Exact non-Markovian cavity dynamics strongly coupled to a reservoir

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The exact non-Markovian dynamics of a microcavity strongly coupled to a general reservoir at arbitrary temperature are studied. With the exact master equation for the reduced density operator of the cavity system, we analytically solve the time evolution of the cavity state and the associated physical observables. We show that the non-Markovian dynamics are completely determined by the propagating (retarded) and correlation Green functions. Comparing the non-Markovian behavior at finite temperature with those at the zero-temperature limit or Born-Markov limit, we find that the non-Markovian memory effect can dramatically change the coherent and thermal dynamics of the cavity. We also numerically study the dissipation dynamics of the cavity through the mean mode amplitude decay and the average photon number decay in the microwave regime. It is shown that the strong coupling between the cavity and the reservoir results in a long-time dissipationless evolution to the cavity field amplitude, and its noise dynamics undergo a critical transition from the weak to the strong coupling due to the non-Markovian memory effect.

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I. INTRODUCTION

The dissipation quantum dynamics of optical cavities have been well investigated and deeply understood under the Born-Markov (BM) approximation [1]. The BM approximation is valid when the coupling between the system and the environment is weak enough so that the perturbation is applied, and in the meantime, the characteristic time of the environment is sufficiently shorter than that of the system so that the non-Markovian memory effect is negligible. However, in many situations in the recent development of optical microcavities, the strong coupling effect or the long-time memory effect has become an important factor in controlling cavity dynamics. Typical examples include optical fields propagating in cavity arrays or in an optical fiber [2–4], trapped ions subjected to artificial colored noise [5–8], and microcavities interacting with a coupled resonator optical waveguide (CROW) or photonic crystals [9–16], and so on. Specifically, for the trapped ions coupled with an engineered reservoir, the change of the characteristic frequency of the reservoir can be accomplished simply by applying a random electric field through a band-pass filter defining the frequency spectrum of the reservoir [5]. While, for a cavity interacting with CROW or photonic crystals, the coupling between them is controllable by changing the geometrical parameters of the defect cavity and the distance between the cavity and the CROW [15]. Both of them provide non-Markovian dissipation and decoherence channels [6–8,16]. These strong coupling or long-time memory effects result in a complicated non-Markovian process in cavity systems that have become a crucial concern for the rapid development of quantum information and quantum computation in terms of photons [17]. The non-Markovian behavior of the trapped ions has been discussed in many works [6–8]. In this article, we shall investigate the non-Markovian dynamics for the second case, the cavity strongly coupled with its environment (CROW or photonic crystals).

Quantum dynamics of cavity systems are completely described by the master equation of the reduced density operator by taking the cavity as an open system. The master equation under the BM approximation can be found in many textbooks [1,18,19]. However, an exact master equation beyond the BM approximation is only derived in a few works. The first exact master equation, called the Hu-Paz-Zhang master equation for quantum Brownian motion, was found almost two decades ago [20] (also see [21] with some exact solutions [22]) in terms of the Wigner distribution function in phase space via the Feynman-Vernon influence functional [23]. The exact master equation for a damped harmonic oscillator at finite temperature was further reproduced later by Yu [24] using the approach of averaging non-Markovian quantum trajectories based on the stochastic Schrödinger equation [25]. Recently, our group developed exact master equations of the reduced density operator for both the fermion systems [26,27] and the bosonic systems [28,29] by extending the Feynman-Vernon influence functional in terms of the coherent-state path integrals. These exact master equations can fully depict the quantum dissipative and decoherence dynamics in various open systems in the strong coupling regime. For the cavity system, the master equation obtained in [28,29] is for zero temperature. Here we shall extend it to a finite temperature. In reality, the temperature effect is also unavoidable and nonnegligible. It has been pointed out [1] that for a cavity frequency lying in the microwave regime, thermal photons are presented even at liquid helium temperatures [30]. In this article, we shall use the exact master equation, which is valid at arbitrary temperature, to investigate the exact non-Markovian dynamics of a general cavity strongly coupled with its reservoir and to find the general features of the coupling and temperature dependence in non-Markovian dynamics.

The article is organized as follows. In Sec. II, we introduce the exact master equation we developed recently for the reduced density operator of the cavity system coupled to a general reservoir, from which the second-order perturbation
master equation and the BM master equation are reproduced at well-defined limits. In Sec. III, we then solve analytically the exact master equation via the coherent-state representation, including the exact solutions of the mean-field amplitude and the average photon number inside the cavity that are experimentally measurable. We also obtain explicitly the reduced density matrix for three different initial states, the vacuum state, the coherent state, and the mixed state, to show the different non-Markovian behaviors. The decoherence dynamics of the cavity field is explicitly analyzed. In Sec. IV, we numerically demonstrate the exact non-Markovian behavior through the time dependence of the mean mode amplitude and the average photon number in the cavity in both the weak and Strong coupling regimes for three typical spectral densities, the Ohmic, sub-Ohmic, and super-Ohmic cases, with the cavity’s frequency being focused in the microwave regime. We find that the strong coupling between the cavity and the reservoir results in a long-time dissipationless evolution to the cavity field amplitude and the noise dynamics undergo a critical transition from the weak to the strong coupling due to the non-Markovian memory effect. Finally a conclusion is given in Sec. V.

II. EXACT MASTER EQUATION FOR A CA VITY IN A GENERAL RESERVOIR

We consider a cavity with a single mode coupled to a general reservoir. The Hamiltonian of the total system is given by

\[ H = \hbar \omega_0 a^\dagger a + \sum_k \hbar \omega_k b^\dagger_k b_k + \sum_k \hbar V_k (b^\dagger_k a + a^\dagger b_k). \]

in which the first term is for the single cavity mode with \( a^\dagger, a \) being the creation and annihilation operators of the cavity field, \( \omega_0 \) is its frequency; the second term is the Hamiltonian \( H_R \) for a general reservoir modeled as a collection of infinite harmonic oscillators, where \( b_k^\dagger \) and \( b_k \) are the corresponding creation and annihilation operators of the \( k \)th oscillator with the frequency \( \omega_k \). The coupling between the cavity and the environment is described by the third term and \( V_k \) is the coupling strength between them, which is tunable for the reservoir being CROW or photonic crystals [9–16].

A. Exact master equation

The exact master equation for the cavity field is given in terms of the reduced density operator, which is defined from the density operator of the total system by tracing over entirely the environmental degrees of freedom \( \rho(t) \equiv Tr_R(\rho_{tot}(t)) \), where the total density operator is governed by the quantum Liouville equation \( \rho_{tot}(t) = e^{-i H(t-t_0)} \rho_{tot}(t_0) e^{i H(t-t_0)} \). As usual \([31]\), assuming that the cavity field is uncorrelated with the reservoir before the initial time \( t_0 \), \( \rho_{tot}(t_0) = \rho(t_0) \otimes \rho_R(t_0) \), and the reservoir is initially in the equilibrium state \( \rho_R(t_0) = \frac{1}{Z_R} e^{-\beta H_R} \) where \( Z_R \) is the Boltzmann constant and \( T \) the reservoir’s initial temperature. Then tracing over all the environmental degrees of freedom can be easily carried out using the Feynman-Vernon influence functional approach \([23]\) in the framework of coherent-state-path-integral representation \([32]\). The resulting master equation for the reduced density operator has a standard form similar to the master equation for electrons in the nanostructure we developed recently \([26,27]\) (with some sign difference due to the different statistical property between fermions and bosons)

\[
\dot{\rho}(t) = -i [H_0(t), \rho(t)] + \kappa(t) \{ 2 \rho(t) a^\dagger a - a^\dagger a \rho(t) + \rho(t) a^\dagger a \} + \tilde{\kappa}(t) \{ a^\dagger \rho(t) a + a \rho(t) a^\dagger - a^\dagger a \rho(t) - \rho(t) a a^\dagger \},
\]

where the time-dependent coefficient \( \omega(t) \) is the renormalized frequency of the cavity, while \( \kappa(t) \) and \( \tilde{\kappa}(t) \) describe the dissipation and noise to the cavity field due to the coupling with the reservoir. These coefficients are nonperturbatively determined by the following relations:

\[
\begin{align*}
\omega(t) &= -\text{Im}[\dot{u}(t) u^{-1}(t)], \\
\kappa(t) &= -\text{Re}[\dot{u}(t) u^{-1}(t)], \\
\tilde{\kappa}(t) &= \dot{v}(t) - 2v(t)\text{Re}[\dot{u}(t) u^{-1}(t)],
\end{align*}
\]

and \( u(t) \) and \( v(t) \) satisfy the integrodifferential equations of motion

\[
\begin{align*}
\dot{u}(\tau) + i \omega_0 u(\tau) + \int_{t_0}^\tau d\tau' g(\tau - \tau') u(\tau') &= 0, \\
\dot{v}(\tau) + i \omega_0 v(\tau) + \int_{t_0}^\tau d\tau' g(\tau - \tau') v(\tau') &= \int_{t_0}^\tau d\tau' \tilde{g}(\tau - \tau') \bar{\omega}(\tau).
\end{align*}
\]
Then these time-dependent coefficients in the exact master equation can be simplified as

\[ \omega_0(t) = \omega_0 + \text{Im}[w(t)], \quad \kappa(t) = \text{Re}[w(t)], \quad (7a) \]

\[ \tilde{\kappa}(t) = v(t) + 2\nu(t)\text{Re}[w(t)], \quad (7b) \]

with \( w(t) = \int_{t_0}^{t} dt' g(t - t') u(t) u^{-1}(t), \) which can be calculated by solving \( u(t) \) nonperturbatively from Eq. (4a). Thus, the non-Markovian memory structure is nonperturbatively built into the integral kernels in these equations. The expression of the integrodifferential Eq. (4) shows that \( u(t) \) is just the propagating function of the cavity field (the retarded Green function in nonequilibrium Green function theory [33]), and \( v(t) \) is the corresponding correlation (Green) function which is also determined by \( u(t) \) [see Eq. (6b)]. Therefore, the exact master equation for the cavity reduced density operator depicts the full nonequilibrium dynamics of the cavity system.

In fact, the exact master equation presented here simply covers the exact master equation at zero-temperature we derived very recently [28,29]. Taking the zero-temperature limit \( T = 0 \), then \( \bar{g}(\tau - \tau') = 0. \) As a result, we have \( v(t) = 0 \) and therefore the coefficient \( \tilde{\kappa}(t) = 0. \)

The master equation is simply reduced to

\[ \dot{\rho}(t) = -i\omega_0(t)[a^\dagger a, \rho(t)] + \kappa(t)[2\alpha_0 a^\dagger a^\dagger \rho(t) - \rho(t) a^\dagger a], \quad (8) \]

with the temperature-independent coefficients \( \omega_0(t), \kappa(t) \) obeying the same Eqs. (3a) and (3b) through Eq. (4a).

Equation (8) is the exact master equation for cavity fields coupled to the vacuum fluctuation (i.e., the zero temperature limit) [28,29].

B. Reduce to the BM master equation without and with the long-time limit

Interestingly, the exact master equation (2) has the same form as the BM master equation in the literature [1]. The difference is the coefficients in the master equation. Here all the coefficients are time dependent and determined nonperturbatively by integrodifferential equations of motion (4) or (6), which take into account the back-reaction effects of the reservoir on the cavity field. While the coefficients in the BM master equation are all time independent, which ignore the memory effects between the cavity and the reservoir and are determined under the perturbation approximation up to the second order of the coupling of \( V(\omega) \) and then taking the long-time Markov limit.

Explicitly, since the time correlation functions \( g(\tau - \tau') \) and \( \bar{g}(\tau - \tau') \) are already proportional to \( V(\omega)^2 \), taking approximately the time-dependent coefficients in Eq. (7) up to the second order of the coupling of \( V(\omega) \) means that the propagating functions \( u(t) \) and \( u^{-1}(t) \) in the right-hand side of Eq. (6) should be approximated only up to the zero order \( u_0(t) = e^{-i\omega_0(t-t_0)} \) and \( u_0^{-1}(t) = e^{i\omega_0(t-t_0)}. \) This leads to

\[ \dot{u}(t) u^{-1}(t) \simeq -i\omega_0 - \int_{t_0}^{t} dt' \int_{0}^{\infty} \frac{d\omega}{2\pi} J(\omega) e^{-i(\omega - \omega_0)(t-t')}, \quad (9a) \]

\[ \dot{v}(t) \simeq 2 \int_{t_0}^{t} dt' \int_{0}^{\infty} \frac{d\omega}{2\pi} J(\omega) \tilde{n}(\omega, T) \cos(\omega - \omega_0)(t-t'), \quad (9b) \]

The term \( 2\nu(t)\text{Re}[w(t)] \) in (7b) is proportional to \( V(\omega)^4 \) and should be ignored in the same approximation. Then the coefficients of Eqs. (3) or (7) are reduced to

\[ \omega_0(t) \simeq \omega_0 - \int_{t_0}^{t} dt' \int_{0}^{\infty} \frac{d\omega}{2\pi} J(\omega) \sin(\omega - \omega_0)(t-t'), \quad (10a) \]

\[ \kappa(t) \simeq \int_{t_0}^{t} dt' \int_{0}^{\infty} \frac{d\omega}{2\pi} J(\omega) \cos(\omega - \omega_0)(t-t'), \quad (10b) \]

\[ \tilde{\kappa}(t) \simeq 2 \int_{t_0}^{t} dt' \int_{0}^{\infty} \frac{d\omega}{2\pi} J(\omega) \tilde{n}(\omega, T) \cos(\omega - \omega_0)(t-t'). \quad (10c) \]

Substituting these coefficients into Eq. (2) results in the master equation of the cavity field in the perturbation approximation up to the second order of the coupling constant between the cavity and the reservoir. This perturbative master equation up to the second order in the coupling constant is indeed the same as the master equation under the Born and Markov approximations without taking the long-time Markov limit [1]. This master equation is a good approximation only for the dissipation and noise dynamics of the cavity mode in the very weak coupling regime.

To reproduce the standard BM master equation with the time-independent coefficients, one needs to take further the long-time Markov limit where \( t \) is the typical time scale of the cavity so that the \( t' \) integration is dominated by a much shorter time characterizing the decay of the reservoir correlations [1,29]. In other words, one can take the \( t' \) integration to infinity in Eq. (10)

\[ \lim_{t \to \infty} \int_{0}^{t} dt' e^{\pm i(\omega - \omega_0)t'} = \pi \delta(\omega - \omega_0) \mp i \frac{\mathcal{P}}{\omega - \omega_0}, \quad (11) \]

where \( \mathcal{P} \) denotes the principle value of the integral. As a result, all the coefficients in the master equation, given by (10), become time independent.

\[ \omega_0' = \omega_0 + \delta_0, \quad \kappa = \pi g(\omega_0)|V(\omega_0)|^2 = J(\omega_0)/2, \quad (12a) \]

\[ \tilde{\kappa} = 2\pi g(\omega_0)|V(\omega_0)|^2 \tilde{n}(\omega_0, T) = 2\kappa \tilde{n}(\omega_0, T), \quad (12b) \]

and the frequency shift \( \delta_0 = \mathcal{P} \int_{0}^{\infty} d\omega \frac{g(\omega)|V(\omega)|^2}{\omega - \omega_0}. \) Then the exact master equation (2) is reduced to

\[ \dot{\rho}(t) = -i\omega_0 [a^\dagger a, \rho(t)] + \kappa [2\alpha_0 a^\dagger a^\dagger \rho(t) - \rho(t) a^\dagger a] \]

\[ + [a^\dagger \rho(t) a + \alpha_0 a^\dagger a^\dagger \rho(t) - \rho(t) a^\dagger a] + 2\kappa \tilde{n}(\omega_0, T) \times [a^\dagger \rho(t) a + \alpha_0 a^\dagger a^\dagger \rho(t) - \rho(t) a^\dagger a]. \quad (13) \]

Equation (12) with Eq. (13) reproduces exactly the BM master equation in quantum optics for a single cavity mode interacting with a thermal field [1].

It is worth pointing out that the BM master equation without and with the long-time limit, given by the time-dependent and time-independent coefficients, Eqs. (10) and (12), respectively, are only valid for the weak coupling regime. As a result, the BM master equation with the time-dependent coefficients, which is the same as the perturbative master equation up to the second order in the coupling constant between the system and the
reservoir, cannot reveal the non-Markovian effect since the memory effect has already been dropped. This conclusion will become clearer in Sec. IV where we numerically show that in the weak coupling regime the exact master equation and the BM master equation give the qualitatively same dynamics, namely the damping dynamics are a simple exponential decay and the non-Markovian dynamics are negligible. In other words, the perturbative master equation up to the second order can only show the Markov-like dynamics for an open system, although quantitatively it may have a slightly different result obtained from the master equation with time-independent coefficients in the long-time Markov limit.

III. EXACT SOLUTION OF THE MASTER EQUATION

In this section, we shall present the exact solution of the master equation for some physical observables and also for the reduced density operator of the cavity.

A. Exact solution for some physical observables

The main physical observables for a cavity are the decays of the mean mode amplitude and the average photon number inside the cavity. The mean mode amplitude of the cavity field is defined by \(\langle a(t)\rangle = \text{tr}[a\rho(t)]\). From the exact master equation (2), it is easy to find that \(\langle a(t)\rangle\) obeys the equation of motion

\[
\dot{\langle a(t)\rangle} = -[i\omega_0(t) + \kappa(t)]\langle a(t)\rangle = \frac{\dot{u}(t)}{u(t)}\langle a(t)\rangle,
\]

which has the exact solution

\[
\langle a(t)\rangle = u(t)\langle a(t_0)\rangle.
\]

That is, the time evolution and the decay behavior of the mean mode amplitude is totally determined by \(u(t)\). Equation (15) clearly indicates that \(u(t)\) is the propagating function characterizing the time evolution of the cavity field.

Another important physical observable is the average particle number inside the cavity, which is defined by \(n(t) = \text{tr}[a^2\rho(t)]\). From the exact master equation, it is also easy to find that

\[
\dot{n}(t) = -2\kappa(t)n(t) + \bar{\kappa}(t).
\]

On the other hand, Eq. (3b) can be rewritten as

\[
\dot{v}(t) = -2\kappa(t)v(t) + \bar{\kappa}(t),
\]

with \(-2\kappa(t) = [\dot{u}/u(t) + \text{H.c.}]\). Combining these equations together, we obtain the exact solution of \(n(t)\) in terms of \(u(t)\) and \(v(t)\)

\[
\dot{n}(t) = u(t)n(t_0)u^*(t) + v(t).
\]

This relationship is similar to the fermion case we derived recently [34]. In fact, the above solution is a result of the correlated Green function in nonequilibrium Green function theory [33]. Since \(v(t)\) is also determined by \(u(t)\) as we can see from Eq. (6b), both the mean mode amplitude and the average photon number inside the cavity are completely solved by the propagating function \(u(t)\).

If we take the BM limit in which all the coefficients in the master equation are reduced to \(\omega_0 = \omega_0 + \delta\omega_0, \kappa = J(\omega_0)/2, \bar{\kappa} = 2\kappa\bar{\pi}(\omega_0, T)\) with \(\delta\omega_0 = P \int_0^\infty \frac{dw}{2\pi} \frac{J(\omega)}{\omega - \omega_0}\) [see Eq. (12)], then the resolution of \(u(t)\) simply becomes

\[
u_{BM}(t) = e^{-\bar{\kappa}(t_0 + \kappa)(t - t_0)}.
\]

Correspondingly, the evolution of the mean mode amplitude in the BM limit is

\[
\langle a(t)\rangle_{BM} = e^{-\bar{\kappa}(t_0 + \kappa)(t - t_0)}\langle a(t_0)\rangle,
\]

which shows an exponential decay in time. Substituting (19) into (4b), we have the BM solution of \(v(t)\)

\[
v_{BM}(t) = \bar{\pi}(\omega_0, T)[1 - e^{-2\kappa(t - t_0)}].
\]

Thus the average photon number in the BM limit is simply given by

\[
n_{BM}(t) = n(t_0)e^{-2\kappa(t - t_0)} + \bar{\pi}(\omega_0, T)[1 - e^{-2\kappa(t - t_0)}].
\]

These results reproduce all the BM solutions in the weak coupling regime [1]. However, the exact solutions of Eqs. (15) and (18) allow us to explore the non-Markovian dynamics of the cavity systems not only in the weak coupling regime but also in the strong coupling regime. The explicit difference between the non-Markovian and Markov dynamics can be seen by comparing the solution of Eqs. (15) and (18) with (20) and (22).

B. Exact solution of the reduced density operator

In fact, the reduced density operator can be also explicitly obtained through the coherent-state representation. The reduced density matrix in the coherent-state representation is given by

\[
\rho(t) = \int d\mu(\alpha)\rho(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha^*|,
\]

where \(d\mu(\alpha) = \frac{d\alpha d\alpha^*}{2\pi} e^{-|\alpha|^2}\) is the integrable measure in the complex space of the coherent state \(|\alpha\rangle = e^{\alpha\hat{a}^\dagger}|0\rangle\) [32], and \(\rho(\alpha,\alpha^*,\alpha^*,\alpha',t) = |\langle\alpha|\rho(t)|\alpha^*\rangle|\) can be obtained from the exact master equation

\[
\dot{\rho}(\alpha,\alpha^*,\alpha^*,\alpha',t) = \Lambda(t)\int d\mu(\alpha)\int d\mu(\alpha^*)\rho(\alpha,\alpha^*,\alpha^*,\alpha',t_0)\exp[\alpha^*\hat{B}(t)\alpha_i + \alpha^*\hat{C}(t)\alpha_j + \alpha^*\hat{D}(t)\alpha_i + \alpha^*\hat{B}^*(t)\alpha'_j],
\]

where \(\rho(\alpha,\alpha^*,\alpha^*,\alpha',t_0) = |\langle\alpha|\rho(t_0)|\alpha^*\rangle|\) is the initial state \(\rho(t_0)\) in the coherent-state representation. The exponential function in the integral is the propagating function of the reduced density operator, which fully determines the time evolution of the reduced density matrix in the coherent-state representation. The time-dependent coefficients \(\Lambda(t), \hat{B}(t), \hat{C}(t),\) and \(\hat{D}(t)\) in the propagating function are listed in Table I for various cases. Thus, for a given initial state \(\rho(t_0)\), the corresponding analytical solution of the reduced density operator for an arbitrary coupling to the reservoir at an arbitrary temperature can be obtained from Eqs. (23) and (24). Here we shall consider a few typical initial states: the vacuum state, the coherent state, and the mixed state.
Eq. (29) in the strong coupling regime.

that the solution of Eq. (26) can be very different from that of
unchanged. On the other hand, in the BM limit, we have
between the cavity and the reservoir so that the cavity remains
consequence since at zero temperature, the reservoir is also
In other words, the vacuum state remains unchanged if
First, we consider the cavity initially in the vacuum state
This is a completely mixed state in terms of the Fock space of
the cavity. It indicates that the coupling to the reservoir makes
is initially empty and also no external driving field is applied to

that is, there is no existence of a coherent field in the cavity if it
remains unchanged if\( T \to 0 \).

\[
\rho(t) = \sum_{n=0}^{\infty} \frac{\langle n | v(t) \rangle \langle v(t) | n \rangle}{1 + [v(t)]^2 T^2}.
\]

This is a completely mixed state in terms of the Fock space of
the cavity, with the average photon number

\[
n(t) = v(t).
\]

That is, there is no existence of a coherent field in the cavity if it
is initially empty and also no external driving field is applied to
the cavity. It indicates that the coupling to the reservoir makes
the cavity continuously but randomly gain the energy (photons)
from the reservoir, and eventually reaches a steady mixed state
with a constant average photon number \( n(t) = v(t \to \infty) \).
However, we must point out that this steady limit is generally
different from the BM limit since \( v(t \to \infty) \neq \bar{n}(\omega_0, T) \). The
difference is a manifestation of the non-Markovian effect that we
shall demonstrate numerically in the next section.

At the zero-temperature limit, \( v(t) = 0 \), then

\[
\rho(t)|_{T=0} = |0\rangle \langle 0| = \rho(t_0).
\]

In other words, the vacuum state remains unchanged if the
reservoir temperature is zero. This is a trivial physical consequence
since at zero temperature, the reservoir is also in the vacuum state.
Then no photon can be exchanged between the cavity and the reservoir so that the cavity remains
unchanged. On the other hand, in the BM limit, we have

\[
\rho_{\text{BM}}(t) = \sum_{n=0}^{\infty} \frac{[\bar{n}(\omega_0, T)]^n}{[1 + \bar{n}(\omega_0, T)]^{n+1}} |n\rangle \langle n|,
\]

which is a thermal equilibrium state with the average photon
number \( n_{\text{BM}}(t) \to \bar{n}(\omega_0, T) \) at temperature \( T \). It is expected
that the solution of Eq. (26) can be very different from that of
Eq. (29) in the strong coupling regime.

Next, we consider an initial coherent state

\[
\rho(t_0) = e^{-|\alpha(t_0)|^2} |\alpha(t_0)\rangle \langle \alpha(t_0) |.
\]

From Eqs. (23) and (24), we find that the reduced density
operator at time \( t \) becomes

\[
\rho(t) = \exp\left\{ \frac{|\alpha(t)|^2}{1 + v(t)} \sum_{n=0}^{\infty} \frac{[v(t)]^n}{[1 + v(t)]^{n+1}} \right\} \alpha(t) \langle \alpha(t) | \left( 1 + v(t) \right)^n, \]

where \( |\alpha(t)| \equiv \exp\left[ \frac{|\alpha(t)|^2}{1 + v(t)} \right] |\alpha(t)\rangle \langle \alpha(t)| \) is defined as a generalized
coherent state, and \( \alpha(t) = u(t) \alpha_0 \). It is interesting to see that
Eq. (31) is indeed a mixed state of generalized coherent states
\( |\alpha(t)| \equiv |\alpha(t)| \langle \alpha(t)| \left( 1 + v(t) \right)^n \). The average photon number in this state is given by

\[
n(t) = |\alpha(t)|^2 + v(t),
\]

as we expected.

In the weak coupling regime, \( u(t) \) generally decays to zero.
The steady-state limit of the above state will be the same as
that in case 1, and the asymptotic average photon number is
also given by \( n(t) = v(t \to \infty) \). The corresponding reduced density
operator asymptotically becomes a completely mixed state
of Eq. (26). This solution shows an exact decoherence
process in the cavity. The decoherence arises from two sources,
the mean mode amplitude damping characterized by the decay
behavior in terms of the propagating function through the
solution \( \alpha(t) = u(t) \alpha_0 \) and the thermal-fluctuation-induced
noise effect characterized by \( v(t) \), as we can see from Eq. (31).
The latter describes a process of the random loss or gain of
thermal energy from the reservoir, up to the initial temperature
of the reservoir.

However, in the strong coupling regime, \( u(t) \) may not decay
to zero, as we shall show explicitly in the numerical calculation
in the next section. Then the reduced density operator remains
as a mixed coherent state. On the other hand, at the zero-
temperature limit \( v(t) = 0 \), the reduced density operator at
time \( t \) is given by

\[
\rho(t)|_{T=0} = e^{-|\alpha(t)|^2} |\alpha(t)\rangle \langle \alpha(t) |.
\]

In other words, the cavity can remain in a coherent state in the
zero-temperature limit. These two features \( |u(t)| \) may not decay
to zero in the strong coupling regime and \( v(t) = 0 \) at \( T = 0 \)
indicate that enhancing the coupling between the cavity and the
reservoir, and in the meantime lowering the initial temperature
of the reservoir, can significantly reduce the decoherence effect
in the cavity system.

On the other hand, in the BM limit the reduced density
operator is always reduced to a thermal state after a long time
has passed

\[
\rho_{\text{BM}}(t \to \infty) = \sum_{n=0}^{\infty} \frac{[\bar{n}(\omega_0, T)]^n}{[1 + \bar{n}(\omega_0, T)]^{n+1}} |n\rangle \langle n|.
\]

This is because \( u_{\text{BM}}(t \to \infty) \) must approach zero, see Eq. (19).
It shows that the results in the BM limit can be qualitatively
different from the exact solution. These analytical results reveal
the underlying mechanism of reservoir-induced decoherence.
in cavity dynamics, which may provide some insight into how to control the decoherence dynamics in such systems.

3. Initial mixed state

The last special case we shall consider is the cavity in an initially mixed state,
\[
\rho(t_0) = \sum_{n=0}^{\infty} \frac{[n(t_0)]^n}{[1 + n(t_0)]^{n+1}} |n\rangle\langle n|, \tag{35}
\]
with the average initial photon number \( n(t_0) \). Using the solution of the master equation, we have
\[
\rho(t) = \frac{A(t)}{1 + n(t_0) \Delta(t)} \times \sum_{n=0}^{\infty} \left[ \frac{n(t_0)}{1 + n(t_0) - n(t_0) \Delta(t)} + C(t) \right]^n |n\rangle\langle n|
= \sum_{n=0}^{\infty} \frac{[n(t)]^n}{[1 + n(t)]^{n+1}} |n\rangle\langle n|, \tag{36}
\]
where
\[
n(t) = |u(t)|^2 n(t_0) + v(t), \tag{37}
\]
is the average photon number of the cavity during time evolution. It shows that if the cavity is initially in a mixed state, it will remain in a mixed state, with a different particle number distribution varying in time. In the weak coupling regime, \( u(t) \) approaches zero so that the reduced density matrix at a steady limit will reach the same state as in the other two cases. In the strong coupling regime, \( u(t) \) may not decay to zero so that the average photon number in cavity \( n(t) \) can be very different from the weak coupling regime, namely very different from the BM limit, even though both the exact reduced density matrix and its BM limit are mixed states.

In the next section, we will numerically demonstrate these dynamics for some specifically given spectral density \( J(\omega) \).

IV. NUMERICAL ANALYSIS OF THE EXACT NON-MARKOVIAN DYNAMICS

To explicitly see the non-Markovian memory effect in the cavity system when it strongly couples with a reservoir, we consider a general spectral density of the bosonic environment
\[
J(\omega) = 2\pi \eta \omega \left( \frac{\omega}{\omega_0} \right)^{s-1} e^{-\omega/\omega_0}, \tag{38}
\]
where \( \eta \) is the dimensionless coupling strength between the system and reservoir and \( \omega_0 \) is the cutoff frequency of the spectrum. The parameter \( s \) classifies the environment as sub-Ohmic \((0 < s < 1)\), Ohmic \((s = 1)\), and super-Ohmic \((s > 1)\). In the following, we set the value of \( s \) to 1/2, 1, and 3, respectively.

For a single-mode cavity with the frequency in the optical regime, it is well known that the BM approximation works very well for the usual thermal environment. However, for a structured reservoir, such as the CROW, the coupling strength between the cavity and reservoir can be controlled by changing the geometrical parameters of the defect cavity and the distance between the cavity and the CROW [15]. Then the BM approximation or the perturbative approximation must be re-examined. Furthermore, when the cavity frequency lies in the microwave regime, the temperature of the environment also becomes nonnegligible. Specifically, we take the cavity frequency to be \( \omega_0 = 21.5 \) GHz \((i.e., \hbar \omega_0 = 13.83 \mu \text{eV})\) and the temperature is taken at \( T = 2 \) K so that \( k_B T = 172.3 \mu \text{eV} \approx 12.5 \hbar \omega_0 \) [30]. With these experimental parameters as input, we numerically calculate the exact dissipative dynamics of the cavity for three different spectral densities with different coupling strength \( \eta \) in Eq. (38). In addition, the cutoff frequency \( \omega_c \) in Eq. (38) is taken roughly to be the same order as the cavity frequency \( \omega_0 \) \((i.e., \omega_c \approx \omega_0)\). The detailed numerical results are plotted in Figs. 1 through 6. Note that the BM limits are the same for the three different spectral densities when we use the cutoff frequency \( \omega_c = \omega_0 \) since \( n_{BM}(t) \), \( v_{BM}(t) \), and \( n_{BM}(t) \) only depend on \( \kappa = J(\omega_0)/2 = \pi \eta \omega_0 e^{-1} \) for all three different spectral densities \([see \ Eq. \ (38)]\). From Eqs. (19), (21), and (22), we can analytically know that \( u(t), v(t), \) and \( n(t) \) in the BM limit monotonously change in time as the coupling \( \eta \) increases. However, in the exact cases, we will show that the results have qualitative changes from the weak to the strong coupling regimes.

In Fig. 1, we show the exact solutions for the absolute value of \( |u(t)| \) \([i.e., \text{the amplitude of the propagating function} \ u(t) \text{which characterizes the time evolution of the amplitude of the cavity field through the relation} \ a(t) = u(t) |a(t_0)| \text{] for Ohmic, sub-Ohmic, and super-Ohmic spectral densities in the weak and strong coupling cases with a comparison to the BM limit. \text{For a given coupling strength} \ \eta, \text{comparing the behavior of the exact} \ u(t) \text{of different spectral densities with its BM limit, we see that in the very weak coupling limit} \ (\eta \approx 0.02), \text{the difference between the exact amplitude of} \ u(t) \text{and the BM limit is very small for all three spectral densities. With the increase of} \ \eta, \text{the difference becomes more and more visible. The largest difference between the exact result and the BM limit for a strong coupling is a significant manifestation of the non-Markovian memory effect.}\]

For a given spectral density, comparing the behavior of \( |u(t)| \) among different coupling strengths, we find that roughly for \( \eta < 0.3, |u(t)| \) decays almost monotonously for all three spectral densities, except for a short-time oscillation in the beginning. In general, a weak coupling to the reservoir always induces an amplitude damping to the cavity field. However, when \( \eta \gg 0.3, \text{besides a short-time oscillation and decay,} \ |u(t)| \text{may approach a nonzero stationary value. In other words, in the strong coupling regime, the non-Markovian memory effect can result in a long-time qualitative change to the time evolution of the cavity field, namely it changes the cavity field from a pure damping process in the weak coupling regime to a dissipationless process in the strong coupling regime. In particular, this qualitative change in the exact solution of} \ u(t) \text{becomes most significant for the super-Ohmic case, and then} \text{for the sub-Ohmic and the Ohmic cases, in comparison with the BM limit, as shown in Fig. 1.}\]

To see more clearly the short-time and long-time behaviors of \( u(t) \), we plot \( \kappa(t) = -\text{Re} |u(t)/u(t_0)| \) in Fig. 2, which is the decay coefficient in the master equation (2) that describes the energy dissipation of the cavity. The short-time rapid increase of \( \kappa(t) \) in the beginning indicates a fast decay of the cavity field, in agreement with the results shown in Fig. 1. However, by
increasing the coupling $\eta$, in particular for $\eta \geq 0.3$, we see that the decay coefficient $\kappa(t)$ shows a very different time dependence. It has a large oscillation between an equal positive and negative bounded value and then approaches zero. The oscillation indicates that the cavity dissipates energy into the reservoir first and gains the energy back from the reservoir. Then the zero steady value indicates that after the oscillation, the cavity becomes dissipationless. This is why $|u(t)|$ has a nonzero steady value, as we see in Fig. 1. $|u(t)|$ of the super-Ohmic reservoir maintains the largest nonzero steady value in the strong coupling regime because its short-time oscillation occurs in the shortest time within which only the smallest energy is dissipated. While, in the BM limit, $\kappa$ keeps a nonzero constant, namely the cavity is always in a dissipation state.

Therefore, we can conclude that the super-Ohmic reservoir contains the strongest non-Markovian memory effect shown in the long-time dissipationless process for the cavity field, while the sub-Ohmic reservoir involves the strongest short-time oscillating non-Markovian dynamics. In the previous investigations on the non-Markovian effect, the short-time oscillation behavior in the dissipation processes in the weak coupling regime was observed [8]. The long-time dissipationless stationary behavior in the strong coupling regime may only be revealed from a nonperturbation theory, as we show here.

To see clearly the significance of the temperature effect of the non-Markovian memory dynamics, we show in Fig. 3 the time evolution of the correlation function $\nu(t)$ in the weak and strong coupling regimes and compare the results with the BM solution again. The correlation function $\nu(t)$ characterizes the temperature-dependent noise effect (the average photon correlation through the reservoir). Meanwhile, $\nu(t)$ is also the main contribution to the average photon number inside the cavity induced mainly by thermal noise dynamics, see Eq. (18).

It is shown in Table I that in the zero-temperature limit $T \rightarrow 0$, $\nu(t) = 0$. However, at a finite temperature it will behave very differently for different couplings.

FIG. 1. (Color online) Comparison of the exact solution of $|u(t)|$ for sub-Ohmic (dashed black line), Ohmic (dotted red line), and super-Ohmic (dash-dotted blue line) with the corresponding BM limit (solid magenta line) in the weak coupling ($\eta \leq 0.3$) and the strong coupling ($\eta \geq 0.4$) regimes. Here we have taken the parameters $\omega_0 = 21.5$ GHZ, $\omega_c = \omega_0$.

FIG. 2. (Color online) Comparison of the exact solution of $\kappa(t)$ for sub-Ohmic (dashed black line), Ohmic (dotted red line), and super-Ohmic (dash-dotted blue line) with the corresponding BM limit (solid magenta line) in the weak coupling ($\eta < 0.3$) and the strong coupling ($\eta \geq 0.3$) regimes. Here we have taken the parameters $\omega_0 = 21.5$ GHZ, $\omega_c = \omega_0$. 

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FIG. 3. (Color online) Exact solution of \( v(t) \) for sub-Ohmic (dashed black line), Ohmic (dotted red line), and super-Ohmic (dash-dotted blue line) with the corresponding BM limit (solid magenta line) in the weak coupling (\( \eta < 0.3 \)) and the strong coupling (\( \eta \geq 0.3 \)) regimes. Here \( \omega_0 = 21.5 \) GHz, \( \omega_c = \omega_0 \), and \( T = 2 \) K.

In the very weak coupling limit (\( \eta = 0.02 \)), the exact \( v(t) \) is almost the same as the BM solution except for the sub-Ohmic spectral density in which \( v(t) \) shows a long-time oscillation, as a weak non-Markovian memory effect. With the increasing of \( \eta \), the quantitative difference between the exact solution and its BM limit is enlarged for all three spectral densities. Comparing the behavior of \( v(t) \) among different \( \eta \) for a given spectral density, we find that, similar to the behavior of \( |u(t)| \), a short-time oscillation (almost invisible in Fig. 3) of \( v(t) \) exists for all three spectral densities. In the long-time limit, for different coupling strengths, the stationary values of the exact \( v(t) \) for all three different spectral densities can be very different from the BM result. The numerical result in Fig. 3 clearly shows that \( v(t \to \infty) \) is far away from the BM limit \( v_{\text{BM}}(t \to \infty) = \overline{\eta}(\omega_0, T) \) in the strong coupling regime, as we discussed analytically in the last section.

In particular, continuously increasing the coupling strength \( \eta \) shows that the sub-Ohmic case manifests a significant qualitative difference from the BM limit. The time-oscillation behavior of \( v(t) \) for the sub-Ohmic case is very strong and is maintained for a longer period of time when the coupling \( \eta \) becomes larger. This long-time oscillation behavior actually comes from the long-time periodical oscillation of \( u(t) \), as we show in Fig. 4. However, in the strong coupling regime, \( v(t) \) of the super-Ohmic spectral density increases more slowly than the BM limit with increasing \( \eta \), but the overall time evolution is qualitatively the same as in the BM limit. In other words, the temperature-induced noise effect becomes very important for the sub-Ohmic case and then for the Ohmic case while it may be minor for the super-Ohmic reservoir in the strong coupling regime, which is quite different in comparison with the solution in the weak coupling cases, see Fig. 3.

To show the totally non-Markovian memory effect distributing in the amplitude damping and the noise dynamics, we examine the average photon number inside the cavity, which is analytically given by Eq. (18). From Eq. (18), we see that the average photon number contains two terms, the first term is the decay of the initial average photon number determined by the propagating function \( u(t) \) and the second term is the correlation function \( v(t) \) as a noise effect which sensitively depends on the initial temperature of the reservoir. The numerical solutions based on the exact solution (18) for three different spectral densities are plotted in Fig. 5 with a comparison to the BM result.

FIG. 4. (Color online) Long-time behavior of \( u(t) \), \( v(t) \), and \( n(t) \) for sub-Ohmic spectral density with \( \eta = 0.1 \). Figure (a) shows \( \text{Re}[u(t)] \) (blue solid line) and \( \text{Im}[u(t)] \) (dotted red line). Figure (b) shows \( v(t) \) (blue solid line) and \( n(t) \) (dotted red line). Here \( \omega_0 = 21.5 \) GHz, \( \omega_c = \omega_0 \), and \( T = 2 \) K and \( n(t_0) = 50 \).

FIG. 5. (Color online) Time evolution of the average particle number \( n(t) \) for \( \eta = 0.1 \) (dashed black line), \( \eta = 0.3 \) (dotted red line), \( \eta = 0.4 \) (dash-dotted blue line), \( \eta = 0.6 \) (dash-dot-dotted dark yellow line), and \( \eta = 1.0 \) (solid magenta line). Here \( \omega_0 = 21.5 \) GHz, \( \omega_c = \omega_0 \), and \( T = 2 \) K, and \( n(t_0) = 50 \).
In fact, the plots in Fig. 5 show that the time evolution of the average photon number undergoes a critical transition (or no oscillation), it revives till it reaches a steady value. This behavior does not show up in the BM approximation. In the very beginning and then after a short-time oscillation in the very low temperature regime, as we have also discussed in the analytical solution presented numerically demonstrated. Moreover, the numerical results indicate that the initial temperature of the reservoir may serve as a sensitive control parameter to control the coherence photon number in cavity dynamics in the very low temperature regime, as we have also discussed in the analytical solution presented in the last section.

Putting all these analyses together, we find that the non-Markovian memory effect can qualitatively change the dissipation dynamics of the cavity field in the strong coupling regime, in particular for the super-Ohmic reservoir. In the meantime, the non-Markovian memory effects play a significant role in the thermal noise dynamics, in particular for the sub-Ohmic reservoir. These interesting phenomena are worth further investigation in other open systems.

V. CONCLUSION

In summary, we have solved analytically the exact master equation and obtained the general expression for the reduced density operator of the cavity as well as the general solution of the mean model amplitude and the average photon numbers in the cavity. We take three different initial states, the vacuum state, the coherent state, and the mixed state, to show the different non-Markovian time evolution of the cavity state. We find that (i) the solution of the exact master equation is very different from the BM approximation due to the non-Markovian memory effect. In the exact non-Markovian case, different initial states may result in different evolution states and different steady states. While the BM solution at the steady-state limit is the same, independent of the initial states. (ii) For the exact non-Markovian evolution process, temperature effect can play an important role, and it drastically changes the thermal noise dynamics. (iii) The decoherence of a cavity-coherent field arises from both the reservoir-induced damping effect (energy dissipation) and the temperature-dependent noise effect. Both of them can be controlled by varying the coupling between the cavity and the reservoir and lowering the temperature of the reservoir, as shown in our exact solution. These exact analytical cavity dynamics show the extreme importance of the non-Markovian effect and are numerically demonstrated. Moreover, the numerical results show that the non-Markovian memory effect qualitatively changes the amplitude damping behavior of the cavity field as well as the thermal noise dynamics from the weak coupling regime to the strong coupling regime, which does not occur in the BM limit. In particular, we show that the non-Markovian memory effect leads to a long-time dissipationless process to the cavity field in the strong coupling regime, and in the meantime it results in a critical transition dynamic when we vary the cavity system from the weak to strong couplings with the reservoir. Further investigations of all these general non-Markovian phenomena in other open systems are in progress.

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