subspace [31,32] which does not decohere for reasons of symmetry; and active approaches, such as quantum error correction [33] and dynamical decoupling (DD) techniques [34–45]. In Ref. [3], the author used a decoherence-free subspace to suppress the collective dephasing of \( N \) ions which are stored in a linear Paul trap and proved that quantum enhancement can readily be achieved in the presence of noise. DD strategies, as another protocol to protect quantum information, aim at averaging the unwanted interaction with the environment to zero by means of a dynamical control field. DD methods have been studied in connection with a wide range of applications and have become particularly popular in the area of quantum information. In Ref. [10], the authors investigated how to extract the maximum information from a noisy quantum system; in their work the DD scheme was considered for recovering the lost information from a single qubit in a heat bath of bosons.

In this paper we propose a scheme to enhance the PPE in \( N \)-qubit noisy systems by employing DD pulses. By the use of the transfer matrix and exact time-dependent Kraus operators, we attain analytic expressions for the QFI and error precision in the presence of DD pulses. We show that the use of the DD pulse sequences is very effective in protecting the PPE, and it is found that the HL can be achieved in the presence of noise as long as the number of pulses is large enough.

The paper is organized as follows. In Sec. II, we derive the time-dependent Kraus operators for the ADC case in the presence of DD pulses, and present exact solution of \( N \)-qubit reduced density matrix. In Sec. III, we study the effects of the DD pulses on protection of the PPE, and indicate that precision at the HL can be achieved even in noisy systems. Finally, a summary is provided in the last section.
II. DYNAMICS OF N QUBITS IN A NOISY SYSTEM UNDER DD PULSE SEQUENCES

In this section, we investigate the dynamics of N qubits in a noisy system under DD pulse sequences. We consider the independent-reservoir case in which each qubit interacts with a reservoir. Supposing that there is no interaction between the N pairs of “qubit + reservoir” systems, the dynamics of the whole system can be obtained simply from the evolution of the individual pairs.

A. Controlled Hamiltonian

The Hamiltonian $H(t)$ of one single qubit interacting with its own reservoir with controlled pulses is given by

$$H = H_S(t) + H_B + H_I,$$

with

$$H_B = \sum_j \omega_j a_j^\dagger a_j, \quad H_I = \sum_j g_j (\sigma_- a_j^\dagger + \sigma_+ a_j)$$

the Hamiltonian of reservoir and qubit-reservoir interaction. And the Hamiltonian of the qubit is

$$H_S(t) = H_S + H_c(t) = \frac{\omega_0}{2} \sigma_z - \frac{\lambda}{2} \sum_{n=1}^\infty \delta(t - nT) \sigma_z,$$

which consists of two parts. The first term is the free Hamiltonian; the second term is the control part, which comprises a train of instantaneous $\pi$ pulses (the width of each pulse is sufficiently short), where $T$ is the time interval between two consecutive pulses. The effect of each pulse on the qubit is simply a rotation around the $z$ axis by $\pi$, which is described by the operator $U_c = -i \sigma_z$.

Choosing $U(t) = \tilde{T} \exp[-i \int_0^t dt' H_S(t')]$, then the effective Hamiltonian $H_{\text{eff}}$ of the total system in the presence of control pulses is given by

$$H_{\text{eff}} = U^\dagger (H_B + H_S + H_I) U(t)$$

$$= \omega_0 |e\rangle \langle e| + \sum_j \omega_j a_j^\dagger a_j + \sum_j g_j (-1)^j (\sigma_- a_j^\dagger + \sigma_+ a_j),$$

where $n = \lfloor t/T \rfloor$ is the number of pulses denoted by the integer part of $t/T$. To get the above equation, we have used the relation $\sigma_\pm, \sigma_z, \sigma_\mp = -\sigma_\mp$ and omitted a constant for convenience. From this equation we can see clearly that the control pulses only change the sign of $g_j$ periodically, leading to $\langle H_I \rangle = 0$.

B. Model solution

At zero temperature, the Hamiltonian (6) can be exactly solved under the single-excitation approximation for the environment. Here, we first assume that the initial state of the system plus environment is of the form

$$|\Psi(0)\rangle = |C_e(0)|e\rangle + |C_g(0)|g\rangle |0\rangle_E,$$

which evolves over time $t$ into the state

$$|\Psi(t)\rangle = |C_e(t)|e\rangle + |C_g(t)|g\rangle |0\rangle_E + \sum_j C_j(t)|j\rangle |1_j\rangle_E,$$

where $|1_j\rangle$ denotes that only the $j$th mode of the bath is excited. Note that the basis $|g\rangle |0\rangle_E$ does not evolve under the rotating-wave approximation.

Substituting Eqs. (6) and (8) into the Schrödinger equation, we can obtain the following coupled equations:

$$\dot{C}_e(t) = -i \omega_0 C_e(t) - i \sum_j g_j (-1)^{j+1} C_j(t),$$

$$\dot{C}_g(t) = -i \omega_0 C_g(t) - i \sum_j g_j (-1)^{j+1} C_e(t).$$

To obtain $\dot{C}_e(t)$ and $\dot{C}_g(t)$, we can go to the rotating frame, define $c_e(t) = C_e(t) e^{i \omega_0 t}$ and $c_g(t) = C_g(t) e^{i \omega_0 t}$ [44], and get

$$\dot{c}_e(t) = -i \sum_j g_j (-1)^{j+1} e^{i (\omega_j - \omega_0) t} c_j(t),$$

$$\dot{c}_g(t) = -i \sum_j g_j (-1)^{j+1} e^{i (\omega_j + \omega_0) t} c_j(t).$$

Assuming that $c_j(0) = C_j(0) = 0$, we can get a closed equation for $c_e(t)$, namely,

$$\dot{c}_e(t) = - \int_0^t dt' f(t - t') c_e(t'),$$

and the correlation function $f(t - t')$ is related to the spectral density $J(\omega)$ of the reservoir. Assuming that the qubit is in resonance with the cavity mode, the spectral density has the Lorentzian spectral distribution [46]

$$J(\omega) = \frac{\gamma_0 \omega_0^2}{2 \pi (\omega_0^2 - \omega^2 + \lambda^2)}.$$

where $\lambda$ reflects the spectral width of the coupling, which is connected to the reservoir correlation time $\tau_R$ by $\tau_R = \lambda^{-1}$, and $\gamma_0$ is related to the decay of the excited state of the atom in the Markovian limit, connected to the relaxation time by $\tau_R = \gamma_0^{-1}$. Then we have

$$f(t - t_1) = \frac{1}{2} (-1)^{t/T + \lfloor t/T \rfloor} \lambda e^{-\lambda (t - t_1)}.$$

The factor $(-1)^{t/T + \lfloor t/T \rfloor}$ is induced by the sequence of $\pi$ pulses. If $n = 0 (T \rightarrow \infty)$, we have $\lim_{T \rightarrow \infty} (-1)^{t/T + \lfloor t/T \rfloor} = 1$.

When $t \in [nT, (n + 1)T)$, the general solution of Eq. (11) can be derived as (see Appendix A for details):

$$c_e(t) = \left\{ \begin{array}{ll}
\frac{e^{-\lambda t/2} \Delta_n F_1(n) + (1 + \lambda \Delta_n) F_2(n) c_e(0)}{1 + \lambda \Delta_n}, & \lambda = 2 \gamma_0, \\
\frac{e^{-\lambda t/2} A_n \cos(\Delta_n d) + B_n \sinh(\Delta_n d) c_e(0)}{\Delta_n}, & \lambda \neq 2 \gamma_0,
\end{array} \right.$$

where $d = \sqrt{\lambda^2 - 2 \gamma_0 \lambda}$ and $\Delta_n = (t - nT)/T$. The coefficients $F_1$ and $F_2$ are given by

$$F_1 = \frac{\lambda^2 T (p_+^n - p_-^n)}{4 \sqrt{(\lambda T)^2 + 4}}, \quad F_2 = \frac{p_+^n + p_-^n}{2} + \frac{(\lambda T)^2}{4} F_1,$$

with $p_\pm = \left\{ 1 \pm \sqrt{(\lambda T)^2 + 4} \right\}$. Later, we mainly study the case of $\lambda \neq 2 \gamma_0$. In this case the constant coefficients $A_n$ and
$B_n$ ($n \geq 1$) in the presence of control pulses can be attained as

$$
(A_n) = M^n (A_0),
$$

with the initial values $A_0 = 1$ and $B_0 = \lambda / d$ [46], corresponding to the case in the absence of pulses. The transfer matrix in the presence of control pulses is given by

$$
M = \begin{pmatrix}
\cosh(\tau) & \sinh(\tau)
\frac{\lambda}{d} \cosh(\tau) - \sinh(\tau) & \frac{\lambda}{d} \sinh(\tau) - \cosh(\tau)
\end{pmatrix},
$$

where we have introduced $\tau = Td/2$. By diagonalizing the transfer matrix, we can obtain

$$
A_n = \alpha_n m_n^+ + \alpha_- m_n^-, \quad B_n = \beta_+ m_n^+ + \beta_- m_n^-,
$$

where

$$
\alpha_\pm = \frac{1}{2}[1 \pm \cosh(\tau) / \Theta], \quad m_\pm = \frac{\lambda}{d} \sinh(\tau) / \Theta,
$$

$$
\beta_\pm = \frac{\alpha_\pm}{2}[m_\pm - \cosh(\tau)] / \sinh(\tau),
$$

with $\Theta = \sqrt{1 + [\frac{\lambda}{d} \sinh(\tau)]^2}$. Furthermore, for finite time $t$ and $\lambda$ in the limit $n \to \infty$ ($T \to 0$), we have $\cosh(\tau) \approx 1$ and $\sinh(\tau) \approx \tau$; thus we arrive at $A_\pm \approx (1 + \frac{\lambda}{2d} t)^\pm$ and $B_\pm \approx \frac{\lambda}{2d} [A_\pm - \frac{\lambda}{2d} (\frac{\lambda}{2d} - 1)^\pm]$. Therefore, we can obtain $c_n(t) \approx c_n(0)$, which means that the decoherence effect can be nearly completely suppressed in this case.

By defining the decay rate $\kappa(t) \equiv \frac{c(t)}{c(0)} \in [0, 1]$, we can now express the reduced density matrix $\rho(t)$ of the qubit system in the form of Kraus operators [24] (see Appendix B),

$$
\rho_S(\varphi, t) = \sum_i K_i(\varphi, t) \sqrt{\rho}_S(0) K_i^\dagger(\varphi, t) = \rho_{\varphi}(t)[\rho_S(0)],
$$

where $\varphi = \varphi_0 t$. The time-dependent Kraus operators can be expressed as

$$
K_1(\varphi, t) = e^{-i\varphi/2}[k(t)]e|e\rangle + e^{i\varphi/2}|g\rangle \langle g|,
$$

$$
K_2(\varphi, t) = \sqrt{1 - \kappa(t)^2}e^{i\varphi/2}|e\rangle \langle g|,
$$

which corresponds to the ADC model. When $\kappa(t) \to 1$, we have $K_1(\varphi, t) \to e^{-i\varphi\sqrt{\kappa(t)}}$ and $K_2(\varphi, t) \to 0$. With the help of these Kraus operators, the time evolution of the $N$-qubit reduced density operator can be given by

$$
\rho(t) = \sum_{\mu_1, \ldots, \mu_N} [\otimes_{i=1}^N K_{\mu_i}(t)] \rho(0) [\otimes_{i=1}^N K_{\mu_i}^\dagger(t)],
$$

where $K_{\mu_i}(t)$ denotes the Kraus operators for the $i$th qubit.

**III. PPE ENHANCEMENT BY PULSE SEQUENCES**

Now we study how to protect the QFI and improve the estimation precision of an unknown parameter $\varphi$, induced by the noisy channel $E_\varphi$. The schematic we propose is shown in Fig. 1, which consists of $N$ qubits in independent reservoirs. Each qubit interacts with a reservoir which leads to decoherence. In order to suppress decoherence and enhance the precision of estimation, a sequence of $\pi$ pulses is applied to each qubit simultaneously. The Hamiltonian of each qubit-reservoir system has been given in Eq. (3).

![Fig. 1. (Color online) Schematic representation of parameter estimation for an $N$-qubit noisy system in the presence of dynamical decoupling. The total evolution procedure can be described by the tensor product $E_\varphi^{\otimes N}$.]

To obtain the maximal QFI, the input state is initially prepared in the GHZ state

$$
|\psi_\text{in}(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}),
$$

where $\sigma_1|0\rangle = |0\rangle$ and $\sigma_1|1\rangle = -|1\rangle$.

Such a state is a maximally entangled state, which can improve the precision to $1/N$, i.e., the HL.

According to Eq. (22), the reduced density matrix of the total system at time $t$ reads [4]

$$
\rho_{2S}(t) = \frac{1}{2} E_{\varphi}(t)(|0\rangle^{\otimes N} + \rho(t)(|1\rangle^{\otimes N})
$$

$$
+ E_{\varphi}(t)(|1\rangle^{\otimes N} + \rho(t)(|0\rangle^{\otimes N}) = \varrho_1 \otimes \varrho_2,
$$

where $E_{\varphi}(t)$ represents the noisy channel with DD pulses for a single qubit. Also,

$$
\varrho_1 = \frac{1}{2} \sum_{m=1}^{N-1} \kappa(t)^{2(N-m)}[1 - \kappa(t)^2]^{m} |0\rangle^{\otimes(N-m)}|0\rangle^{\otimes m},
$$

$$
\varrho_2 = \frac{1}{2} \kappa(t)^N |0\rangle^{\otimes N} + [1 + (1 - \kappa(t)^2)^N]|1\rangle^{\otimes N}
$$

$$
+ \kappa(t)^N(e^{-iN\varphi}|0\rangle^{\otimes N} + e^{iN\varphi}|1\rangle^{\otimes N}).
$$

The diagonal matrix $\varrho_1$ is independent of the parameter $\varphi$; thus we need to consider only $\varrho_2$ when estimating the value of $\varphi$. In order to estimate $\varphi$, we calculate the QFI. To do this, we first diagonalize the $\varphi$-dependent density matrix $\varrho_2$ as $\varrho_2 = \sum_i p_i(t)|\psi_i\rangle\langle \psi_i|$, where $|\psi_i\rangle$ are the eigenstates of $\varrho_2$ with eigenvalues $p_i$. In this diagonal representation, the explicit expression for the QFI is given by (as shown in Appendix C)

$$
F[\rho(\varphi, t)] = \frac{4N^2\kappa(t)^2}{1 + [1 - \kappa(t)^2]N + \kappa(t)^{2N}}.
$$

According to the quantum Cramér-Rao bound, the minimal variance of the estimation of the parameter $\varphi$ can be obtained as

$$
\Delta \varphi_{\text{min}}(t) \equiv \frac{1}{\sqrt{F[\rho(\varphi, t)]}} = \frac{1 + [1 - \kappa(t)^2]N + \kappa(t)^{2N}}{2N\kappa(t)^N}.
$$

Here, we have set the measurement time $\nu = 1$. Equations (26) and (27) are functions of the decay rate $\kappa(t)$ and the number $N$. 

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of qubits $N$. From the above equations, we can find at the initial time, $\kappa(0) = 1$, $F[\rho(\varphi,0)] = N^2$, and $\Delta\varphi_{\text{min}}(t) = 1/N$, which is the HL on the precision of estimation. In this case, the large number of particles $N$ helps to improve the precision of estimation. However, if $\kappa(t) \to 0$ (i.e., without DD pulses and $t \to \infty$), $F[\rho(\varphi,t)] \to 0$, the QFI-based parameter $\varphi$ is lost completely, and the parameter $\varphi$ cannot be estimated in this case, i.e., $\Delta\varphi_{\text{min}}(t) \to \infty$. Worst of all, with increase of $N$, the values of $[\kappa(t)]^N$ decay rapidly. Thus, there exists a competitive relation between $[\kappa(t)]^N$ and $N$. To take advantage of the large number of particles $N$, we need to suppress the decay of $\kappa(t)$. Luckily, a nearly unit value of $\kappa(t)$ can be obtained in the present of DD pulses when $n \to \infty$ ($T \to 0$), as analyzed in the previous section, and the precision of the HL can be achieved in this limit.

In order to clearly observe the effect of DD pulses on the estimation precision, in Figs. 2 and 3 we have plotted the mean QFI $\bar{F}$ = $F/N$ and the estimation precision $\Delta\varphi_{\text{min}}$ with respect to the evolution time $t$ and the number of pulses $n$ and qubits $N$, with different values of $\lambda$ corresponding to the correlation time ($\tau_B = \lambda^{-1}$) of the reservoir.

$F(t)$ and $\Delta\varphi_{\text{min}}(t)$ vs. $n$

$\lambda = \gamma_0$

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Figure 3(a) indicates that Δφ_{min} reflects a different dependence on the number of qubits N for different values of λ. The precision of the HL level can be achieved with increase of N for λ = γ_0 at fixed time γ_0T = 10 and n = 20. However, the behavior is completely different for λ = γ_0. In the case of λ = γ_0, precision higher than the SQL can be reached only for the case of N < 16, and the error value increases rapidly when N > 16. This means that large N may increase the error value of the estimation at fixed n.

The values of Δφ_{min} at fixed time γ_0T = 10 as a function of pulse number n are also given in Fig. 3(b). We can see that as long as the number of control pulses is large enough the precision can be improved to the nearby HL for different λ. However, the larger λ the more pulses are needed to reach the same precision of estimation. In a word, the DD scheme is fully effective as long as T ≪ τ_B (= λ^{−1}).

IV. CONCLUSION

In conclusion, we have proposed a scheme to enhance the PPE in noisy systems by employing dynamical decoupling pulses. In our scheme N qubits are embedded into independent reservoirs. The unknown parameter φ to be estimated is induced by the channel E_φ(t). Resorting to the transfer matrix method and time-dependent Kraus operators, an exact analytical expression for the estimation precision of φ has been derived. Using this expression, we have demonstrated that the PPE in N-qubit noisy systems can be preserved at the HL by control of the dynamical decoupling pulses. It has been found that a larger number of pulses and longer reservoir correlation time can protect the PPE more effectively.

Finally, it should be pointed out that the results we have obtained in this paper are based on ideal π pulses, which can be treated as δ functions. This means that the effects of the duration time and errors of the pulses are neglected. However, experimentally, this idealized situation may not be realistic. Real pulses are always of finite amplitude and of finite length [41,42]. These imperfect pulses will accumulate an extra phase, which increases with the number of pulses and affects the PPE. To reduce the error as much as possible, we can apply optimized π pulse sequences [40]. Although it is more sophisticated in form, the same PPE can be attained with a smaller number of pulses. Therefore, fewer phase errors are accumulated. A detailed consideration of these effects will be interesting. We hope that the scheme proposed in the present paper might have promising applications in quantum information processing and quantum metrology.

ACKNOWLEDGMENTS

Q.S.T. thanks W. Zhong and X. Xiao for valuable discussions. X.W. acknowledges support from the NFRPC through Grant No. 2012CB921602 and the NSFC through Grants No. 11025527 and No. 10935103. L.M.K. acknowledges support from the 973 Program under Grant No. 2013CB921804, the NSF under Grant No. 11075050, the PCSIRTU under Grant No. IRT0964, and the HPNSF under Grant No. 11JJ7001.

APPENDIX A: DERIVATION OF EQ. (14)

In this Appendix, we present details of the derivation of Eq. (14). When t ∈ [nT,(n+1)T), Eq. (11) can be rewritten as

\[ \dot{c}_e(t) = -\frac{\gamma_0\lambda}{2} \int_0^t (-1)^{t/2+(t'/T)} e^{-\lambda(t-t')} c_e(t') dt' \]

\[ = -\frac{\gamma_0\lambda}{2} (-1)^n \left\{ \sum_{k=1}^n (-1)^{k-1} \int_{(k-1)T}^{kT} e^{-\lambda(t-t')} c_e(t') dt' \right\} + (-1)^n c_e(t) \]

\[ = -\lambda \dot{c}_e(t) - \frac{\gamma_0\lambda}{2} c_e(t). \] (A2)

This equation is an ordinary differential equation which is local in time, and contains only \( \dot{c}_e(t), c_e(t), \) and \( c_e(t) \).

In the following, we solve for \( c_e(t) \) in two different cases: λ = 2γ_0 and λ ≠ 2γ_0.

(i) The case of \( \lambda = 2\gamma_0 \). In this case, we have d = \( \sqrt{\lambda^2 - 2\gamma_0\lambda} = 0 \), and when t ∈ [nT,(n+1)T] the general solution of \( \dot{c}_e(t) \) can be derived as

\[ c_e(t) = (C_1 t + C_2) e^{-\lambda t/2}, \] (A3)

and C_1 and C_2 are given by

\[ C_1 = e^{\lambda t/2} \left[ \dot{c}_e(nT) + \frac{\lambda}{2} c_e(nT) \right], \]

\[ C_2 = e^{\lambda t/2} \left[ -nT \dot{c}_e(nT) + \left( 1 - \frac{\lambda nT}{2} \right) c_e(nT) \right]. \] (A4)

Then we have

\[ \begin{pmatrix} c_e(t) \\ \dot{c}_e(t) \end{pmatrix} = e^{-\lambda t(nT)/2} \begin{pmatrix} 1 + \frac{\lambda(t-nT)}{2} & t-nT \\ \frac{\lambda^2(t-nT)}{4} & 1 - \frac{\lambda(t-nT)}{2} \end{pmatrix} \times \begin{pmatrix} c_e(nT) \\ \dot{c}_e(nT) \end{pmatrix} \]

\[ = e^{-\lambda t(nT)/2} \begin{pmatrix} 1 + \frac{\lambda(t-nT)}{2} & t-nT \\ \frac{\lambda^2(t-nT)}{4} & 1 - \frac{\lambda(t-nT)}{2} \end{pmatrix} \times \begin{pmatrix} c_e(nT) \\ \dot{c}_e(nT) \end{pmatrix}, \] (A5)
The transfer matrix

\[
\begin{pmatrix}
{c}_e(nT) \\
{\hat{c}_e}(nT)
\end{pmatrix} = e^{-\lambda T/2} \begin{pmatrix} 1 + \frac{\lambda (t-nT)}{2} & T \\
-\frac{\lambda}{2} & 1 - \frac{\lambda T}{2}
\end{pmatrix}
\times \sigma_z \begin{pmatrix}
{c}_e[(n-1)T] \\
{\hat{c}_e}[(n-1)T]
\end{pmatrix}.
\] (A6)

Here we have used the boundary conditions \( {c}_e(nT) = {c}_e(nT_0) \) and \( {\hat{c}_e}(nT) = -{\hat{c}_e}(nT_0) \) and \( \sigma_z \) is the Pauli matrix. Using the recurrence relation, we can easily obtain the following expression after an \( n \)-pulse sequence:

\[
\sigma_z \begin{pmatrix}
{c}_e[(n-1)T] \\
{\hat{c}_e}[(n-1)T]
\end{pmatrix} = e^{-\lambda T/2} (\sigma_3)^n \begin{pmatrix} 1 + \frac{\lambda (t-nT)}{2} & T \\
-\frac{\lambda}{2} & 1 - \frac{\lambda T}{2}
\end{pmatrix}^n
\times \begin{pmatrix}
{c}_e(0) \\
0
\end{pmatrix};
\] (A7)

here we have used the initial condition \( {\hat{c}_e}(0) = 0 \). Thus, we have

\[
\begin{pmatrix}
{c}_e(t) \\
{\hat{c}_e}(t)
\end{pmatrix} = e^{-\lambda t/2} \begin{pmatrix} 1 + \frac{\lambda (t-nT)}{2} & T \\
-\frac{\lambda}{2} & 1 - \frac{\lambda T}{2}
\end{pmatrix}
\times M^n \begin{pmatrix}
{c}_e(0) \\
0
\end{pmatrix}.
\] (A8)

The transfer matrix

\[
M = \begin{pmatrix} 1 + \frac{\lambda T}{2} & T \\
\frac{\lambda}{2} & 1 - \frac{\lambda T}{2}
\end{pmatrix}
\] (A9)

can be diagonalized as \( P^{-1}MP = \text{Diag}\{p_+, p_-\} \) with \( p_{\pm} = \frac{1}{2}[\lambda T \pm \sqrt{\lambda^2 T^2 + 4}] \), where the matrices \( P \) and \( P^{-1} \) are given as

\[
P = \begin{pmatrix}
\frac{1}{2} \sqrt{\lambda^2 T^2 + 4} + 1 & \frac{1}{2} \sqrt{\lambda^2 T^2 + 4} + 1 \\
-\frac{1}{2} \sqrt{\lambda^2 T^2 + 4} + 1 & -\frac{1}{2} \sqrt{\lambda^2 T^2 + 4} + 1
\end{pmatrix},
\] (A10)

\[
P^{-1} = \frac{1}{T \sqrt{\lambda^2 T^2 + 4}} \begin{pmatrix}
\frac{1}{2} \sqrt{\lambda^2 T^2 + 4} + 1 & T \\
-\frac{1}{2} \sqrt{\lambda^2 T^2 + 4} - 1 & T
\end{pmatrix}.
\]

Thus we have

\[
M^n = P \begin{pmatrix}
0 & 0 \\
p_+ & p_-
\end{pmatrix} P^{-1} = \begin{pmatrix} m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix}.
\] (A11)

where

\[
m_{11} = \frac{p_+^n + p_-^n}{2} + \frac{p_+^n - p_-^n}{\sqrt{\lambda^2 T^2 + 4}}, \quad m_{12} = \frac{T(p_+^n - p_-^n)}{\sqrt{\lambda^2 T^2 + 4}} + \frac{\lambda^2 T(p_+^n - p_-^n)}{4 \sqrt{\lambda^2 T^2 + 4}}.
\]

\[
m_{22} = \frac{p_+^n + p_-^n}{2} - \frac{p_+^n - p_-^n}{\sqrt{\lambda^2 T^2 + 4}}, \quad m_{21} = \frac{\lambda^2 T(p_+^n - p_-^n)}{4 \sqrt{\lambda^2 T^2 + 4}}.
\] (A12)

Hence, Eq. (A8) can be rewritten as

\[
\begin{pmatrix}
{c}_e(t) \\
{\hat{c}_e}(t)
\end{pmatrix} = e^{-\lambda t/2} \begin{pmatrix} 1 + \frac{\lambda (t-nT)}{2} & T \\
-\frac{\lambda}{2} & 1 - \frac{\lambda T}{2}
\end{pmatrix}
\times \begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix} \begin{pmatrix}
{c}_e(0) \\
0
\end{pmatrix}.
\] (A13)

Therefore, the population of the excited state in the present of decoupling pulses can be obtained as

\[
{c}_e(t) = e^{-\lambda t/2} \left\{ (t-nT)m_{21} + \left[ 1 + \frac{\lambda (t-nT)}{2} \right] m_{11} \right\} {c}_e(0),
\] (A14)

where \( m_{21} \) and \( m_{11} \) are replaced, respectively, by \( F_1(n) \) and \( F_2(n) \) in Eq. (14).

(ii) The case of \( \lambda \neq 2 \gamma_0 \). In this case, we have \( d = \sqrt{\lambda^2 - 4 \gamma_0^2} \neq 0 \), and the general solution for \( c_e(t) \) can be derived as

\[
{c}_e(t) = e^{-\lambda t/2} \left[ A_n \cosh \left( \frac{(t-nT)d}{2} \right) + B_n \sinh \left( \frac{(t-nT)d}{2} \right) \right] {c}_e(0),
\] (A15)

with

\[
A_n = e^{\lambda t/2} {c}_e(nT_+), \quad B_n = e^{\lambda t/2} \left[ \frac{\lambda {c}_e(nT_+)}{d} + \frac{2\dot{c}_e(nT_+)}{d} \right].
\] (A16)

When \( t \in [(n-1)T, nT) \), we also have

\[
{c}_e(t) = e^{-\lambda t/2} \left\{ A_{n-1} \cosh \left( \frac{[(t-(n-1)T)d}{2} \right) \\
+ B_{n-1} \sinh \left( \frac{[(t-(n-1)T)d}{2} \right) \right\}.\] (A17)

Using the boundary conditions \( {c}_e(nT_-) = {c}_e(nT_+) \) and \( {\hat{c}_e}(nT_-) = -{\hat{c}_e}(nT_+) \), we have

\[
A_n = A_{n-1} \cosh(\tau) + B_{n-1} \sinh(\tau), \quad B_n = \frac{2 \lambda}{d} \left[ A_{n-1} \cosh(\tau) + B_{n-1} \sinh(\tau) \right] \\
- [A_{n-1} \sinh(\tau) + B_{n-1} \cosh(\tau)].
\] (A18)

Thanks to the recurrence relations of the constant coefficients \( A_n \) and \( B_n \), we can obtain Eq. (16). The transfer matrix \( \tilde{M} \) can be diagonalized as \( \tilde{P}^{-1} \tilde{M} \tilde{P} = \text{Diag}\{m_+, m_-\} \) . The matrices \( \tilde{P} \) and \( \tilde{P}^{-1} \) are given as

\[
\tilde{P} = \begin{pmatrix} \sinh(\tau) & \sinh(\tau) \\
m_+ - \cosh(\tau) & m_- - \cosh(\tau)
\end{pmatrix}, \quad \tilde{P}^{-1} = \frac{1}{|\tilde{P}|_{\text{det}}} \begin{pmatrix} m_- - \cosh(\tau) & - \sinh(\tau) \\
\cosh(\tau) & -m_+ \sinh(\tau)
\end{pmatrix}.
\] (A19)

Thus we have

\[
\begin{pmatrix} A_n \\
B_n
\end{pmatrix} = \tilde{P} \begin{pmatrix} m_+ & 0 \\
0 & m_-
\end{pmatrix} \tilde{P}^{-1} \begin{pmatrix} A_0 \\
B_0
\end{pmatrix},
\] (A20)

and hence Eq. (18) is attained.

### APPENDIX B: NOISE CHANNEL \( \mathcal{E}_p(t) \)

Here, we will give a derivation of Eq. (20). Corresponding to Eq. (8), we can rewrite the final state of the system plus
Then the reduced density matrix of the qubit system can be read as

$$\rho_2(t) = \text{Tr}_E[|\Psi(t)\rangle\langle\Psi(t)|]$$

$$= \left(e^{i\omega t}C_e(0)C_g(0)\rho_e(0)C_g(0)C_e(0)t^2 \right) + e^{-i\omega t}C_e(0)C_g(0)\rho_e(0)C_g(0)C_e(0)t^2$$

$$= e^{-i\omega t/2} \left(e^{i\omega t}C_e(0)\rho_e(0)C_g(0)t^2 \right) + e^{-i\omega t/2} \left(e^{-i\omega t}C_e(0)\rho_e(0)C_g(0)t^2 \right)$$

$$= \sum_i K_i(\varphi, t)\rho(0)K_i^\dagger(\varphi, t) \equiv E_i(\varphi, t)\rho(0),$$

where

$$E_i(t) = \kappa(t)|e\rangle\langle e| + |g\rangle\langle g|, \quad E_2(t) = \sqrt{1 - \kappa(t)^2}|g\rangle\langle e|,$$

and $K_i(\varphi, t)$ have been given in Eq. (21).

### APPENDIX C: QUANTUM FISHER INFORMATION

In this Appendix, we will calculate the QFI, which was given in Eq. (26). In the basis of $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$, $\varphi_2(t)$ can be written as

$$\varphi_2(t) = \frac{1}{2} \left(e^{i\kappa N\varphi}e^{i\kappa N\varphi} \right) \left(1 + (1 - \kappa^2)N \right).$$

In order to calculate the QFI, we first diagonalize $\varphi_2(t)$ as

$$\varphi_2(t) = \sum_i p_i(t)|\psi_i(t)\rangle\langle\psi_i(t)|.$$ (C2)

The corresponding eigenvalues and eigenvectors are given by

$$p_{1,2}(t) = \frac{1}{2} \left(1 + \kappa^2N + (1 - \kappa^2)N \right) \pm \sqrt{\left(1 + \kappa^2N + (1 - \kappa^2)N \right)^2 - 4\kappa^2N(1 - \kappa^2)N}$$ (C3)

and

$$|\psi_1(t)\rangle = \sin\alpha(t)|1\rangle^{\otimes N} + e^{-iN\varphi}\cos\alpha(t)|0\rangle^{\otimes N},$$

$$|\psi_2(t)\rangle = \cos\alpha(t)|1\rangle^{\otimes N} - e^{-iN\varphi}\sin\alpha(t)|0\rangle^{\otimes N},$$

where

$$\alpha(t) = \arctan\left(\frac{2\kappa}{N^2 - 1 - (1 - \kappa^2)N + \Xi} \right)$$ (C5)

with $\Xi = \sqrt{\left(1 + \kappa^2N + (1 - \kappa^2)N \right)^2 - 4\kappa^2N(1 - \kappa^2)N}$. In this diagonal representation, the matrix elements of the symmetric logarithmic derivative are

$$L_{ij} = \frac{2\langle\psi_i|\partial_\varphi\psi_j\rangle|\psi_j\rangle}{p_i + p_j}.$$ (C6)

Then $L(t)$ is obtained explicitly as

$$L(t) = \frac{2iN\kappa(t)}{1 + \kappa^2N + (1 - \kappa^2)N} [\langle\psi_1|\langle\psi_2| - |\psi_2\rangle\langle\psi_1|].$$ (C7)

Thus the QFI can be calculated as

$$F = \frac{1}{2} \text{Tr}[\varphi_2 L^2 + L^2 \varphi_2] = \frac{4N^2\kappa(t)^2N}{[1 + [1 - \kappa(t)^2]N + \kappa(t)^2N]^2}.$$ (C8)

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