Sudden vanishing of spin squeezing under decoherence

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In order to witness multipartite correlations beyond pairwise entanglement, spin-squeezing parameters are analytically calculated for a spin ensemble in a collective initial state under three different decoherence channels. It is shown that, in analogy to pairwise entanglement, the spin squeezing described by different parameters can suddenly become zero at different vanishing times. This finding shows the general occurrence of sudden vanishing phenomena of quantum correlations in many-body systems, which here is referred to as spin-squeezing sudden death (SSSD). It is shown that the SSSD usually occurs due to decoherence and that SSSD never occurs for some initial states in the amplitude-damping channel. We also analytically obtain the vanishing times of spin squeezing.

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I. INTRODUCTION

Quantum entanglement [1] plays an important role in both the foundations of quantum physics and quantum-information processing [2]. Moreover, various entangled states have been produced in many experiments for different goals when studying various nonclassical phenomena and their applications [3–11]. Thus, entanglement is a quantum resource, and how to measure and detect entanglement is very crucial for both theoretical investigations and potential practical applications.

For a system of two spin-1/2 particles or a composite system of a spin-1/2 and a spin-1, there are operationally computable entanglement measures such as concurrence [12] and negativity [13,14], but no universal measures have been found for general many-body systems. To overcome this difficulty, entanglement witnesses are presented to detect some kinds of entanglement in many-body systems [14,15]. Now it is believed that spin squeezing [16,17] may be useful for this task [18–20]. In a general sense, spin-squeezing parameters are multipartite entanglement witnesses. For a class of many-particle states, it has been proved that the concurrence is linearly related to some squeezing parameters [21]. In fact, spin-squeezing parameters [16–19] could be calculated also in a simple operational fashion, which characterizes multipartite quantum correlations beyond the pairwise entanglement. Another important reason for choosing spin-squeezing parameters as indicators of multipartite correlations is that spin squeezing is relatively easy to generate [17,22] and measure experimentally [23,24].

Besides being a parameter characterizing multipartite correlations, spin squeezing is physically natural for controlling many-body systems. It is difficult to control a quantum many-body system since its constituents cannot be individually addressed. In this sense, one needs to use collective operations, and spin squeezing is one of the most successful approaches for controlling such systems. For example, creating spin squeezing of an atomic ensemble could result in precision measurements based on many-atom spectroscopy [17]. Therefore, we can also regard spin squeezing as a quantum resource since for more than two particles it behaves as two-particle entanglement in controlling and detecting quantum correlations. On this quantum resource, we need to further consider the effects of decoherence [25,26]. Thus, it is important to study the environment-induced decoherence effects on both spin squeezing and multiparticle entanglement [27–37]. A decaying time evolution of the spin squeezing under decoherence [27,38–40] can be used to analyze whether this quantum resource is robust.

In this article we address this problem by calculating three spin-squeezing parameters for a spin ensemble in a collective excited state. We study the time evolution of spin squeezing under local decoherence, acting independently and equally on each spin. Here, the irreversible processes are modelled as three decoherence channels: the amplitude damping, pure dephasing, and depolarizing channels. We find that, similar to the sudden death of pairwise entanglement [41], spin squeezing can also suddenly vanish with different lifetimes for some decoherence channels, showing in general different vanishing times in multipartite correlations in quantum many-body systems. Thus, similar to the discovery of pairwise entanglement sudden death (ESD) [41], the spin-squeezing sudden death (SSSD) occurs due to decoherence. We will see that for some initial states, the SSSD never occurs under the amplitude-damping channel. We also give analytical expressions for the vanishing time of spin squeezing and pairwise entanglement. The ESD has been tested experimentally [39,42] and we also expect that the SSSD can also be realized experimentally.

This article is organized as follows. In Sec. II, we introduce the initial state from the one-axis twisting Hamiltonian and then, in Sec. III, the decoherence channels. In Sec. IV, we list three parameters of spin squeezing and discuss the relations among them. For a necessary comparison, the concurrence is also calculated. We also study initial-state squeezing. In Sec. V, we study three different types of spin squeezing and concurrence under three different decoherence channels. Both...
analytical and numerical results are given. We conclude in Sec. VI.

II. INITIAL STATE

We consider an ensemble of $N$ spin-1/2 particles with ground state $|1\rangle$ and excited state $|0\rangle$. This system has exchange symmetry, and its dynamical properties can be described by the collective operators

$$J_{\alpha} = \sum_{k=1}^{N} j_{k\alpha} = \frac{1}{2} \sum_{k=1}^{N} \sigma_{k\alpha}$$

(1)

for $\alpha = x, y, z$. Here, $\sigma_{k\alpha}$ are the Pauli matrices for the $k$th qubit. To study the decoherence of spin squeezing, we choose a state which is initially squeezed. One typical class of such spin-squeezed states is the one-axis twisting collective spin state [16],

$$|\Psi(\theta_0)\rangle_0 = e^{-i\theta_0 J_{z}/2}|1\rangle^\otimes N = e^{-i\theta_0 J_{z}/2}|1\rangle$$

(2)

which could be prepared by the one-axis twisting Hamiltonian

$$H = \chi J_{x}^{2}$$

(3)

where

$$\theta_0 = 2\chi t$$

(4)

is the one-axis twist angle and $\chi$ is the coupling constant. For this state, it was proved [21] that the spin squeezing $\xi_{z}^{2}$ [16] and the concurrence $C_{0}$ [12] are equivalent since there exists a linear relation

$$\xi_{z}^{2} = 1 - (N - 1)C_{0}$$

between them. Physically, they occur and disappear simultaneously. The spin squeezing of this state can be generated and stored in, e.g., a two-component Bose-Einstein condensate [43].

A. Initial-state symmetry

The initial state has an obvious symmetry resulting from Eq. (2), the so-called even-parity symmetry, which means that only even excitations of spins occur in the state. Since $J_{x}$, $J_{y}$, and $J_{z}$ can work well by identifying $N/2$ with the highest weight $J_{z}$, which corresponds to the collective ground state

$$|J_{z} = J\rangle = |1\rangle^\otimes N \equiv |1\rangle,$$

(5)

indicating that all spins are in the ground state. The symmetric space is generated by the collective operator

$$J_{z} = \frac{1}{2} \sum_{k=1}^{N} \sigma_{kz}$$

acting on the collective ground state. Here,

$$\sigma_{k\pm} = \frac{i}{2}(\sigma_{kx} \pm i\sigma_{ky}).$$

In other words, the state is in the maximally symmetric space spanned by the Dicke states. So, the $N$ spin-1/2 system behaves like a larger spin-$N/2$ system. It can be proved that any pure state with exchange symmetry belongs to the above-mentioned symmetric space, but for mixed states the state space can be extended to include a space beyond the symmetric one [44]. In the following discussions, we focus on such an extended space.

In fact, after decoherence, not only the symmetric Dicke states are populated, but also states with lower symmetry. So, it is not sufficient to describe the system in only $(N + 1)$-dimensional space. Although the maximal symmetry is broken, the exchange symmetry is not affected by the decoherence as each local decoherence equally acts on each spin. In other words, a state with exchange symmetry does not necessarily belong to the maximally symmetric space.

With only the exchange symmetry, from Eq. (1), the global expectations or correlations of collective operators are obtained as

$$\langle J_{z}^{2} \rangle = \frac{N}{4} + \frac{N(N - 1)}{4} \langle \sigma_{1z} \sigma_{2z} \rangle,$$

(6)

$$\langle J_{x}^{2} \rangle = \frac{N}{2} + \frac{N(N - 1)}{4} \langle \sigma_{1x} \sigma_{2x} \rangle - \frac{N(N - 1)}{4} \langle \sigma_{1y} \sigma_{2y} \rangle,$$

(7)

$$\langle [J_{z}, J_{x}]_{+} \rangle = \frac{N(N - 1)}{4} \langle [\sigma_{1x}, \sigma_{2y} + \sigma_{2x}] \rangle.$$

(8)

Furthermore, it follows from Eq. (6) that

$$\langle J_{z}^{2} + J_{x}^{2} \rangle = \frac{N}{2} + \frac{N(N - 1)}{2} \langle \sigma_{1x} \sigma_{2x} \rangle + \langle \sigma_{1y} \sigma_{2y} \rangle,$$

(9)

$$\langle J_{z}^{2} + J_{x}^{2} + J_{y}^{2} \rangle = \frac{N^{2}}{4} \left[ 3 + \frac{1}{N} \right] \langle \sigma_{1z} \sigma_{2z} \rangle - \frac{1}{N} \langle \sigma_{1y} \sigma_{2y} \rangle.$$

(10)

These equations show the relations between the global and local expectations and correlations, which are useful in the following calculations.

III. DECOHERENCE CHANNELS AND EXAMPLES OF THEIR IMPLEMENTATIONS

Having introduced the initial state, now we discuss three typical decoherence channels: the amplitude-damping channel (ADC), the phase-damping channel (PDC), and the depolarizing channel (DPC).

These channels are prototype models of dissipation relevant in various experimental systems. They provide “a revealing caricature of decoherence in realistic physical situations, with all inessential mathematical details stripped away” [45]. But yet this “caricature of decoherence” leads to theoretical predictions being often in good agreement with experimental data. Examples include multiphoton systems, ion traps, atomic ensembles, or a solid-state spin systems such as quantum dots or NV diamonds, where qubits are encoded in electron or nuclear spins.

Here, we briefly describe only a few of such implementations.

A. Amplitude-damping channel

The ADC is defined as

$$\mathcal{E}_{ADC}(\rho) = E_{0}\rho E_{0}^{\dagger} + E_{1}\rho E_{1}^{\dagger},$$

(11)

where

$$E_{0} = \sqrt{s}|0\rangle\langle 0| + |1\rangle\langle 1|, \quad E_{1} = \sqrt{p}|1\rangle\langle 0|$$

(12)
are the Kraus operators, $p = 1 - s$, $s = \exp(-\gamma t/2)$, and $\gamma$ is the damping rate. In the Bloch representation, the ADC squeezes the Bloch sphere into an ellipsoid and shifts it toward the north pole. The radius in the $xy$ plane is reduced by a factor $\sqrt{s}$, while in the $z$ direction it is reduced by a factor $s$.

The ADC is a prototype model of a dissipative interaction between a qubit and its environment. For example, the ADC model can be applied to describe the spontaneous emission of a photon by a two-level system into an environment of photon or phonon modes at zero (or very low) temperature in (usually) the weak Born-Markov approximation. The ADC can also describe processes contributing to $T_1$-relaxation in spin resonance at zero temperature. Note that by introducing an “upward” decay (i.e., a decay toward the south pole of the Bloch sphere), in addition to the standard “downward” decay, the ADC can be used to describe dissipation into the environment also at finite temperature.

The ADC acting on a system qubit in an unknown state $\rho$ can be implemented in a two-qubit circuit performing a rotation $R_\theta(0)$ of an ancilla qubit (initially in the ground state) controlled by the system qubit and followed by a controlled-NOT (CNOT) gate on the system qubit controlled by the ancilla qubit [2]. The parameter $\theta$ is simply related to the probability $p$ in Eq. (11). The ancilla qubit, which models the environment, is measured after the gate operation.

The ADC-induced sudden vanishing of entanglement was first experimentally demonstrated for polarization-encoded qubits [42]. For this reason let us shortly describe this optical implementation of the ADC. It is based on a Sagnac-type ring interferometer composed of a polarizing beam splitter and a half-wave plate at an angle corresponding to the parameter $p$ in Eq. (11). The beam splitter separates an incident beam (being in a superposition of states with horizontal, $|H\rangle$, and vertical, $|V\rangle$, polarizations) into spatially distinct counterpropagating light beams. The $H$ component leaves the interferometer unchanged. But the $V$ component is rotated in the wave plate, which corresponds to probabilistic damping into the $H$ component. Then, at the exit from the interferometer, this component is probabilistically transmitted or reflected from the beam splitter. So it is cast into two orthogonal spatial modes corresponding the reservoir states with and without excitation.

The action of the ADC can be represented by an interaction Hamiltonian [2]: $H \sim ab^\dagger + a^\dagger b$, where $a$ ($a^\dagger$) and $b$ ($b^\dagger$) are annihilation (creation) operators of the system and environment oscillators, respectively. In more general models of damping, a single oscillator $b$ of the reservoir is replaced by a finite or infinite collection of oscillators $\{b_n\}$ coupled to the system oscillator with different strengths (see, e.g., Refs. [46,47]). For the example of quantum states of motion of ions trapped in a radiofrequency (Paul) trap, the amplitude damping can be modeled by coupling an ion to the motional amplitude reservoir described by the above multiscillator Hamiltonian [47]. The high-temperature reservoir is possible to simulate by applying (on trap electrodes) a random uniform electric field with spectral amplitude at the ion motional frequency [48,49]. The zero-temperature reservoir can be simulated by laser cooling combined with spontaneous Raman scattering [50].

B. Phase-damping channel

The PDC is a prototype model of dephasing or pure decoherence, i.e., loss of coherence of a two-level state without any loss of system’s energy. The PDC is described by the map

$$E_{\text{PDC}}(\rho) = s\rho + p (\rho_{00}|0\rangle\langle0| + \rho_{11}|1\rangle\langle1|),$$

and obviously the three Kraus operators are given by

$$E_0 = \sqrt{s}\mathbb{1}, \quad E_1 = \sqrt{p}|0\rangle\langle0|, \quad E_2 = \sqrt{p}|1\rangle\langle1|,$$

where $\mathbb{1}$ is the identity operator. For the PDC, there is no energy change and a loss of coherence occurs with probability $p$. As a result of the action of the PDC, the Bloch sphere is compressed by a factor $(1 - 2p)$ in the $xy$ plane.

In analogy to the ADC, the PDC can be considered as an interaction between two oscillators (modes) representing system and environment as described by the interaction Hamiltonian: $H \sim a^\dagger a(b^\dagger + b)$ [2]. In more general phase-damping models, a single environmental mode $b$ is usually replaced by an infinite collection of modes $b_n$ coupled, with various strengths, to mode $a$.

It is evident that the action of the PDC is nondissipative. It means that, in the standard computational basis $|0\rangle$ and $|1\rangle$, the diagonal elements of the density matrix $\rho$ remain unchanged, while the off-diagonal elements are suppressed. Moreover, the qubit states $|0\rangle$ and $|1\rangle$ are also unchanged under the action of the PDC, although any superposition of them (i.e., any point in the Bloch sphere, except the poles) becomes entangled with the environment.

The PDC can be interpreted as elastic scattering between a (two-level) system and a reservoir. It is also a model of coupling a system with a noisy environment via a quantum nondemolition (QND) interaction. The spin squeezing of atomic ensembles can be generated via QND measurements [10,24,51–55]. So modeling the spin-squeezing decoherence via the PDC can be relevant in this context. The PDC is also a suitable model to describe $T_2$ relaxation in spin resonance. This in contrast to modeling $T_1$ relaxation via the ADC.

A circuit modeling the PDC can be realized as a simplified version of the circuit for the ADC, discussed in the previous subsection, obtained by removing the CNOT gate [2]. Then, the angle $\theta$ in the controlled rotation gate $R_\theta(0)$ is related to the probability $p$ in Eq. (13).

The sudden vanishing of entanglement under the PDC was first experimentally observed in Ref. [42]. This optical implementation of the PDC was based on the same system as the above-mentioned Sagnac interferometer for the ADC but with an additional half-wave plate at a $\pi/4$ angle in one of the outgoing modes.

Some specific kinds of PDCs can be realized in a more straightforward manner. For example, in experiments with trapped ions, the motional PDC can be implemented just by modulating the trap frequency, which changes the phase of the harmonic motion of ions [48,49] (for a review see Ref. [47] and references therein).
C. Depolarizing channel

The definition of the DPC is given via the map

$$\mathcal{E}_{\text{DPC}}(\rho) = \sum_{i=0}^{3} E_i \rho E_i^\dagger,$$

$$= (1 - p')\rho + \frac{p'}{3}(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z),$$  (15)

where

$$E_0 = \sqrt{1 - p' \mathbb{1}}, \quad E_1 = \sqrt{\frac{p'}{3}} \sigma_x,$$

$$E_2 = \sqrt{\frac{p'}{3}} \sigma_y, \quad E_3 = \frac{\sqrt{p'}}{3} \sigma_z,$$

are the Kraus operators. By using the following identity

$$\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z + \rho = 2 \mathbb{1},$$

we obtain

$$\mathcal{E}_{\text{DPC}}(\rho) = s \rho + p \frac{\mathbb{1}}{2},$$  (17)

where $p = 4p'/3$. We see that for the DPC, the spin is unchanged with probability $s = 1 - p$ or it is depolarized to the maximally mixed state $\mathbb{1}/2$ with probability $p$. It is seen that due to the action of the DPC, the radius of the Bloch sphere is reduced by a factor $s$, but its shape remains unchanged.

Formally, the action of the DPC on a qubit in an unknown state $\rho$ can be implemented in a three-qubit circuit composed of two CNOT gates with two auxiliary qubits initially in mixed states

$$\rho_1 = \frac{\mathbb{1}}{2}, \quad \rho_2 = (1 - p)\ket{00}\bra{00} + p\ket{11}\bra{11},$$  (18)

which model the environment. Qubit $\rho_2$ controls the other qubits via the CNOT gates [2].

The DPC map can also be implemented by applying each of the Pauli operators $[1, \sigma_x, \sigma_y, \sigma_z]$ at random with the same probability. Using this approach, optical DPCs have been realized experimentally both in free space [56] and in fibers [57], where qubits are associated with polarization states of single photons. In Ref. [56], the DPC was implemented by using a pair of equal electro-optical Pockels cells. One of them was performing a $\sigma_x$ gate and the other a $\sigma_y$ gate. The simultaneous action of both $\sigma_x$ and $\sigma_y$ corresponds to a $\sigma_z$ gate. The cells were driven (with a mutual delay of $\tau/2$) by a continuous-wave periodic square-wave electric field with a variable pulse duration $\tau$, so the total depolarizing process lasted $2\tau$ for each period.

Analogous procedures can be implemented in other systems, including collective spin states of atomic ensembles. The coherent manipulation of atomic spin states by applying off-resonantly coherent pulses of light is a basic operation used in many applications [58]. We must admit that the standard methods enable rotations in the Bloch sphere of only classical spin states (i.e., coherent spin states). Nevertheless, recently [24] an experimental method has been developed to rotate also spin-squeezed states.

It is worth noting that in experimental realizations of decoherence channels (e.g., in ion-trap systems [59]), sufficient resources for complete quantum tomography are provided even for imperfect preparation of input states and the imperfect measurements of output states from the channels.

IV. SPIN-SQUEEZING DEFINITIONS AND CONCURRENCE

Now, we discuss several parameters of spin squeezing and give several relations among them. To compare spin squeezing with pairwise entanglement, we also give the definition of concurrence. We notice that most previous investigations on ESD of concurrence were only carried out for two-particle system rather than for two-particle subsystem embedded in a larger system. For the initial states, spin-squeezing parameters and concurrence are also given below.

A. Spin-squeezing parameters and their relations

1. Definitions of spin squeezing

There are several spin-squeezing parameters, but we list only three typical and related ones as follows [16–19]:

$$\xi_1^2 = \frac{4(\Delta J_\parallel)^2_{\text{min}}}{N},$$  (19)

$$\xi_2^2 = \frac{N^2}{4\langle J^2 \rangle} \xi_1^2,$$  (20)

$$\xi_3^2 = \frac{\lambda_{\text{min}}}{\langle J^2 \rangle - \frac{N}{2}}.$$  (21)

Here, the minimization in the first equation is over all directions denoted by $\hat{n}_\perp$, perpendicular to the mean spin direction $\langle \vec{J} \rangle/\langle \vec{J}^2 \rangle$; $\lambda_{\text{min}}$ is the minimum eigenvalue of the matrix [19]

$$\Gamma = (N - 1)\gamma + \mathbf{C},$$  (22)

where

$$\gamma_{kl} = C_{kl} - \langle J_k \rangle \langle J_l \rangle$$

for $k, l \in \{x, y, z\} = \{1, 2, 3\},$  (23)

is the covariance matrix and $\mathbf{C} = [C_{kl}]$ with

$$C_{kl} = \frac{1}{N} \langle J_k J_l + J_l J_k \rangle,$$  (24)

is the global correlation matrix. The parameters $\xi_1^2, \xi_2^2$, and $\xi_3^2$ were defined by Kitagawa and Ueda [16], Wineland et al. [17], and Tóth et al. [19], respectively. If $\xi_3^2 < 1$ ($\xi_3^2 < 1$), spin squeezing occurs, and we can safely say that the multipartite state is entangled [18,19]. Although we cannot say that the squeezed state via the parameter $\xi_3^2$ is entangled, it is indeed closely related to quantum entanglement [21].

2. Squeezing parameters for states with parity

We know from Sec. II A that the initial state has an even parity and that the mean spin direction is along the $z$ direction. During the transmission through all the three decoherence channels discussed here, the mean spin direction does not change. For states with a well-defined parity (even or odd), the spin-squeezing parameter $\xi_1^2$ was found to be [21]

$$\xi_1^2 = \frac{2}{N} \langle (J_1^2 + J_3^2) - \langle J_1^2 \rangle \rangle.$$  (25)
Then, the parameter $\xi_2^2$ given by Eq. (20) becomes
\[ \xi_2^2 = \frac{N^2\xi_1^2}{4(\Delta J_z)^2} = \frac{N((J_x^2 + J_y^2) - |J_z^2|)}{2(\Delta J_z)^2}. \] (26)
For the third squeezing parameter (see Appendix A for the derivation), we have
\[ \xi_3^2 = \frac{\min\{\xi_1^2, \xi_2^2\}}{4N^{-2}(\Delta J_z)^2 - 2N^{-1}}, \] (27)
where
\[ \xi^2 = \frac{4}{N^2}(N(\Delta J_z)^2 + (\Delta J_z)^2). \] (28)

Note that the first parameter $\xi_1^2$ becomes a key ingredient for the latter two squeezing parameters ($\xi_2^2$ and $\xi_3^2$).

3. Spin-squeezing parameters in terms of local expectations

For later applications, we now express the squeezing parameters in terms of local expectations and correlations, and also examine the meaning of $\zeta^2$, which will be clear by substituting Eqs. (1) and (6) into Eq. (28),
\[ \zeta^2 = 1 + C_{zz} = 1 + (N - 1)(\sigma_{1z}\sigma_{2z} - \langle \sigma_{1z}\rangle\langle \sigma_{2z}\rangle). \] (29)
Thus, the parameter $\zeta^2$ is simply related to the correlation $C_{zz}$ along the $z$ direction. A negative correlation $C_{zz} < 0$ is equivalent to $\zeta^2 < 1$. It is already known that the spin squeezing parameter $\xi_1^2$ can be written as [60]
\[ \xi_1^2 = 1 + (N - 1)C_{\bar{a}_1,\bar{a}_z}, \] (30)
where $C_{\bar{a}_1,\bar{a}_z}$ is the correlation function in the direction perpendicular to the mean spin direction. So, the spin squeezing $\xi_1^2 < 1$ is equivalent to the negative pairwise correlations $C_{\bar{a}_1,\bar{a}_z} < 0$ [60].

Thus, from the above analysis, spin squeezing and negative correlations are closely connected to each other. The parameter $\zeta^2 < 1$ indicates that spin squeezing occurs along the $z$ direction, and $\xi_1^2 < 1$ implies spin squeezing along the direction perpendicular to the mean spin direction. Furthermore, from Eq. (27), a competition between the transverse and longitudinal correlations is evident.

By substituting Eqs. (7) and (9) to Eq. (25), one can obtain the expression of $\xi_3^2$ in terms of local correlations $\langle \sigma_{1z}\rangle$ and $\langle \sigma_{2z}\rangle$ as follows:
\[ \xi_3^2 = 1 + (N - 1)(\sigma_{1z}\sigma_{2z} - \sigma_{1z}\sigma_{2z}) - 2(N - 1)\langle \sigma_{1z}\sigma_{2z}\rangle = 1 + 2(N - 1)(\sigma_{1z}\sigma_{2z} - \langle \sigma_{1z}\sigma_{2z}\rangle). \] (31)
The second equality in Eq. (31) results from the exchange symmetry. From Eqs. (1), (10), and (29), one finds
\[ \xi_3^2 = \frac{\xi_1^2}{\sigma_{1z}\sigma_{2z}}, \] (32)
\[ \xi_3^2 = \frac{\min\{\xi_1^2, 1 + C_{zz}\}}{(1 - N^{-1})(\sigma_{1z}\sigma_{2z}) + N^{-1}}. \] (33)
Thus, we have reexpressed the squeezing parameters in terms of local correlations and expectations.

4. New spin-squeezing parameters

In order to characterize spin squeezing more conveniently, we define the following squeezing parameters:
\[ \zeta_1^2 = \max(0, 1 - \xi_1^2), \quad k \in \{1, 2, 3\}. \] (34)
This definition is similar to the expression of the concurrence given below. Spin squeezing appears when $\zeta_1^2 > 0$, and there is no squeezing when $\zeta_1^2$ vanishes. Thus, the definition of the first parameter $\zeta_1^2$ has a clear meaning, namely, it is the strength of the negative correlations as seen from Eq. (30). The larger is $\zeta_1^2$, the larger is the strength of the negative correlation, and the larger is of the squeezing. More explicitly, for the initial state, we have $\xi_1^2 = 1 - (N - 1)C_0$ [21], so $\zeta_1^2$ is just the rescaled concurrence $\zeta_1^2 = C_0(0) = (N - 1)C_0$ [61].

Here, we give a few comments on the spin-squeezing parameter $\zeta_2^2$, which represents a competition between $\xi_1^2$ and $\langle \sigma_{1z}\rangle^2$: the state is squeezed according to the definition of $\zeta_2^2$ if $\xi_1^2 < \langle \sigma_{1z}\rangle^2$. We further note that [62]
\[ \langle \sigma_{1z}\rangle^2 = 1 - 2E_L, \] (35)
where $E_L$ is the linear entropy of one spin and it can be used to quantify the entanglement of pure states [14]. So, there is a competition between the strength of negative correlations and the linear entropy $2E_L$ in the parameter $\zeta_2^2$, and $\zeta_1^2 > 2E_L$ implies the appearance of squeezing.

B. Concurrence for pairwise entanglement

It has been found that the concurrence is closely related to spin squeezing [21]. Here, we consider its behavior under various decoherence channels. The concurrence quantifying the entanglement of a pair of spin-1/2 can be calculated from the reduced density matrix. It is defined as [12]
\[ C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \] (36)
where the quantities $\lambda$ are the square roots of the eigenvalues in descending order of the matrix product
\[ \rho_{12} = \rho_{12}(\sigma_{1y} \otimes \sigma_{2y})\rho_{12}^*(\sigma_{1y} \otimes \sigma_{2y}). \] (37)
In (37), $\rho_{12}^*$ denotes the complex conjugate of $\rho_{12}$.

The two-spin reduced density matrix for a parity state with the symmetry exchange can be written in a block-diagonal form [63]
\[ \rho_{12} = \begin{pmatrix} v_+ & u^* \\ u & v_- \end{pmatrix} \otimes \begin{pmatrix} w & y \\ y & w \end{pmatrix}, \] (38)
in the basis \{00\}, \{11\}, \{01\}, \{10\}, where
\[ v_\pm = \frac{1}{2}(1 \pm 2\langle \sigma_{1z}\rangle + \langle \sigma_{1z}\sigma_{2z}\rangle), \] (39)
\[ w = \frac{1}{4}((1 - \langle \sigma_{1z}\sigma_{2z}\rangle), \] (40)
\[ u = \langle \sigma_{1z}\sigma_{2z}\rangle, \] (41)
\[ y = \langle \sigma_{1z}\sigma_{2z}\rangle. \] (42)
The concurrence is then given by [64]
\[ C = \max(0, 2(|u| - w), 2(y - \sqrt{v_+ v_-})). \] (43)
From the above expressions of the spin-squeezing parameters and concurrence, we notice that if we know the expectation $\langle \sigma_{1z}\rangle$, and the correlations $\langle \sigma_{1z}\sigma_{2z}\rangle$, $\langle \sigma_{1z}\sigma_{2z}\rangle$, and $\langle \sigma_{1z}\sigma_{2z}\rangle$,
all the squeezing parameters and concurrence can be determined. Below, we will give explicit analytical expressions for them subject to three decoherence channels.

C. Initial-state squeezing and concurrence

We will now investigate initial spin squeezing and pairwise entanglement by using our results for the spin-squeezing parameters and concurrence obtained in the last subsections. We find that the third squeezing parameter $\xi_3^2$ is equal to the first one $\xi_1^2$. The squeezing parameter $\xi_1^2$ is given by (see Appendix B):

$$\xi_1^2(0) = 1 - C_r(0) = 1 - (N - 1)C_0,$$

$$= 1 - 2(N - 1)(|u_0| - y_0),$$

(44)

where

$$C_0 = \frac{1}{4}[(1 - \cos N^2 \theta_0)^2 + 16 \sin^2 (\theta_0/2) \cos^{2N-4} (\theta_0/2)]^\frac{1}{2} - 1 + \cos N^2 \theta_0$$

(45)

is the concurrence [21].

The parameter $\xi_3^2(0)$ is easily obtained, as we know both $\xi_1(0)$ and $|\langle \sigma_1 \rangle_0^2|$. For this state, following from Eq. (10), $|\langle \sigma_1 \rangle_0| = 1$, and thus the third parameter given by Eq. (33) becomes

$$\xi_3^2(0) = \min \{\xi_2^2(0), \xi_2^2(0)\} = \min \{1 - C_r(0), 1 + C_{zz}(0)\},$$

(46)

where the correlation function is

$$C_{zz}(0) = \frac{1}{2}(1 + \cos N^2 \theta_0 - \cos^{2N-2} (\theta_0/2)) \geq 0.$$  

(47)

The proof of the above inequality is given in Appendix C.

As the correlation function $C_{zz}(0)$ and the concurrence $C_r(0)$ are always $\geq 0$, Eq. (46) reduces to

$$\xi_3^2(0) = \xi_1^2(0) = 1 - C_r(0).$$

(48)

So, for the initial state, the spin-squeezing parameters $\xi_1^2(0)$ and $\xi_3^2(0)$ are equal or equivalently, we can write $\xi_1^2(0) = C_r(0)$ according to the definition of parameter $\xi_3^2$ given by Eq. (34). Below we made a summary of results of this section in Table I.

### Table I. Spin-squeezing parameters $\xi_1^2$ [16], $\xi_2^2$ [17], $\xi_3^2$ [19] and concurrence $C$ [12] for arbitrary states (first two columns), states with parity (third column). The squeezing parameters are also expressed in terms of local expectations (fourth column) and in terms of the initial rescaled concurrence $C_r(0)$ for initial states (last column). Also, $C_0$ is the initial concurrence, and other parameters are defined in the text.

<table>
<thead>
<tr>
<th>Squeezing parameters</th>
<th>Definitions</th>
<th>States with parity</th>
<th>In terms of local expectations</th>
<th>Initial state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1^2$</td>
<td>$\frac{4(\Delta J_{ij})_{\min}}{N}$</td>
<td>$\frac{2}{N} &lt; (\langle J_i^2 + J_j^2 \rangle - \langle J_j^2 \rangle)$</td>
<td>$1 + 2(N - 1)(y -</td>
<td>u</td>
</tr>
<tr>
<td>$\xi_2^2$</td>
<td>$\frac{N^2}{4\langle J \rangle^2} \xi_1^2$</td>
<td>$\frac{N^2 \xi_1^2}{4\langle J \rangle^2}$</td>
<td>$\frac{\xi_1^2}{\langle \sigma_1 \rangle^2}$</td>
<td>$1 - C_r(0)$</td>
</tr>
<tr>
<td>$\xi_3^2$</td>
<td>$\frac{\lambda_{\min}}{(\langle J \rangle - \frac{N}{2})^2}$</td>
<td>$\min {\xi_1^2, \xi_2^2}$</td>
<td>$\frac{\min {\xi_1^2, 1 + C_{zz}}}{(1 - N^{-1}) \langle \sigma_1 \rangle \langle \sigma_2 \rangle + N^{-1}}$</td>
<td>$1 - C_r(0)$</td>
</tr>
<tr>
<td>Concurrence $C$</td>
<td>$\max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$</td>
<td>$2\max(0,</td>
<td>u</td>
<td>- w, y - \sqrt{v_1 v_2})$</td>
</tr>
</tbody>
</table>

V. SPIN SQUEEZING UNDER DECOHERENCE

Now we begin to study spin squeezing under three different decoherence channels. From the previous analysis, all the spin-squeezing parameters and the concurrence are determined by some correlation functions and expectations. So, if we know the evolution of them under decoherence, the evolution of any squeezing parameters and pairwise entanglement can be calculated.

A. Heisenberg approach

We now use the Heisenberg picture to calculate the correlation functions and the relevant expectations. A decoherence channel with Kraus operators $K_\mu$ is defined via the map

$$E(\rho) = \sum_\mu K_\mu \rho K_\mu.$$  

(49)

Then, an expectation value of the operator $A$ can be calculated as $\langle A \rangle = \text{Tr}[AE(\rho)]$. Alternatively, we can define the following map,

$$E^\dagger(\rho) = \sum_\mu K_\mu^\dagger \rho K_\mu.$$  

(50)

It is easy to check that

$$\langle A \rangle = \text{Tr}[AE(\rho)] = \text{Tr}[E^\dagger(A)\rho].$$  

(51)

So, one can calculate the expectation value via the above equation (51). This is very similar to the standard Heisenberg picture.

B. Amplitude-damping channel

1. Squeezing parameters

Based on the above approach and the Kraus operators for the ADC given by Eq. (12), we now find the evolutions of the following expectations under decoherence (see Appendix D for details)

$$\langle \sigma_{1z} \rangle = s_{z} \langle \sigma_{1z} \rangle_0 - p,$$

(52a)

$$\langle \sigma_{1z} - \sigma_{2z} \rangle = s_{z} \langle \sigma_{1z} - \sigma_{2z} \rangle_0,$$  

(52b)

$$\langle \sigma_{1z} + \sigma_{2z} \rangle = s_{z} \langle \sigma_{1z} + \sigma_{2z} \rangle_0,$$  

(52c)

$$\langle \sigma_{1z} \sigma_{2z} \rangle = s_{z}^2 \langle \sigma_{1z} \sigma_{2z} \rangle_0 - 2sp \langle \sigma_{1z} \rangle_0 + p^2.$$  

(52d)
To determine the squeezing parameters and the concurrence, it is convenient to know the correlation function $C_{zz}$ and the expectation $\langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle$, which can be determined from the above expectations as follows:

$$\langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle = 1 - s p x_0, \quad (53)$$

$$C_{zz} = s^2 \left( \langle \sigma_{1z} \sigma_{2z} \rangle - \langle \sigma_{1z} \rangle \langle \sigma_{2z} \rangle \right) = s^2 C_{zz}(0), \quad (54)$$

where

$$x_0 = 1 + 2 \langle \sigma_z \rangle - \langle \sigma_{1z} \sigma_{2z} \rangle. \quad (55)$$

Substituting the relevant expectation values and the correlation function into Eqs. (31), (32), and (33) leads to the explicit expression of the spin-squeezing parameters

$$\xi_1^2 = 1 - s C_{r}(0), \quad (56)$$

$$\xi_2^2 = \frac{\xi_1^2}{s^{(\sigma_{1z})} - p \tau^2}, \quad (57)$$

$$\xi_3^2 = \frac{\min \{ \xi_1^2, 1 + s^2 C_{zz}(0) \}}{1 + (N-1) s p x_0}. \quad (58)$$

As the correlation function $C_{zz}(0) > 0$, given by Eq. (47), the third parameter can be simplified as

$$\xi_3^2 = \frac{1 - s C_{r}(0)}{1 + (N-1) s p x_0}. \quad (59)$$

Initially, the state is spin squeezed, i.e., $\xi_1^2(0) < 1$ or $C_r(0) > 0$. From Eq. (56), one can find that $\xi_1^2 < 1$, except in the asymptotic limit of $p = 1$. As we will see below, for the PDC and DPC,

$$\xi_1^2 = 1 - s^2 C_{r}(0).$$

Thus, we conclude that according to $\xi_1^2$, the initially spin-squeezed state is always squeezed for $p \neq 1$, irrespective of both the decoherence strength and decoherence models. In other words, there exists no SSSD if we quantify spin squeezing by the first parameter $\xi_1^2$.  

### 2. Concurrence

In the expression (43) of the concurrence, there are three terms inside the max function. The expression can be simplified to (see Appendix E for details):

$$C_r = 2 (N - 1) \max(0, |u| - w). \quad (60)$$

By using Eqs. (40) and (52c), one finds

$$2(|u| - w) = 2 s |u_0| + s \left[ s - 2 + s \langle \sigma_{1z} \sigma_{2z} \rangle_0 - 2 p \langle \sigma_{1z} \rangle_0 \right] \quad (61)$$

$$= s C_0 - \frac{s p x_0}{2}. \quad (62)$$

So, we obtain the evolution of the rescaled concurrence as

$$C_r = \max(0, s C_r(0) - 2 (N - 1) s p x_0), \quad (63)$$

which depends on the initial concurrence, expectation $\langle \sigma_{1z} \rangle_0$, and correlation $\langle \sigma_{1z} \sigma_{2z} \rangle_0$.

### 3. Numerical results

The numerical results for the squeezing parameters and concurrence are shown in Fig. 1 for different initial values of the twist angle $\theta_0$, defined in Eq. (4). For the smaller value of $\theta_0$, e.g., $\theta_0 = \pi/10$, we see that there is no ESD and SSSD. All the spin squeezing and the pairwise entanglement are completely robust against decoherence. Intuitively, the larger is the squeezing, the larger is the vanishing time. However, here, in contrast to this, no matter how small are the squeezing parameters and concurrence, they vanish only in the asymptotic limit. This results from the complex correlations in the initial state and the special characteristics of the ADC.

For larger values of $\theta_0$, as the decoherence strength $p$ increases, the spin squeezing decreases until it suddenly vanishes, so the phenomenon of SSSD occurs. There exists a critical value $p_c$, after which there is no spin squeezing. The vanishing time of $\xi_1^2$ is always larger than those of $\xi_2^2$ and the concurrence. We note that depending on the initial state, the concurrence can vanish before or after $\xi_2^2$. This means that in our model, the parameter $\xi_2^2 < 1$ implies the existence of pairwise entanglement, while $\xi_2^2$ does not.

### 4. Decoherence strength $p_c$ corresponding to the SSSD

From Eqs. (57), (58), and (63), the critical value $p_c$ can be analytically obtained as

$$p_c^{(k)} = \frac{x_k C_r(0)}{(N - 1) x_0}, \quad (k = 1, 3) \quad (64)$$

$$p_c^{(2)} = \frac{\langle \sigma_{1z} \rangle_0^2 + C_r(0) - 1}{1 + 2 \langle \sigma_{1z} \rangle_0 + \langle \sigma_z \rangle_0^2}. \quad (65)$$

FIG. 1. (Color online) Spin-squeezing parameters $\xi_2^2$ (red curve with squares), $\xi_1^2$ (top green curve with circles), and the concurrence $C_r$ (blue solid curve) versus the decoherence strength $p = 1 - \exp(-\gamma t)$ for the amplitude-damping channel, where $\gamma$ is the damping rate. Here, $\theta_0$ is the initial twist angle given by Eq. (4). In all figures, we consider an ensemble of $N = 12$ spins. Note that for a small initial twist angle $\theta_0$ (e.g., $\theta_0 = 0.1 \pi$), the two squeezing parameters and the concurrence all concur. For larger values of $\theta_0$, the parameters $\xi_2^2$, $\xi_1^2$, and $C$ become quite different and all vanish for sufficiently large values of the decoherence strength.
versus the initial twist angle with circles), and the squeezing parameter $\zeta$ (black dashed curve) versus the initial twist angle $\theta_0$ given by Eq. (4) for the amplitude-damping channel, ADC. Here, $p_c$ is related to the vanishing time $t_v$ via $p_c = 1 - \exp(-\gamma t_v)$. At vanishing times, SSSD occurs. The critical values $p_c^{(1)}$, $p_c^{(2)}$, and $p_c^{(3)}$ correspond to the concurrence, squeezing parameter $\xi_1^2$, and $\xi_2^2$, respectively.

where $x_1 = 2$ for the concurrence and $x_3 = N$ for the squeezing parameter $\xi_2^2$. The critical value $p_c^{(2)}$ is for the second squeezing parameter $\xi_2^2$. Here, $p_c$ is related to the vanishing time $t_v$ via $p_c = 1 - \exp(-\gamma t_v)$.

In Fig. 2, we plot the critical values $p_c$ of the decoherence strength versus $\theta_0$. The initial-state squeezing parameter $\xi_1^2$ is also plotted for comparison. For a range of small values of $\theta_0$, the entanglement and squeezing are robust to decoherence. The concurrence and parameter $\xi_2^2$ intersect. However, we do not see the intersections between $\xi_3^2$ and $\xi_2^2$ or between $\xi_3^2$ and the concurrence. We also see that for the same degree of squeezing, the vanishing times are quite different, which implies that except for the spin-squeezing correlations, other type of correlations exist. For large enough initial twist angles $\pi < \theta_0 < 2\pi$, the behavior of the squeezing parameter $\xi_1^2$ is similar to those corresponding to $p_c^{(1)}$ and $p_c^{(3)}$.

C. Phase-damping channel

1. Squeezing parameters and concurrence

Now, we study the spin squeezing and pairwise entanglement under the PDC. For this channel, the expectation values $\langle \sigma_i^{(0)} \rangle$ are unchanged and the two correlations $\langle \sigma_1-\sigma_2- \rangle$ and $\langle \sigma_1+\sigma_2- \rangle$ evolve as (see Appendix D for details)

$$\langle \sigma_1-\sigma_2- \rangle = s^2(\sigma_1-\sigma_2-),$$

$$\langle \sigma_1+\sigma_2- \rangle = s^2(\sigma_1+\sigma_2-).$$

(66)

From the above equations and the fact $\langle \sigma_1 \sigma_2 \rangle_0 = 1$, one finds

$$\langle \sigma_1 \sigma_2 \rangle = s^2(\sigma_1 \sigma_2 + \sigma_1 \sigma_2)_0 + \langle \sigma_1 \sigma_2 \rangle_0$$

$$= s^2(1 - \langle \sigma_1 \sigma_2 \rangle_0) + \langle \sigma_1 \sigma_2 \rangle_0,$$

(67)

$$C_{zz}(p) = C_{zz}(0).$$

(68)

Therefore, from the above properties, we obtain the evolution of the squeezing parameters,

$$\xi_1^2 = 1 - s^2 C_r(0),$$

$$\xi_2^2 = \frac{\xi_1^2}{(\sigma_1-\sigma_2-)^2},$$

(69)

(70)

and the third parameter becomes

$$\xi_3^2 = \frac{N \min \left[ \xi_1^2, 1 + C_{zz}(0) \right]}{(N-1)[s^2 + (1-s^2)(\sigma_1-\sigma_2-)_0] + 1}$$

$$= N \frac{\xi_1^2}{(N-1)[s^2 + (1-s^2)(\sigma_1-\sigma_2-)_0] + 1}.$$  

(71)

(72)

where we have used Eqs. (67) and (68), and the property $C_{zz}(0) > 0$.

From Eq. (66) and the simplified form of the concurrence given by Eq. (60), the concurrence is found to be

$$C_r = \max \left[ 0, 2(N-1)[s^2|a_0| - 4^{-1}(1 - \langle \sigma_1 \sigma_2 \rangle_0)]] \right]$$

$$= \max \left[ 0, s^2 C_r(0) + \frac{a_0(s^2 - 1)}{2} \right].$$

(73)

where

$$a_0 = (N-1)(1 - \langle \sigma_1 \sigma_2 \rangle_0).$$

(74)

Thus, we obtained all time evolutions of the spin-squeezing parameters and the concurrence. To study the phenomenon of SSSD, we below examine the vanishing times.

2. Decoherence strength $p_c$ corresponding to the SSSD

The critical decoherence strengths $p_c$ can be obtained from Eqs. (70), (71), and (73) as follows:

$$p_c^{(k)} = 1 - \left[ \frac{a_0}{\chi_k C_r(0) + a_0} \right]^\frac{1}{k},$$

$$p_c^{(2)} = 1 - \left[ \frac{1 - \langle \sigma_1 \rangle_0^2}{C_r(0)} \right]^\frac{1}{3},$$

(75)

(76)

where $k = 1, 3$ and $x_1 = 2, x_3 = N$.

In Fig. 3, we plot the decoherence strength $p_c$ versus the twist angle $\theta_0$ of the initial state for the PDC. For this decoherence channel, the critical value $p_c$ first decrease until they reach zero. Also, it is symmetric with respect to $\theta_0 = \pi$, which is in contrast to the ADC. There are also intersections between the concurrence and parameter $\xi_2^2$, and the critical value $p_c^{(3)}$ is always larger than $p_c^{(1)}$ and $p_c^{(3)}$.

D. Depolarizing channel

1. Squeezing parameters and concurrence

The decoherence of the squeezing parameter defined by Sørensen et al. [18] has been studied in Ref. [27] for the DPC. It is intimately related to the second squeezing parameter $\xi_2^2$. For the DPC, the evolution of correlations $\langle \sigma_1-\sigma_2- \rangle$ and $\langle \sigma_1+\sigma_2- \rangle$ are the same as those of the DPC given by Eq. (66), and the expectations $\langle \sigma_1 \rangle$ and $\langle \sigma_1 \sigma_2 \rangle$ change as (see Appendix D).

$$\langle \sigma_1 \rangle = s \langle \sigma_1 \rangle_0,$$

$$\langle \sigma_1 \sigma_2 \rangle = s^2 \langle \sigma_1 \rangle_0 \langle \sigma_2 \rangle_0.$$

(77)

(78)
The squeezing parameter \( \xi^2 \) is given by Eq. (69), and the other two squeezing parameters are obtained as

\[
\xi^2 = \frac{\xi^2}{s^2 (\sigma_{1z})^2}, \\
\xi^3 = \frac{N \min \{\xi^2, 1 + s^2 C_{zz}(0)\}}{(N-1)s^2 + 1}.
\]

By making use of Eqs. (66) and (78) and starting from the simplified form of the concurrence (60), we obtain

\[
C_r = \max \left\{ 0, 2(N-1) \left[ s^2 |\alpha_0| \left[ 1 - s^2 (\sigma_{1z})_0 \right] \right] \right\} = \max [0, s^2 C_r(0) + 2^{-1} (N-1) (s^2 - 1)].
\]

We observe that the concurrence is dependent only on the initial value itself, not other ones.

2. Decoherence strength \( p_c \) corresponding to the SSSD

From Eqs. (83), (81), and (82), the vanishing times are analytically calculated as

\[
p_c^{(1)} = 1 - \left[ \frac{N - 1}{x_1 C_r(0) + N - 1} \right]^{\frac{1}{2}},
\]

\[
p_c^{(2)} = 1 - \left[ \frac{1}{C_r(0) + (\sigma_{1z})^2} \right]^{\frac{1}{2}},
\]

where \( k = 1, 3 \) and \( x_1, x_3 = N \).

### Table II: Analytical results for the time-evolutions of all relevant expectations, correlations, spin-squeezing parameters, and concurrence, as well as the critical values \( p_c \) of the decoherence strength \( p \). This is done for the three decoherence channels considered in this work. For the concurrence \( C_r \), we give the expression for \( C_{r_c} \), which is related to the rescaled concurrence \( C_r \) via \( C_r = \max(0, C_{r_c}) \).

<table>
<thead>
<tr>
<th>( \langle \sigma_{1} \rangle )</th>
<th>( \langle \sigma_{1z} \rangle )</th>
<th>( \langle \sigma_{1z} \rangle )</th>
<th>( \langle \sigma_{1z} \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude-damping channel ( (\text{ADC}) )</td>
<td>Phase-damping channel ( (\text{PDC}) )</td>
<td>Depolarizing channel ( (\text{DPC}) )</td>
<td></td>
</tr>
<tr>
<td>( \langle \sigma_{1} \rangle )</td>
<td>( \sigma_{1z} \rangle )</td>
<td>( \sigma_{1z} \rangle )</td>
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<td>( \sigma_{1z} \rangle )</td>
</tr>
</tbody>
</table>

The table above shows the analytical results for the time-evolutions of various expectations, correlations, spin-squeezing parameters, and concurrence for the three decoherence channels considered in this work. The critical value \( p_c \) for decoherence is also provided for each channel.
Our investigations show the widespread occurrence of sudden death phenomena in many-body quantum correlations. Since there exists different vanishing times for different squeezing parameters, spin squeezing offers a possible way to detect the total spin correlation and their quantum fluctuations with distinguishable time scales. The discovery of different lifetimes for various spin-squeezing parameters means that, in some time region, there still exists another quantum correlation when other quantum correlations suddenly vanish. However, to determine which kind of correlations will vanish, one possible approach is to further invoke irreducible multiparty correlations \cite{66}, where the multipartite correlations are classified in a series of irreducible $k$-party ones. If we could obtain the time evolution behaviors of such irreducible multiparty correlations in various decoherence channels, we could classify lifetimes for the spin-squeezing sudden death of various multipartite correlations order by order.

\section*{VI. CONCLUSIONS AND REMARKS}

To summarize, for a spin ensemble in a typical spin-squeezing initial state under three different decoherence channels, we have studied spin squeezing with three different parameters in comparison with the pairwise entanglement quantified by the concurrence. When the subsystems of the correlated system decay asymptotically in time, the spin-squeezing parameter $\xi_2^2$ also decays asymptotically in time for all three types of decoherence. However, for the other two squeezing parameters $\xi_2^0$ and $\xi_3^0$, we find the appearance of spin-squeezing sudden death and entanglement sudden death. The global behaviors of the correlated state are markedly different from the local ones. The spin-squeezing parameter $\xi_2^0$ can vanish before, simultaneously, or after the concurrence, while the squeezing parameter $\xi_3^0$ is always the last to vanish. This means that this parameter is more robust to decoherence, and it can detect more entanglement than $\xi_2^2$.

Our analytical approach for the vanishing times can be applied to any initial quantum correlated states, not restricted to the present one-axis twisted state. Moreover, for more complicated channels, such as the amplitude-damping channel at finite temperatures \cite{31} or the channel discussed in Ref. \cite{65}, the method developed in this article can be readily applied to study spin squeezing under these decoherence channels.

\section*{ACKNOWLEDGMENTS}

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\section*{APPENDIX A: SPIN-SQUEEZING PARAMETER $\xi_2^2$ FOR STATES WITH PARITY SYMMETRY}

Here, we calculate the spin-squeezing parameter $\xi_2^2$ for collective states with either even or odd parity symmetry. For such states, we immediately have
\begin{equation}
\langle J_z \rangle = \langle J_y \rangle = \langle J_x J_z \rangle = \langle J_y J_z \rangle = 0
\end{equation}
as the operators change the parity of the state. Then, the mean spin direction is along the $z$ direction and the correlation matrix given by Eq. (24) is simplified to
\begin{equation}
C = \begin{pmatrix}
\langle J_x^2 \rangle & C_{xy} & 0 \\
C_{xy} & \langle J_y^2 \rangle & 0 \\
0 & 0 & \langle J_z^2 \rangle
\end{pmatrix},
\end{equation}
where $C_{xy} = \langle [J_x, J_y]_z \rangle/2$. From the correlation matrix $C$ and the definition of covariance matrix $\gamma$ given by Eq. (23), one
finds

\[
\Gamma = \begin{pmatrix}
N\langle J_1^2 \rangle & NC_{xy} & 0 \\
NC_{xy} & N\langle J_2^2 \rangle & 0 \\
0 & 0 & N(\Delta J_+^2 + \langle J_+^2 \rangle)
\end{pmatrix}.
\] (A3)

This matrix has a block-diagonal form and the eigenvalues of the 2 × 2 block are obtained as

\[
\lambda_{\pm} = \frac{N}{2} \left( (\langle J_1^2 \rangle + \langle J_2^2 \rangle) \pm |\langle J_+^2 \rangle| \right).
\] (A4)

In deriving the above equation, we have used the relation

\[
\langle J_+^2 \rangle = \langle J_1^2 \rangle - \langle J_2^2 \rangle - i\langle J_+, J_+ \rangle.
\] (A5)

Therefore, the smallest eigenvalue \(\lambda_{\text{min}}\) of \(\Gamma\) is obtained as

\[
\lambda_{\text{min}} = \min (\lambda_-, N(\Delta J_+^2 + \langle J_+^2 \rangle)),
\] (A6)

where \(\lambda_-\) differs from the squeezing parameter \(\xi_1^2\) given by Eq. (25) by only a multiplicative constant, as seen by comparing Eqs. (25) and (A6). From Eqs. (A6) and (21), one finds that the squeezing parameter \(\xi_1^2\) is given by Eq. (27).

**APPENDIX B: SPIN-SQUEEZING PARAMETERS FOR THE ONE-AXIS TWISTED STATE**

Here, we will use the Heisenberg picture to derive the relevant expectations and spin-squeezing parameters for the initial state [67,68]. To determine the spin-squeezing parameter \(\xi_1^2\) given by Eq. (31), one needs to know the expectation \(\langle \sigma_{1z} \rangle_0\), and correlations \(\langle \sigma_{1+}\sigma_{2-} \rangle_0\) and \(\langle \sigma_{1-}\sigma_{2+} \rangle_0\). We first consider the expectation \(\langle \sigma_{1z} \rangle_0\). For simplicity, we omit the subscript 0 in the following formulas.

1. **Expectation \(\langle \sigma_{1z} \rangle\)**

The evolution operator can be written as,

\[
U = \exp \left( -i x t J_+^2 \right) = \exp \left( -i \theta \sum_{k \neq l} j_{kx} j_{lx} \right)
\] (B1)

up to a trivial phase, where \(\theta = 2 \pi t\) given by Eq. (4). From this form, the evolution of \(j_{1z}\) can be obtained as

\[
U^i j_{1z} U = j_{1z} \cos \left( \theta j_{1z}^{(2)} \right) + j_{iy} \sin \left( \theta j_{1y}^{(2)} \right),
\] (B2)

where

\[
j_{ik}^{(k)} = \sum_{l=1}^{N} j_{il}.
\] (B3)

Therefore, the expectations are

\[
\langle j_{1z} \rangle = -2^{-1} \langle \hat{1}' \rangle \cos \left( \theta j_{1z}^{(2)} \right) \langle \hat{1}' \rangle
\] (B4)

since \(\langle \hat{1}' \rangle j_{1z} |\hat{1}'\rangle = 0\). Here, \(\langle \hat{1}\rangle = |1\rangle_2 \otimes \cdots \otimes |1\rangle_N\). So, one can find the following form for the expectation values

\[
\langle \hat{1} \rangle \cos[\theta J_{1z}] |\hat{1}\rangle = \left( \langle \hat{1} \rangle \langle \hat{1} \rangle \right) / 2
\]

\[
= \left( \prod_{i=1}^{N} \langle 1 \rangle \langle 1 \rangle \langle 0 \rangle \langle 1 \rangle + \text{c.c.} / 2
\]

\[
= \cos^N \langle \theta' \rangle,
\] (B5)

where \(\theta' = \theta/2\) and \(\langle \hat{1} \rangle = |1\rangle \otimes |N\rangle\).

By using Eqs. (B4) and (B5), one gets

\[
\langle \sigma_{1z} \rangle = -\cos^{N-1} \langle \theta' \rangle.
\] (B6)

2. **Correlation \(\langle \sigma_{1+}\sigma_{2-} \rangle\)**

Since the operator \(\sigma_{1+}\sigma_{2-}\) commutes with the unitary operator \(U\), we easily obtain

\[
\langle \sigma_{1+}\sigma_{2-} \rangle = 0.
\] (B7)

We now compute the correlations \(\langle \sigma_{1+}\sigma_{2+} \rangle\) and \(\langle \sigma_{1-}\sigma_{2+} \rangle\). From the unitary operator,

\[
U^1 j_{1z} j_{2z} U
\]

\[
= \left[ j_{1z} \cos \left( \theta j_{1z}^{(2)} \right) + j_{iy} \sin \left( \theta j_{1y}^{(2)} \right) \right]
\]

\[
- j_{1z} j_{2z} \sin \theta \sin \left( \theta j_{1y}^{(2)} \right),
\]

\[
+ j_{1y} \sin \left( \theta j_{2y}^{(2)} \right) \cos \left( \theta j_{1y}^{(2)} \right) + j_{2y} \cos \left( \theta j_{1y}^{(2)} \right) \sin \left( \theta j_{2y}^{(2)} \right)
\]

\[
+ j_{1y} \sin \left( \theta j_{2y}^{(2)} \right) \cos \left( \theta j_{2y}^{(2)} \right) - j_{2z} \sin \left( \theta j_{1y}^{(2)} \right) \sin \left( \theta j_{2y}^{(2)} \right)
\]

\[
+ j_{2y} \sin \left( \theta j_{1y}^{(2)} \right) \cos \left( \theta j_{2y}^{(2)} \right) + j_{2y} \cos \left( \theta j_{1y}^{(2)} \right) \sin \left( \theta j_{2y}^{(2)} \right).
\]

Although there are 16 terms after expanding the above equation, only 4 terms survive when calculating \(\langle \sigma_{1+}\sigma_{2+} \rangle\). We then have

\[
\langle j_{1z} j_{2z} \rangle = \langle \hat{1}' \rangle \cos \left( \theta j_{1z}^{(2)} \right) \langle \hat{1}' \rangle
\]

\[
- j_{1z} j_{2z} \sin \theta \sin \left( \theta j_{1y}^{(2)} \right)
\]

\[
+ 4 j_{1y} j_{1x} j_{2x} j_{2y} \sin \left( \theta j_{2y}^{(2)} \right) \cos \left( \theta j_{2y}^{(2)} \right)
\]

\[
- j_{1y} j_{1z} j_{2z} \sin \theta \sin \left( \theta j_{1y}^{(2)} \right)
\]

\[
= 4^{-1} \langle \hat{1}' \rangle \cos \left( \theta j_{1z}^{(2)} \right) \langle \hat{1}' \rangle
\]

\[
= 8^{-1} \langle \hat{1}' \rangle \left( 1 + \cos \left( 2 \theta j_{1z}^{(2)} \right) \right) \langle \hat{1}' \rangle
\]

\[
= 8^{-1} \left( 1 + \cos^{N-2} \langle \theta' \rangle \right),
\] (B8)

where \(\langle \hat{1}' \rangle = |1\rangle \otimes \cdots \otimes |1\rangle_N\). The second equality in Eq. (B8) is due to the property \(j_{1z} j_{2z} = -j_{1z} j_{2z} = j_{1z} j_{2z} = j_{1z} j_{2z}\) and the last equality from Eq. (B5). Finally, from the above equation, one finds

\[
\langle \sigma_{1+}\sigma_{2+} \rangle = 2^{-1} (1 + \cos^{N-2} \langle \theta' \rangle).
\] (B9)

Due to the relation \(\langle \sigma_{1+}\sigma_{2+} + \sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+} \rangle = 1\) for the initial state, the correlation \(\langle \sigma_{1+}\sigma_{2-} \rangle\) is obtained from Eqs. (B7) and (B9) as

\[
\langle \sigma_{1+}\sigma_{2-} \rangle = 2^{-1} (1 - \cos^{N-2} \langle \theta' \rangle)
\] (B10)

Substituting Eqs. (B7) and (B10) into the following relations

\[
\sigma_{1+}\sigma_{2+} + \sigma_{1+}\sigma_{2-} = 2 (\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+})
\]

leads to one element of the two-spin reduced density matrix,

\[
y_0 = \langle \sigma_{1+}\sigma_{2-} \rangle = 8^{-1} (1 - \cos^{N-2} \langle \theta' \rangle),
\] (B11)

where the relation \(\langle \sigma_{1+}\sigma_{2-} \rangle = \langle \sigma_{1-}\sigma_{2+} \rangle\) is used due to the exchange symmetry.
3. Correlation \( \langle \sigma_1, \sigma_2 \rangle \)

To calculate the correlation \( \langle \sigma_1, \sigma_2 \rangle \), due to the following relations

\[
\sigma_1 \sigma_2 - \sigma_1 \sigma_2 = 2(\sigma_1 \sigma_2 + \sigma_1 \sigma_2), \quad (B12)
\]

\[
i(\sigma_1 \sigma_2 + \sigma_1 \sigma_2) = 2(\sigma_1 \sigma_2 - \sigma_1 \sigma_2), \quad (B13)
\]

we need to know the expectations \( \langle j_1, j_2 \rangle \). The evolution of

\[
U^1 s_1 s_2 U = j_1 \left\{ j_2 \cos \left[ \theta (j_1 + j_2) \right] \right. \\
- j_2 \sin \left[ \theta (j_1 + j_2) \right] \right\},
\]

and the expectation is obtained as

\[
\langle j_1 j_2 \rangle = 2^{-1}\langle j_1 \sin \left[ \theta (j_1 + j_2) \right] \rangle^1 \langle 1 \rangle.
\]

Moreover, \( \langle j_1 j_2 \rangle = 2^{-1} \sin(\theta) \cos^{N-2}(\theta) \).

For the initial state \( (2) \), we obtain the following expectations

\[
\langle \sigma_1 \sigma_2 \rangle = 2 \sin(\theta) \cos^{N-2}(\theta). \quad (B14)
\]

The combination of Eqs. (B7), (B10), (B12), (B13), and (B14) leads to the correlation

\[
u_0 = \langle \sigma_1, \sigma_2 \rangle = -8^{-1}(1 - \cos^{N-2}\theta) \\
- i 2^{-1} \sin(\theta) \cos^{N-2}(\theta). \quad (B15)
\]

Substituting Eqs. (B11) and (B15) to Eq. (31) leads to the expression of the squeezing parameter \( \xi_1 \) given by Eq. (44).

APPENDIX C: PROOF OF \( C_{\alpha}(0) \geq 0 \)

To prove this, we will not use this specific function of the initial twist angle \( \theta \) as given by Eq. (47), but use the positivity of the reduced density matrix \( (38) \). We first notice an identity

\[
C_{\alpha} = 4(u_+ v_+ - w_+^2),
\]

which results from Eqs. (39) and (40). This is a key step. Also there exists another identity

\[
u_0 = \gamma_0 \quad (C1)
\]

as \( \langle \sigma_1, \sigma_2 \rangle = 1 \). From the positivity of the reduced density matrix \( (38) \), one has

\[
v_0 + v_0 \geq |u_0|^2 \geq \gamma_0^2 = w_0^2,
\]

where the second inequality follows from Eq. (40) and the last equality results from Eq. (C1). This completes the proof.

APPENDIX D: DERIVATION OF THE EVOLUTION OF THE CORRELATIONS AND EXPECTATIONS UNDER DECOHERENCE

For an arbitrary matrix

\[
A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),
\]

from the Kraus operators \( (12) \) for the ADC, it is straightforward to find

\[
E(A) = \left( \begin{array}{cc} sa & \sqrt{5b} \\ \sqrt{3c} & d + pa \end{array} \right),
\]

\[
E(A) = \left( \begin{array}{cc} sa + pd & \sqrt{5b} \\ \sqrt{3c} & d \end{array} \right).
\]

The above equations imply that

\[
E(\sigma_\mu) = \sqrt{5} \sigma_\mu \quad \text{for} \quad \mu = x, y,
\]

\[
E(\sigma_z) = \sigma_z - p.
\]

As we considered independent and identical decoherence channels acting separately on each spin, the evolution correlations and expectations in Eqs. (52b), (52c), and (52d) are obtained directly from the above equations.

From the Kraus operators \( (14) \), the evolution of the matrix \( A \) under the PDC is obtained as

\[
E(A) = E(A) = \left( \begin{array}{cc} a & sb \\ sc & d \end{array} \right),
\]

from which one finds

\[
E(\sigma_\mu) = s \sigma_\mu \quad \text{for} \quad \mu = x, y,
\]

\[
E(\sigma_\zeta) = \sigma_\zeta - p.
\]

So expectations \( \langle \sigma_\zeta \rangle \) are unchanged and Eq. (66) is obtained.

From the Kraus operators \( (16) \) of the DPC, the evolution of the matrix \( A \) is given by

\[
E(A) = E(A) = \left( \begin{array}{cc} as + \frac{p}{2}(a + d) & sb \\ sc & ds + \frac{p}{2}(a + d) \end{array} \right)
\]

from which one finds

\[
E(\sigma_\zeta) = s \sigma_\zeta \quad \text{for} \quad \zeta \in \{x, y, z\}.
\]

Then, Eq. (78) is obtained.

APPENDIX E: SIMPLIFIED FORM OF THE CONCURRENCE

For our three kinds of decoherence channels, the concurrence \( (43) \) can be simplified and given by

\[
C = \max \{0, 2(|u| - w), 2(y - \sqrt{v_+v_-}) \}
\]

\[
= \max \{0, 2(|u| - w) \}. \quad (E1)
\]

If one can prove

\[
|u| - y \geq 0, \quad (E2)
\]

\[
w - \sqrt{v_+v_-} \leq 0 \quad (E3)
\]
then we obtain the simplified form shown in Eq. (E1). The last inequality can be replaced by
\[ w^2 - v_+v_- \leq 0 \]  
(E4)
as \( w \) and \( v_+v_- \) are real.

We first consider the ADC channel. From Eqs. (52b), (52c), and (54), one obtains
\[ |u| - y = s(|u_0| - y_0) \geq 0, \]  
(E5)
\[ w^2 - v_+v_- = -\frac{1}{4}C_{zz}(0) \leq 0. \]  
(E6)
where the inequalities result from Eqs. (44) and (47), respectively. So, the inequality (E4) follows.

For the PDC, from Eq. (66) and fact that \( \sigma^zz \) is unchanged under decoherence, the concurrence can also be simplified due to the following properties:
\[ |u| - y = s^2(|u_0| - y_0) \geq 0, \]  
(E7)
\[ w^2 - v_+v_- = -\frac{1}{4}C_{zz}(0) \leq 0. \]  
(E8)
So, again, the concurrence can be simplified to the form shown in Eq. (E1). This completes the proof.