Global versus local quantum squeezing in composite systems

Yang Yang,1 Wanfang Liu,1,2 Zhe Sun,3 and Xiaoguang Wang1,*
1Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China
2School of Physics and Electric Engineering, Anqing Teachers College, Anqing 246011, China
3Department of Physics, Hangzhou Normal University, HangZhou 310036, China

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We investigate relations between the global squeezing of composite systems and the local squeezing of subsystems. For the pure symmetric product states, the global squeezing parameter is found to be equal to the local one for both spin and bosonic systems. Hence, a pure symmetric state is entangled if the global parameter is not equal to the local one. Two origins of the global squeezing are identified: one is from the local squeezing and the other from quantum correlations. For both spin and bosonic systems, we find that the entanglement can lead to a smaller global squeezing parameter; namely, the global squeezing is enhanced.

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I. INTRODUCTION

The concept of spin squeezing is closely connected to quantum correlations [1], and spin squeezed states for an ensemble of atoms [2–10] have recently attracted much attention. In the absence of decoherence, the reduced uncertainty of a transverse spin component directly improves the sensitivity of interferometry [11] with potential applications in atomic interferometers and high-precision atomic clocks [12]. The spin-squeezing models with one-axis and two-axis twistings are proposed on the basis of the nonlinear interaction between N spins. Possible realizations of the one-axis twisting Hamiltonian have been studied in a two-component Bose-Einstein condensate [13–16] and an atomic ensemble system in the dispersive regime. Sørensen et al. [13] also proposed that the spin squeezing can be used as an indicator of many-particle quantum entanglement.

For quantum information applications [13,17–19], the close relation between the atomic spin squeezing and quantum entanglement [13,20–24] enhances the importance of atomic spin squeezing. In Ref. [25], Sørensen and Mølmer studied relations between squeezing and entanglement related to collective spin operators $J_z$ and $J_x$. Recently, it has been found that, for a two-qubit symmetric state, spin squeezing is equivalent to its bipartite entanglement [23]. In Ref. [26], Wang and Sanders showed that the spin squeezing implies pairwise entanglement for arbitrary symmetric multiqubit states. If the squeezing parameter $\xi^2 < 1$, a quantitative relation between the squeezing parameter and the concurrence (pairwise entanglement measure) [27] was given for the even and odd states.

On the other hand, relations between the global and local geometric phases have been investigated [28–30]. Being inspired by this idea, we explore further on connections between global and local quantum properties in a composite system. We study the quantum squeezing and find relations between the global squeezing of the whole system and the local squeezing of subsystems. As the pure symmetric states are usually considered in the investigations of spin squeezing, we focus on this class of states. We find that, for a pure symmetric product states, the global squeezing parameter is found to be equal to the local one for both spin and bosonic systems. Thus, a pure symmetric state is entangled if the global parameter is not equal to the local one. We then give several examples to study effects of entanglement on the quantum squeezing.

II. GLOBAL AND LOCAL SQUEEZING FOR SEPARABLE STATES

We consider the spin squeezing for an ensemble of $N$ spin-$j$ particles, where $j$ is not necessarily to be 1/2. The ensemble can be described by collective spin operators $\vec{J} = \sum_{\mu=1}^{N} \vec{J}^{(\mu)}$, where $\vec{J}^{(\mu)}$ denotes the angular momentum operator for the $\mu$th spin. We first study the global and local squeezing for the pure symmetric product states, which is given by

$$|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle.$$  \hspace{1cm} (1)

This is an unentangled state. For an Hermitian operator $F = \sum_{\mu} F^{(\mu)}$, one can easily verify the covariance ($\mu \neq \nu$),

$$\text{Cov}(F^{(\mu)}, F^{(\nu)}) = (F^{(\mu)})(F^{(\nu)}) - (F^{(\mu)})(F^{(\nu)}) = 0,$$  \hspace{1cm} (2)

for state (1).

Spin squeezing can be quantified by the following parameter [1]:

$$\xi^2 = \frac{2(\Delta J_{\vec{n}})^2}{J},$$  \hspace{1cm} (3)

where $J=\sum_{\mu} j^{(\mu)}$ and subscript $\vec{n}$ refers to an arbitrary axis perpendicular to the mean spin ($\vec{J}$), where the minimum value of $(\Delta J)^2$ is obtained. The inequality $\xi^2 < 1$ indicates that the state is spin squeezed. By using Eq. (2), we find the following equality for the symmetric product state:

$$\xi^2 = \frac{\sum_{\mu=1}^{N} \text{Var}(j^{(\mu)}_{x_{\vec{n}}}) + \sum_{\mu \neq \nu} \text{Cov}(j^{(\mu)}_{x_{\vec{n}}}, j^{(\nu)}_{x_{\vec{n}}})}{J} \bigg/ \frac{1}{J} = \frac{2N(\Delta j^{(\mu)}_{x_{\vec{n}}})^2}{J} = \frac{\xi^2}{\xi^2_{\text{g}}}.$$  \hspace{1cm} (4)

Here, the variance $\text{Var}(X)=\langle X^2 \rangle - \langle X \rangle^2$ and the subscript $g$ and $l$ refers to global and local, respectively. The above result
suggests that in the composite spin system with identical subsystems, if the global squeezing parameter $\xi_g$ is not equal to the local one, the state is entangled.

We then study the bosonic squeezing, which is characterized by one parameter [31],

$$\xi^2 = \min_{\theta \in [0, 2\pi]} (\Delta X_\theta)^2,$$

in which $X_\theta = X \cos \theta + P \sin \theta$ and $X$ and $P$ represent displacement and momentum operators, respectively. The definition of $\xi$ provides an atomic squeezing counterpart to bosonic squeezing. We now construct a bosonic composite system containing $N$ modes and define $X_\theta = 2N X_\theta^{(1)} / \sqrt{N}$. Similar as the spin case, by using Eq. (2), we can obtain the global squeezing parameter for the pure symmetric product state,

$$\xi^2_g = \min_{\theta \in [0, 2\pi]} (\Delta X_\theta^{(1)})^2 = \xi^2_l.$$  

We can see that for both the spin and bosonic squeezings, the global squeezing parameter is equal to the local one. Then, the state is doomed to be entangled if the above equalities break; namely, we find a sufficient condition for entanglement. It is interesting to further study effects of the entanglement on the squeezing character of the composite system. We next investigate this issue.

III. EFFECTS OF ENTANGLEMENT

For clarity, we consider a class of pure states, namely, two-spin states with parity, for which the global squeezing parameter can be obtained as [24]

$$\xi^2_g = \xi^2_l + \frac{1}{2j} \left[ \text{Cov}(j_{1+}, j_{2-}) + \text{Cov}(j_{1-}, j_{2+}) \right] - 2 \left[ \left| \text{Var}(j_{1+}) + \text{Cov}(j_{1+}, j_{2-}) \right| - \left| \text{Var}(j_{1-}) \right| \right].$$

We see that the global squeezing parameter differs from the local one by an additional factor which comes from quantum correlations between the two spins.

We suppose the two-spin state be a product state at initial time, $|j_{1+}, j_{2-}\rangle \otimes |j_{1-}, j_{2+}\rangle$, which evolves via the Hamiltonian

$$H_1 = \chi j_{1+}^2 + \chi j_{2-}^2 + \lambda j_{1-} \otimes j_{2+}.$$  

Let $\chi = 1$, and we find

$$\text{Cov}(j_{1+}, j_{2-}) = \frac{j^2}{2} \left[ \cos^{4j-2} \left( \frac{\lambda t}{2} - t \right) - \cos^{4j-2} \left( \frac{\lambda t}{2} + t \right) \right],$$

$$\text{Cov}(j_{1-}, j_{2+}) = \frac{j^2}{2} \left[ \cos^{4j-2} \left( \frac{\lambda t}{2} + t \right) - \cos^{4j-2} \left( \frac{\lambda t}{2} - t \right) \right] + i2j^2 \cos^{2j-1} t \cos^{2j-1} \frac{\lambda t}{2} \sin \frac{\lambda t}{2}. $$

FIG. 1. (Color online) (a) and (b) Global spin-squeezing parameter (solid line) and the local one (red dashed dotted line) versus time $t$. Parameters $\chi=1$, $\lambda=0.4$, and $j=2$ in (a) and (b). (c) and (d) For the entangled spin coherent state the spin-squeezing parameters $\Delta \xi^2$ (solid line) and linear entropy (dashed line) are plotted as a function of $\theta$. Parameters: $\eta=0.5, j=2$. Linear entropy is also plotted in (a) and (b).

$$\text{Var}(j_{1+}) = \frac{1}{2} \left( j - \frac{1}{2} \right) \left\{ \cos^{2j-2} (2t) \cos^{2j} (\lambda t) - 1 + 4i \cos^{2j-2} t \sin t \cos^{2j} \frac{\lambda t}{2} \right\},$$

and the local spin-squeezing parameter is given by

$$\xi^2_l = 1 + j - \frac{1}{2j} \left[ \left( j + \frac{1}{2} \right) + \left( j - \frac{1}{2} \right) \cos^{2j-2} (2t) \cos^{2j} (\lambda t) \right] - \frac{1}{2j} \left( \cos^{2j-2} (2t) \cos^{2j} (\lambda t) - 1 \right) - 4i \cos^{2j-2} t \sin t \cos^{2j} \frac{\lambda t}{2}.$$

Thus, we have obtained analytical exact results for both global and local squeezing parameters.

In Fig. 1, we plot the global and local spin-squeezing parameters for different strengths of coupling. When no interaction is involved [Fig. 1(a)], the global squeezing parameter is equal to the local one, as we expected. Since there is no entanglement between two spins in this case, the global squeezing completely originates from the local squeezing. For the interaction strength $\lambda \neq 0$ [Fig. 1(b)], we observe that the minimal global squeezing parameter (maximal squeezing) is less than the minimal local one, and the average global squeezing strength is evidently larger than the local one. In this sense, the local squeezing is suppressed while the global squeezing is enhanced. However, it should be noted that a state can be entangled with a larger global parameter at some special times.

We see that if the difference between the global and local squeezing parameters is not zero, the entanglement is present, and if there is no difference, there may have entanglement or no entanglement. In these cases, together with the plots of the linear entropy, the global squeezing origi-
nates from both the local squeezing and the correlations between two spins. However, for spin-half systems, there is no local squeezing, and thus the global squeezing is completely from correlations among spins.

We further consider another class of states, namely, entangled spin coherent state,

\[ |\theta\rangle = C_2^{-1}(\cos \theta |\eta\rangle_1 |\eta\rangle_2 + \sin \theta |\eta\rangle_1 |-\eta\rangle_2), \]  

where \( C_1 = \sqrt{1 + \sin 2\theta} \gamma, \) \( \gamma = (1 - |\eta|^2)/(1 + |\eta|^2) \in (0, 1), \) and \(|\eta\rangle \) denotes the spin coherent state \(|\eta\rangle = (1 + |\eta|^2)^{-1/2} |\eta\rangle_n \) with the parameter \( \eta \) being complex.

As a function of \( \theta \), in Figs. 1(c) and 1(d), we numerically compute the spin-squeezing parameters, the difference between the squeezing parameters, i.e., \( \Delta \xi = \xi_E - \xi_I \), and the linear entropy. It can be seen that both the whole system and subsystems become squeezed in some regions of \( \theta \). And in the squeezing regions, the global squeezing is stronger than the local one. The global and local spin-squeezing parameters achieve the minima at \( \theta = (n + \frac{1}{2})\pi \). Considering Eq. (4), we have \( \Delta \xi = 0 \) for pure symmetric product states. Then, the inequality \( \Delta \xi \neq 0 \) implies entanglement for the pure state. Equivalently, we may say that \( \Delta \xi > 0 \) is criteria of entanglement for the pure symmetric states. The linear entropy and \( \Delta \xi \) are plotted in Fig. 1(d). It is evident that \( \Delta \xi \) behaves similarly as the linear entropy and it can be regarded as an indicator of quantum entanglement.

Now we study the bosonic squeezing of two bosons. As shown in [31], the bosonic squeezing parameter [Eq. (5)] can be further expressed as

\[ \xi_0 = 1 + 2 \text{Cov}(a^\dagger a, a^\dagger a^\dagger) - 2 |\text{Var}(a^\dagger a)|. \]  

We construct a bosonic composite system containing \( N \) modes and define an annihilation operator corresponding to this composite system by \( a = \sum_{\mu=1}^{\infty} a_\mu / \sqrt{N} \). From Eq. (12), we find the global squeezing parameter

\[ \xi_g^2 = \xi_j^2 + 2(N - 1) \text{Cov}(a_\mu^\dagger a_\nu, a^\dagger a) - 2 |\text{Var}(a^\dagger a)| (N - 1) \text{Cov}(a_\mu a_\nu) - |\text{Var}(a^\dagger a)|, \]  

in which \( \mu \neq \nu \). Similar as the spin cases, we first study a product state, \(|a\rangle \otimes |a\rangle \) (\(|a\rangle \) is the coherent state), which evolves via the Hamiltonian

\[ H_2 = \chi (a^\dagger a)^2 + \lambda (a^\dagger a^\dagger)^2 + \lambda a^\dagger a^\dagger a^\dagger a. \]  

The relevant variances and covariances are obtained as

\[ \text{Var}(a^\dagger a) = \alpha^2 [e^{-i4x}\cos^2\lambda e^{-i2x\lambda - 2\lambda}] - e^{-i2x\lambda + 2}\alpha^2 |e^{-i2x\lambda e^{-i2\lambda - 2\lambda}}|, \]

\[ \text{Cov}(a^\dagger a^\dagger) = \alpha^2 [e^{-i2x\lambda + 2}\alpha^2 |e^{-i2x\lambda e^{-i2\lambda - 2\lambda}}| - e^{-i2x\lambda + 2}\alpha^2 |e^{-i2x\lambda e^{-i2\lambda - 2\lambda}}|), \]

\[ \text{Cov}(a^\dagger a^\dagger) = |\alpha|^2 [e^{-i4x}\cos^2\lambda e^{-i2x\lambda - 2\lambda}] - e^{-i2x\lambda + 2}|\alpha|^2 |e^{-i2x\lambda e^{-i2\lambda - 2\lambda}}|, \]  

and the local squeezing parameter is given by

\[ \xi_l^2 = 1 + 2 |\alpha|^2 \sin^2 2\theta (1 - e^{-8|\alpha|^2})/2C_2^4, \]  

\[ \text{Var}(a^\dagger a) = \alpha^2 [1 - \cos(2\theta)^2/C_2^2], \]

\[ \text{Cov}(a^\dagger a^\dagger) = \alpha^2 [1 - \cos(2\theta)^2/C_2^2], \]

and the local squeezing parameter is

\[ \xi_l^2 = 1 + 2 |\alpha|^2 \sin^2 2\theta (1 - e^{-8|\alpha|^2})/2C_2^4 - 2 |\alpha|^2 |1 - \cos^2 2\theta|C_2^4. \]  

From Eq. (13), we can find the global squeezing parameter for the entangled coherent state by combining the following relations,

\[ \text{Cov}(a^\dagger a^\dagger) = |\alpha|^2 \sin^2 2\theta (1 - e^{-8|\alpha|^2})/2C_2^4, \]

\[ \text{Var}(a^\dagger a) = \alpha^2 [1 - \cos(2\theta)^2/C_2^2], \]

\[ \text{Cov}(a^\dagger a^\dagger) = \alpha^2 [1 - \cos(2\theta)^2/C_2^2], \]

\[ \text{Cov}(a^\dagger a^\dagger) = |\alpha|^2 \sin^2 2\theta (1 - e^{-8|\alpha|^2})/2C_2^4 - 2 |\alpha|^2 |1 - \cos^2 2\theta|C_2^4. \]

From Eq. (18), we can see that there will be no difference between the global and local squeezing parameters when \( \theta = n\pi/2 \) \( (n = 0, \pm 1, \ldots) \). When no interaction is involved [Fig. 2(a)], the global squeezing completely originates from the local squeezing and there is no difference between them. The introduction of the interaction [Fig. 2(b)] leads to the difference between the global and local squeezing parameters. In the squeezing regions, the global squeezing is stronger than the local one (global parameter is less than the local one) due to the interactions between two bosons.
For the entangled coherent states, in Figs. 2(c) and 2(d), we numerically compute the squeezing parameters, the difference between them, $\Delta \xi^2$, and the concurrence. We can see when $\theta$ satisfies the condition $2\theta < -e^{-\Delta \varphi}$, there will be no squeezing (i.e., $\xi_1^2, \xi_2^2 > 1$) in both subsystems and composite system. Similar as the spin case, $|\Delta \xi^2|$ plays a similar role as the concurrence and can be regarded as an indicator of the bipartite entanglement. Just as other indicators of quantum entanglement, one defect of our indicator is that one cannot assert that the entanglement is absent in the system. Similar cases also exist in spin models. The facts remind us that $\Delta \xi^2 \neq 0$ is sufficient condition for entanglement of pure states, but not necessary.

IV. CONCLUSIONS

In conclusion, we have studied the global and local squeezing in spin and bosonic systems. Especially, we investigated how the entanglement between the subsystems affects the global squeezing. For pure symmetric product states, the global squeezing parameter is equal to the local one for both spin and bosonic systems. Thus, for pure symmetric states, the state is entangled if the global squeezing is different from the local one. For spin-half systems, the squeezing originates purely from quantum correlations due to the fact that there is no local squeezing. For higher spin systems ($j > 1/2$) and infinite-dimensional systems, the global squeezing originates from both the local squeezing and quantum correlations among subsystems.

For the spin and bosonic interaction models, we find that quantum entanglement can indeed enhance the global squeezing and suppress the local squeezing. For the spin and bosonic entangled coherent states, except some special points, the global squeezing is different from the local one, which implies the presence of quantum entanglement. It will be interesting to consider other kinds of squeezing and study their relations with different types of entanglement, which are under consideration.

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