Operator fidelity susceptibility, decoherence, and quantum criticality

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The extension of the notion of quantum fidelity from the state-space to the operator level can be used to study environment-induced decoherence. The state-dependent operator fidelity susceptibility (OFS), the leading-order term for slightly different operator parameters, is shown to have a nontrivial behavior when the environment is at critical points. Two different contributions to the OFS are identified which have distinct physical origins and temporal dependence. Exact results are obtained for the finite-temperature decoherence caused by a bath described by the Ising model in a transverse field.

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I. INTRODUCTION

Generically, the interaction between a quantum system and its environment results in decoherence, and may lead the system to experience the so-called quantum-classical transition [1]. For this reason, the decoherence process is regarded as the main obstacle for the implementation of quantum-information processing [2]. Generally speaking, the properties of the environment may strongly affect its decohering capabilities [3–5, 7, 8]. This implies that a quantum system can be regarded as a probe to extract useful information about the coupled environment, e.g., quantum phase transitions (QPTs). Very recently a nuclear magnetic resonance experiment of this kind has been performed [9]. There a single qubit was used as a probe to detect the (precursors of the) quantum critical point of the coupled environment.

The relationship between the decoherence of a central spin and the QPTs of the coupled environment can be established through the notion of the Loschmidt echo (LE) [5]. It is well known that the LE can be exploited to measure the stability of a quantum system to perturbations [3, 10, 11]. In Ref. [5], a central spin coupled with an Ising spin chain in a transverse field was considered; the authors found that the decay of the LE is enhanced by quantum criticality. The connection between the LE and quantum criticality has been further clarified in Ref. [6]. There the authors showed that the LE enhancement is a sort of artifact of the short-time Gaussian approximation used in Ref. [5]. A genuine signature of criticality, on the other hand, can be recovered by studying the asymptotic (large-time) behavior of the LE as a function of the environment coupling parameter [6]. More recently, the averaged LE over all states on a Hilbert space with a Haar measure, called the operator fidelity, was proposed to study quantum criticality [12]. This proposal is a direct extension to the operator level of state-space quantum fidelity. This latter notion recently attracted a lot of attention as a new tool to analyze quantum criticality, at both zero and finite temperature [13–18].

Whereas in [5] the initial state of the spin chain was assumed to be the ground state, in this paper we aim at considering a more general situation: the initial state of the environment is a mixture of the eigenstates of its Hamiltonian. The Gibbs thermal state is a special instance of this setup. With this assumption the study of decoherence naturally leads to the state-dependent generalizations of the operator fidelity suggested in [12].

In this paper we will discuss the operator extensions of fidelity on general grounds. In order to do that, one has simply to notice that finite-time quantum evolutions correspond to unitary operators that themselves belong to a Hilbert space i.e., the Hilbert-Schmidt one. It follows that some of the results obtained in the fidelity approach for quantum states apply immediately to the operator level we are now interested in. We will show that the corresponding operator fidelity susceptibility (OFS) contains two different contributions. These two terms arise from variation upon parameter change of the energy levels and eigenstates, respectively, and have very different temporal behavior. We will then get a general expression for the OFS for models with a factorization structure typical of quasifree models. Finally, we will exploit the OFS for studying an environment described by an Ising chain with a transverse field that can be driven to quantum criticality.

This paper is organized as follows. In Sec. II, we give a general description of the models and the relations between decoherence of the central system and the operator fidelity of two-time evolutions of the environment. In Sec. III, we give generalities about the operator metric and fidelities and analyze the general OFS. In Sec. IV A, we consider models with a factorization structure, and give a general expression for the OFS, and then study the Ising model with a transverse field to get the exact solution for the OFS in Sec. IV B. Section V contains the conclusions and outlook.

II. DECOHERENCE AND OPERATOR FIDELITY

As mentioned in the Introduction, an important physical motivation for the operator fidelities we are going to analyze in this paper is provided by the decoherence process. To see
this clearly, let us consider a purely dephasing coupling of a system with its environment. The interaction Hamiltonian has the form

$$H_I = \sum_n |n\rangle\langle n| \otimes B_n.$$  

The entire Hamiltonian is $H = H_0 + H_I$, where $H_0$ consists of the system Hamiltonian $H_S = \sum_n |n\rangle\langle n|$ and the bath Hamiltonian $H_B$. The $H_S$ eigenstates $\{|n\rangle\}$ play here the role of preferred pointer states [1].

With the initial state $\rho(0) = |\phi(0)\rangle\langle \phi(0)|$ and $\rho_B$, the reduced density matrix of the system at time $t$ can be written as

$$\rho_S(t) = \sum_{n,m} c_n c_m^* e^{-i(E_n - E_m)t} |n\rangle\langle m| \text{Tr}_B[\rho_B V_m^\dagger(t)V_n(t)],$$  

where $c_n = \langle n| \phi(0)\rangle$ and $V_n(t) = \exp[-i t(H_B + B_n)]$

Obviously, in the generic case $\rho_S$ evolves from initial pure states to mixed states. The decay of the off-diagonal elements of $\rho_S$ means a reduction from a pure state to a classical mixture of the preferred pointer states—quantum–classical transitions. The temporal behavior of the off-diagonal element is decided by two factors: one is $c_n c_m^*$, just relating to the initial state, and the other, $\text{Tr}_B[\rho_B V_m^\dagger(t)V_n(t)]$, is unrelated to the initial state of the system, reflecting the dephasing effect induced by the environment. The latter can be considered as a fidelity for two operators $V_m$ and $V_n$, which is defined by

$$F_\rho(V_m, V_n) = \langle [V_m, V_n^\dagger]\rangle,$$

where the inner product $\langle V_m, V_n^\dagger\rangle = \text{Tr}(\rho V_m^\dagger V_n)$.

This fidelity can be obtained from the density matrix of the central system and encodes information about the bath state. Notice that one might choose $B_n = \lambda_n B$; in this case different $V_n$ simply correspond to different values of the coupling strength $\lambda$ in front of the “perturbation” $B$. This is the scenario we mostly have in mind in this paper (see Sec. IV).

Let us now specialize to a two-level system coupled with a bath. If interaction is weak enough, the two effective Hamiltonians are slightly different. For an example in Sec. IV, we will consider the important case where the initial state of the bath is a thermal equilibrium state $\rho_B = \exp(-B\beta)/Z(\beta)$, where $Z(\beta) = \text{Tr}[\exp(-B\beta)]$ is the partition function, and $T = \beta^{-1}$ is the temperature of the bath. In such a case $\left|\text{Tr}_B[V_1, V_2]\right|^2$ will become the Loschmidt echo for $\beta^{-1} = 0$ (for a nondegenerate ground state), and it coincides with the operator fidelity defined in [12] for $\beta = 0$.

### III. METRICS OVER MANIFOLDS OF UNITARIES

In this section we discuss operator fidelity from a rather general mathematical point view. Let $\mathcal{H}$ be a quantum state space and $\{U_\lambda\} \subset U(\mathcal{H})$ a smooth family of unitaries over $\mathcal{H}$ parametrized by elements of a manifold $\mathcal{M}$. Given a state $\rho \in \mathcal{S}(\mathcal{H}) = \{\rho \in L(\mathcal{H})| \rho \geq 0, \text{Tr} \rho = 1\}$ one can define the following Hermitian product over $L(\mathcal{H})$:

$$\langle X,Y\rangle_\rho := \text{Tr}(\rho X^\dagger Y),$$

If $\rho > 0$ then (4) is nondegenerate and $|X|_\rho := \sqrt{\langle X,X\rangle_\rho}$ defines a norm over $L(\mathcal{H})$. In general, if $\text{ker} \rho \neq \{0\}$, $\|X\|_\rho$ is just a seminorm [if the range of $X$ is included in $(\text{supp} \rho)^{\perp}$ then $\|X\|_\rho = 0$]. Notice that (i) $\|X\|_\rho \leq \|X\|$ and (ii) the unitaries $U$ are normalized, i.e., $\langle U,U\rangle_\rho = 1$.

**Definition.** The $\rho$ fidelity of the operators $X$ and $Y$ is given by

$$\mathcal{F}_\rho(X, Y) := \langle X, Y\rangle_\rho.$$  

It is immediate yet important to realize that for non-full-rank states $\rho$, having two unitaries with $\rho$ fidelity 1 does not imply their equality (up to a phase). Indeed, from the Cauchy-Schwarz inequality one has $\mathcal{F}_\rho(X, Y) \leq \|X\|_\rho \|Y\|_\rho$; in particular, for unitaries $U$ and $W$, $\mathcal{F}_\rho(U, W) \leq 1$. One has that $\mathcal{F}_\rho(U, W) = 1 \Leftrightarrow U|_{\text{supp} \rho} = W|_{\text{supp} \rho}$, where $\text{supp} \rho := (\ker \rho)^{\perp}$.

In order to unveil the operational meaning of the above definition, it is useful to recall the following fact. If $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is a purification of $\rho = \sum_i \rho_i |i_i\rangle\langle i_i|$, i.e., $|\Psi\rangle = \sum_i \rho_i |i_i\rangle\langle i_i|$, then $\langle X, Y\rangle_{\rho} = \langle \phi(\langle X | \otimes 1) (1 | \otimes \langle Y\rangle_{\rho}) \rangle$. The operator scalar product (4) can be seen as a scalar product of suitable quantum states of a bigger system. This simple remark shows that the operator fidelity (5) quantifies the degree of statistical distinguishability between the two states, $|\Psi(\Lambda) := |A \otimes 1\rangle |\Psi\rangle \quad (A = X, Y)$.

When $U$ and $W$ denote unitary transformations, the quantity (5) has an interpretation as the visibility strength in properly designed interferometric experiments [24]. Another kind of operational relevance of the operator fidelities (5) in the context of decoherence has been discussed in Sec. II.

Now, following the differential-geometric spirit of [19,20], we are going to consider the operator fidelity (5) between infinitesimally different unitaries. The leading term in the expansion of (5) will define a quadratic form over the tangent space of the manifold $U(\mathcal{H})$. For full rank $\rho$ that quadratic form is a metric. The following proposition shows that, and its proof is just a direct calculation analogous to the one performed at the state-space level [19].

**Proposition.** Let $\{U_\lambda\} \subset U(\mathcal{H})$ be a smooth family of unitaries over $\mathcal{H}$ parametrized by elements $\lambda$ of a manifold $\mathcal{M}$. One finds

$$\mathcal{F}_\rho(U_\lambda, U_{\lambda + \delta \lambda}) \approx 1 - \frac{\delta \lambda^2}{2} \chi_\rho(\lambda),$$

where, if $U' = \partial U/\partial \lambda$, one has

$$\chi_\rho(\lambda) := \langle U', U'\rangle_\rho - |\langle U', U\rangle_\rho|^2.$$
\[ H(\lambda) = \sum_n E_n(\lambda) |\psi_n(\lambda)\rangle \langle \psi_n(\lambda) |. \]

The adiabatic intertwiners are unitaries such that \( O(\lambda, \lambda_0) |\psi(\lambda)\rangle = |\psi(\lambda')\rangle \longeq n \). Let \( \rho = |\psi(\lambda_0)\rangle \langle \psi(\lambda_0) | \), \( [|\psi(\lambda_0)\rangle \langle \psi(\lambda_0) |) \) is the ground state of \( H(\lambda_0) \) and \( U(\lambda) = O(\lambda, \lambda_0) \); then
\[ \langle U(\lambda), U(\lambda') \rangle_p = \langle \psi(\lambda_0) | O^{1}(\lambda, \lambda_0) O(\lambda', \lambda_0) |\psi(\lambda_0)\rangle \]
\[ = \langle \psi(\lambda), \psi(\lambda') \rangle. \]

(8)

This shows that the ground-state fidelity is a particular instance of the setting we are discussing.

**Example 1.** Let \( U = e^{-iU(t)} \) and \( \rho = |\phi\rangle \langle \phi | \). Then \( U' = -iH(t) \) where \( \chi(t) = (\phi | H \phi \rangle)^{-2} \langle \phi | H | \phi \rangle^2 \).

**Example 2.** \( U = e^{-iH(t)} \), and \( \rho = |\phi\rangle \langle \phi | \). From time-dependent perturbation theory one obtains
\[ U' = -i \int_0^t d\tau e^{-i\tau H} \langle H \rangle e^{-i\tau H}. \]

(9)

Introducing the superoperator \( L_H = [H, \cdot] \) over \( \mathcal{L}(\mathcal{H}) \) one sees that, if \( |\phi\rangle \)’s and \( E_n \)’s denote the eigenvectors and eigenvalues of \( H(\lambda) \), \( E_n - E_m \) and \( \Psi_{n,m} = \langle \phi | n \rangle | m \rangle \) are the eigenvalues and (normalized) eigenvectors of \( L_H \), respectively. Thus one has \( \langle U', \ U' \rangle = \langle f / d \rangle \langle e^{i\theta} (H'), f / d \rangle \langle e^{i\theta} (H') \rangle \). The projection over the kernel of \( L_H \) and \( Q = 1 - P \) one finds \( \langle H' \rangle - 1 Q(\mathcal{H}) \langle H' \rangle \) \( + i^2 \langle P(\mathcal{H}), P(\mathcal{H}) \rangle = \langle f / d \rangle \langle e^{i\theta} (H') \rangle \), \( L_H \) is the projection on the basis \( \Psi_{n,m} \) of \( L_H \) eigenvectors, one finds the explicit result
\[ \chi_{\varepsilon / d}(\lambda) = \frac{1}{d} \sum_{n,m} \left( \langle n | H^\dagger | n \rangle \right)^2 F_n^2 (E_n - E_m) \]
\[ + i^2 \left( \frac{1}{d} \sum_{n} \left( \langle n | H^\dagger | n \rangle \right)^2 - \frac{1}{d^2} | \langle \varepsilon / d | H^\dagger | \varepsilon / d \rangle |^2 \right). \]

(10)

(notic that \( L_H \) is invertible, by definition, on the range of \( Q \), where \( F_n(x) = \sin(x/2) / (x/2) \).

The first term in the last line of (10) can be rewritten as \( \langle H' | F_n^2 (L_H) | H' \rangle \). Similarly, the term \( \langle U', U' \rangle \) in the metric can be written as \( \langle f / d \rangle \langle e^{i\theta} (H') \rangle \). Putting all together and expanding over the basis \( |\Psi_{n,m} \rangle \) of \( L_H \) eigenvectors, one finds the explicit result
\[ \chi_{\varepsilon / d}(\lambda) = \frac{1}{d} \sum_{n,m} \left( \langle n | H^\dagger | n \rangle \right)^2 F_n^2 (E_n - E_m) \]
\[ + i^2 \left( \frac{1}{d} \sum_{n} \left( \langle n | H^\dagger | n \rangle \right)^2 - \frac{1}{d^2} | \langle \varepsilon / d | H^\dagger | \varepsilon / d \rangle |^2 \right). \]

(11)

The key property here is that
\[ \lim_{t \to \infty} t^{-1} F_n^2 (x) = 2 \pi \delta(x). \]

This asymptotic \( \delta \) function is responsible for the large contribution to (11) given by small \( E_n - E_m \). This shows that all (quasi) level crossings in the spectrum of \( H \) might lead to an analyticity breakdown in \( \chi_{\varepsilon / d}(\lambda) \). This has to be contrasted with the ground-state (GS) fidelity studied, e.g., in [13] where just (quasi) level crossings in the GS play a role. In this sense it is clear why the OFS (11) is a more powerful tool than the corresponding state-space analog. On the other hand, it is worth mentioning that the sensitivity of the OFS to high-energy parts of the spectrum might render the analysis more complicated than in the ground-state fidelity case.

An important generalization of (11) can be obtained by considering \( \rho \) commuting with \( H \). In this case one obtains an expression analogous to (11) with the diagonal elements \( \rho_{n,n} = \langle n | \rho | n \rangle \) suitably inserted (see Sec. 3 A). Later in the paper we will consider a key instance of this commuting case: the Gibbs thermal state, i.e., \( \rho = Z^{-1} \exp(-BH) \) (is the partition function). For \( B \) sufficiently large, just the low-lying excited states of \( H \) play a role. This suppression of the high-energy signals should make the analysis of the OFS less cumbersome than in the \( \rho = 1/d \) case, i.e., \( \beta = 0 \).

**Splitting \( \chi_{\varepsilon / d} \)**

Now we derive an alternative form for the OFS (7) for the case of Example 2 discussed above. This form will make even more manifest the different contributions to operator susceptibility arising from eigenvalue and eigenvector variation with the control parameter \( \lambda \). A unitary operator can be written in diagonal form
\[ \langle U(\lambda) = \sum_i u_i(\lambda) P_i(\lambda). \]

(12)

where the \( u_i \)’s are the eigenvalues of \( U \) satisfying \( |u_i| = 1 \), and \( P_i = |\phi_i\rangle \langle \phi_i | \) is a one-dimensional projective operator. Both eigenvalues and eigenstates are parameter dependent. What we would like to do first is to separate the contributions of these two kinds of parameter dependence to the susceptibility \( \chi_{\varepsilon / d} \). In Ref. [16], the authors succeed in distinguishing the classical and quantum contributions to the Bures metric with consideration of these two kinds of dependence.

The differential of this unitary operator can be divided into two parts:
\[ \partial_\lambda U(\lambda) = D_1(\lambda) + D_2(\lambda), \]
\[ D_1(\lambda) = \sum_i (\partial_\lambda u_i) P_i, \quad D_2(\lambda) = \sum_i u_i (\partial_\lambda P_i), \]

(13)

where \( \partial_\lambda P_i = |\phi_i\rangle \langle \partial_\lambda \phi_i | + |\phi_i\rangle \langle \phi_i | \partial_\lambda \phi_i | \). Next, we assume that the density matrix \( \rho \) can be simultaneously diagonalized with \( U \), so it can be written as the form
\[ \rho = \sum_i \rho_i P_i. \]

(14)

The assumption \( [H, \rho] = 0 \) is motivated by the important example where both the time evolution unitary operators and density operators considered are generated by the same Hamiltonian, i.e., \( \rho \) is the Gibbs thermal state associated with \( H \).

Since
\[ \langle P_j \partial_\lambda P_i \rangle = \delta_{ij} \rho_{j} \langle \partial_\lambda \phi_i \rangle + \langle \partial_\lambda \phi_j | \phi_i \rangle = 0, \]

after substituting Eq. (13) into Eq. (7), we obtain
\[ \chi_{\rho}(\lambda) = \chi_{\rho}^{(1)}(\lambda) + \chi_{\rho}^{(2)}(\lambda), \]

(16)
Here we separate the contributions of $D_1$ and $D_2$. To make them explicit, we consider the time evolution operator $U(\lambda,t) = e^{-itH(\lambda)}$, and assume that the eigenstates changing with $\lambda$ can be connected through a smooth unitary transformation $S(\lambda)$ that is time independent and satisfies $S(\lambda)H(\lambda)S(\lambda) = H_d(\lambda)$, where $H_d$ is a diagonal Hamiltonian in the fixed $\lambda$-independent basis. Therefore, the unitary operators considered can be written

$$U(\lambda,t) = S(\lambda)U_d(\lambda,t)S(\lambda),$$

(18)

with $U_d = \exp(-itH_d)$. After differentiating with respect to $\lambda$, we get

$$D_1(\lambda,t) = -i t S(\lambda)[\partial_\lambda H_d(\lambda)]S(\lambda)U(\lambda,t),$$

$$D_2(\lambda,t) = [A(\lambda), U(\lambda,t)],$$

(19)

where

$$A(\lambda) = [\partial_\lambda S(\lambda)]^\dagger S(\lambda).$$

(20)

Substitution of Eq. (20) into Eq. (17) leads to

$$\chi_\rho^{(1)}(\lambda,t) = t^2[\langle \partial_\lambda H_d \rangle^2 \rho - \langle \partial_\lambda H_d \rangle^2].$$

(21)

This can be considered as the fluctuation of the quantity $\partial_\lambda H_d(\lambda)$ under the state $\rho$, with an extra factor $t^2$.

The second term of Eq. (16) will be

$$\chi_\rho^{(2)}(\lambda,t) = \langle [U(\lambda,t), A^\dagger(\lambda)] [A(\lambda), U(\lambda,t)] \rangle_\rho.$$ 

(22)

Since $U$ and $\rho$ can be diagonalized simultaneously, the above expression can be further rewritten as $\chi_\rho^{(2)}(\lambda,t) = 2\sum_{s=0}^N A_m |A_m|^2 [1 - \cos [(E_m - E_n)t]]$. This form explicitly shows that the time-dependent terms in $\chi_\rho^{(2)}(\lambda,t)$ are circular functions.

Now we would like to make a few general comments about Eqs. (21) and (22).

(i) They correspond one to one to the two terms in Eq. (11) (with suitably inserted $\rho_{l,m}$). In particular, this remark shows that $A_m = (E_m - E_n)^{-1}(n|H^*|m)$. Notice that $A$ is nothing but the infinitesimal generator of the adiabatic intertwiner mentioned in Sec III.

(ii) $\chi_\rho^{(1)}(t=1,2)$ separate apart the contributions of the eigenvalue variations from those of the eigenstates. In Ref. [16], a similar distinction was made for the Bures metric on thermal state manifolds (the corresponding terms there were named the classical and the quantum, respectively).

(iii) Equations (21) and (22) have quite different forms of time dependence. $\chi_\rho^{(1)}$ gives $t^2$ contributions explicitly, while the second term consists of circular functions with a finite period of criticality. When $t$ is large, if $\chi_\rho^{(1)}$ is not vanishing, it dominates the OFS and therefore the decay behavior of the operator fidelity.

(iv) If $\rho$ is pure (implying, since we have assumed the form $\rho = \sum_k \rho_k$, that the initial state is an eigenstate of the Hamiltonian) $\chi_\rho^{(1)}$ vanishes. Moreover, if $A(\lambda)$ commutes with $U(\lambda,t)$, and therefore the eigenvectors of $U(\lambda)$ are $\lambda$ independent, it is obvious from (22) that $\chi_\rho^{(2)}$ vanishes.

IV. THE TRANSVERSE FIELD ISING MODEL

In this section, we apply the general formalism developed so far to the concrete case of the transverse field Ising mode. Before doing that it is useful to discuss some what generally models having a factorization structure.

A. Factorizable models

In order to obtain the exact solution of $\chi_\rho(\lambda,t)$, we analyze the cases where the unitary operator $U(\lambda,t)$ and thermal state $\rho(\lambda,\beta)$ have the same factorization structure. This assumption holds true, for example, in cases where they are both generated by the same Hamiltonian $H(\lambda)$. These two quantities can then be written in the composite space

$$U(\lambda) = \otimes_{l=0}^M U_l(\lambda),$$

$$\rho(\lambda) = \otimes_{k=0}^M \rho_k(\lambda),$$

(23)

where $U_k$ is still a unitary operator and $\rho_k$ is still a density operator, corresponding to the $k$th subspace. Note that

$$\partial_\lambda U(l) = \sum_{l=0}^M \otimes_{k=0}^{l-1} U_k(\lambda) \otimes \partial_\lambda U(l) \otimes \left( \otimes_{k=0}^M U_k(\lambda) \right).$$

(24)

Since $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$ and $\text{Tr}(\rho) = 1$, the first and second terms of Eq. (7) can be expressed, respectively, as

$$\text{Tr}[\rho(\partial_\lambda U)^\dagger(\partial_\lambda U)] = \sum_I \text{Tr}[\rho(\partial_\lambda U)^\dagger(\partial_\lambda U)] + \sum A_{I' I},$$

$$|\text{Tr}[\rho U^\dagger(\partial_\lambda U)]|^2 = \sum_I |\text{Tr}[\rho U^\dagger(\partial_\lambda U)]|^2 + \sum A_{I' I},$$

(25)

where $A_{I' I} = \text{Tr}[\rho(\partial_\lambda U)^\dagger(\partial_\lambda U)]\text{Tr}[\rho U^\dagger(\partial_\lambda U)]$, and the subscript $I$ means the $I$th subspace.

Thus, the OFS $\chi_\rho$ for the factorized unitary operator is

$$\chi_\rho(\lambda) = \sum_I \chi_\rho_I(\lambda),$$

(26)

$$\chi_{\rho I}(\lambda) = \text{Tr}[\rho(\partial_\lambda U)^\dagger(\partial_\lambda U)] - |\text{Tr}[\rho U^\dagger(\partial_\lambda U)]|^2.$$ 

(27)
B. Ising Hamiltonian

We now consider transverse field Ising model. The Hamiltonian is given by

\[ H(\lambda) = -J \sum_{l=1}^{M} (\sigma^x_l \sigma^x_{l+1} + \lambda \sigma^z_l), \]  

(28)

where \(J\) is an exchange constant hereafter assumed to be unity, and \(\lambda\) is the transverse field strength. This model can be calculated analytically by using the Jordan-Wigner transformation \([20,21]\)

\[ \sigma^x_l = \prod_{j < l} (1 - 2c^+_j c_j), \]  

(29)

\[ \sigma^z_l = 1 - 2c^+_l c_l, \]

which maps the spins to fermions. After Fourier transformation, the Hamiltonian in the momentum space is

\[ H(\lambda) = - \sum_{k=1}^{M} \left( \cos \frac{2\pi k}{N} - \lambda \right) (d^+_k d_k + d^+_k d_{-k} - 1) + i \sin \frac{2\pi k}{N} (d^+_k d_{-k} - d_k d_{-k}), \]

(30)

where \(N = 2M + 1\). This Hamiltonian can be exactly solved by a Bogoliubov transformation \([22]\). However, introduction of a set of pseudospin operators is more convenient here. Since \(n_k - n_{-k} (n_k = d^+_k d_k)\) commutes with every term of the Hamiltonian (30), the pseudo Pauli operators can be defined by \([23]\]

\[ s_{kx} = d^+_k d_k + d_{-k} d_{-k}, \]  

\[ s_{ky} = -i (d^+_k d_{-k} - d_k d_{-k}), \]  

\[ s_{kz} = d^+_k d_k + d^+_k d_{-k} - 1. \]

(31)

These give the Pauli matrix in the \(n_k - n_{-k} = 0\) subspace, and become a zero matrix in the \(n_k - n_{-k} = \pm 1\) subspaces. Therefore, \(s_{0x}\), \(s_{0y}\), and \(s_{0z}\) are the standard Pauli matrices \(\sigma_{0x}\), \(\sigma_{0y}\), and \(\sigma_{0z}\).

With these operators, the Hamiltonian can be written as

\[ H(\lambda) = \sum_{k=1}^{M} S_k^0(\lambda) H_{k,d}(\lambda) S_k(\lambda) + (\lambda - 1) s_{0z}, \]

\[ H_{k,d} = \Omega_k s_{kz}, \]  

\[ S_k(\lambda) = \exp \left( -\frac{i}{2} \frac{\theta_k}{\Omega_k} \right), \]

(32)

where

\[ \Omega_k = -2 \sqrt{\left[ \lambda - \cos(2\pi k/N) \right]^2 + \sin^2(2\pi k/N)}, \]

\[ \theta_k = \arcsin \left( \frac{2 \sin(2\pi k/N)}{\Omega_k} \right). \]

(33)

We consider the fidelity of two time evolution operators \(U(\lambda) = \exp[-i t H(\lambda)]\) and \(U(\lambda + \Delta \lambda)\) with thermal state \(\rho\)

\[ \rho = \frac{\exp[-\beta H(\lambda)]}{Z(\beta, \lambda)} = \frac{\text{Tr} \exp[-\beta H(\lambda)]}{Z(\beta, \lambda)}, \]

(34)

where

\[ U_k = S_k^0(\lambda) \exp[-i t H_{k,d}(\lambda)] S_k(\lambda), \]  

\[ \rho_k = S_k^0(\lambda) \exp[-i t H_{k,d}(\lambda)] S_k(\lambda), \]

(35)

and the partition functions are

\[ Z_0 = 2 \cosh(\beta(\lambda - 1)), \quad Z_k = 2 \left[ 1 + \cosh(\beta \Omega_k) \right]. \]

(36)

For \(k > 0\), after substituting \(H_{k,d} = \Omega_k s_{kz}\) into Eq. (21), we have

\[ \chi^1_{\rho,k}(\lambda, t) = 4 t^2 \frac{\cos^2 \theta_k}{\cos(\beta \Omega_k) + 1}. \]

(37)

To calculate \(\chi^2_{\rho,k}\), we should first calculate the related quantities \(A_k\) defined by Eq. (20) and the commutator

\[ [A_k, U_k] = i \frac{\theta_k}{2} s_{kz}, \]

(38)

where \(\theta_k\) denotes \(\partial \theta_k\) for convenience. Substituting them into Eq. (22), we obtain

\[ \chi^2_{\rho,k} = 4 \frac{\cosh(\beta \Omega_k) \sin^2 \theta_k \sin^2(\Omega_k)}{\cosh(\beta \Omega_k) + 1} \Omega_k. \]

(39)

For the \(k = 0\) subspace, the contribution is

\[ \chi^1_{\rho,0}(\lambda, t) = 4 t^2 \left[ 1 - \tanh^2(\beta(\lambda - 1)) \right], \quad \chi^2_{\rho,0} = 0. \]

(40)

Thus, we get the OFS

\[ \chi^1_{\rho} = 4 t^2 \left[ 1 - \tanh^2(\beta(\lambda - 1)) \right] + 4 \sum_{k=1}^{M} r^2 \frac{\cos^2 \theta_k}{\cos(\beta \Omega_k) + 1}, \]

\[ \chi^2_{\rho} = 4 \sum_{k=1}^{M} \frac{\cosh(\beta \Omega_k) \sin^2 \theta_k \sin^2(\Omega_k)}{\cosh(\beta \Omega_k) + 1} \Omega_k^2. \]

(41)

Notice that these expressions, for the XY model, can also be obtained directly from Eq. (11). For a given \(\lambda\), we can consider the time-averaged OFS which is given by

\[ \overline{\chi^1_{\rho} (\lambda)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \chi^1_{\rho}(\lambda, t). \]

(42)

Obviously \(\overline{\chi^1_{\rho} (\lambda)}\) diverges, since \(\chi^1_{\rho}(\lambda)\) is proportional to \(t^2\). The time average of the circular function contributions \(\chi^2_{\rho}\) is
For a fixed dynamical limit, we get results for ground states in Ref. 5. In the short-time LE decay is enhanced at criticality, which is important. However, when \( LU \) et al. 44 find both the quantities have nontrivial behavior at the critical point \( \lambda = 1 \) and, therefore, can be used as indicators of criticality. At \( \lambda = 1 \), \( \chi^{(1)}_\rho \) has a minimum (see Fig. 1), while \( \chi^{(2)}_\rho \) has a maximum, (see Fig. 3).

Notice that \( \chi^{(1)}_\rho \) vanishes only when the inverse temperature \( \beta = 0 \). When the temperature is not zero, for long enough time, \( \chi^{(1)}_\rho \) will be dominated by \( \chi^{(1)}_\rho \) and therefore will have a minimum at \( \lambda = 1 \). This has to be contrasted with the previous results for ground states in Ref. [5] where it is argued that the short-time LE decay is enhanced at criticality (large \( N \)). For a fixed \( t \) and \( \lambda \), as temperature increases, \( \chi^{(1)}_\rho \) grows while \( \chi^{(2)}_\rho \) decays.

The time average of the second term of the OFS will diverge at the critical point \( \lambda = 1 \) under the thermodynamical limit (see Fig. 2). It is caused by the denominator \( \Omega_k \) in Eq. (44), since it will vanish for those very small \( k \)’s at the critical point \( \lambda = 1 \) when \( N \) goes to infinity. Thus, this divergence will be retained for all \( \beta \) (see Fig. 3).

In principle, one obtains \( \chi^{(1)}_\rho (\lambda, t) \) and \( \chi^{(2)}_\rho (\lambda) \) from experiments in different ways. If we use \( \chi^{(1)}_\rho (\lambda, t) \) to investigate the quantum critical point, the measurement of the time interval \( t \) is important. However, when \( \chi^{(2)}_\rho (\lambda, t) \) vanishes, we can use \( \chi^{(2)}_\rho (\lambda) \) to investigate quantum criticality. This means we can get a \( \chi^{(2)}_\rho (\lambda, t) \) for a random \( t \) again and again, and then evaluate the time-averaged value. By this scheme, we can avoid the demand for an exact measurement of the time interval from the coupling instant to the measuring instant.

**V. Conclusion**

To summarize, we have introduced the state-dependent operator fidelity and its susceptibility \( \chi_\rho \) to study environment-induced decoherence. For the important case in which the state \( \rho \) commutes with the Hamiltonian, we derived general expressions for \( \chi_\rho \). These latter allow one to tell apart two different contributions, \( \chi^{(1)}_\rho \) and \( \chi^{(2)}_\rho \). These two terms have different physical origin and temporal behavior. For the transverse field Ising model, we obtained an exact expression for \( \chi_\rho \) and showed that it has nontrivial behavior at the critical point at both zero and nonzero temperature.

Even though operator fidelity does not necessarily provide a better tool than the ground-state one for studying criticality, it might have other interesting applications in addition to the decoherence problem addressed in this paper. For example, from (11) it is clear that the OFS depends crucially on the level spacing distribution of \( H \). This leads to conjecture that the OFS might be an effective tool to investigate the transition to quantum chaos as well. Finally, we believe this type of analysis is directly relevant to experiments aimed at using quantum probes to detect QPTs.

**FIG. 1.** First term of OFS divided by \( t^2 \) as a function of \( \lambda \) with different temperatures in the thermodynamical limit, for the transverse field Ising model.

\[
\chi^{(2)}_\rho (\lambda) = 2 \sum_{k=1}^{M} \frac{\cosh(\beta \Omega_k)}{\cosh(\beta \Omega_k) + 1} \sin^2 \theta_k. \tag{43}
\]

After rescaling \( \chi_\rho \rightarrow \chi_\rho / N, 2\pi k/N \rightarrow k \) and taking the thermodynamical limit, we get

\[
\chi^{(1)}_\rho (\lambda, t) = \frac{2}{\pi} \int_0^\pi dk \frac{1}{\cosh(\beta \Omega_k) + 1} \cos^2 \theta_k.
\]

\[
\chi^{(2)}_\rho (\lambda, t) = \frac{2}{\pi} \int_0^\pi dk \frac{\cosh(\beta \Omega_k) \sin^2 \theta_k}{\cosh(\beta \Omega_k) + 1} \Omega_k^2.
\]

\[
\chi^{(3)}_\rho (\lambda) = \frac{1}{\pi} \int_0^\pi dk \frac{\cosh(\beta \Omega_k) \sin^2 \theta_k}{\cosh(\beta \Omega_k) + 1} \Omega_k^2. \tag{44}
\]

We stress that these equations for the OFS are valid just for the thermal-state case \( \rho = Z^{-1} \exp(-\beta H) \). One finds that both the quantities \( \chi^{(1)}_\rho (\lambda, t) \) and \( \chi^{(2)}_\rho (\lambda) \) have nontrivial behavior at the critical point \( \lambda = 1 \) and, therefore, can be used as indicators of criticality. At \( \lambda = 1 \), \( \chi^{(1)}_\rho \) has a minimum (see Fig. 1), while \( \chi^{(2)}_\rho \) has a maximum, (see Fig. 3).

**FIG. 2.** Time average of the second term of the OFS as a function of \( \lambda \) for a given \( \beta = 1 \) with different \( M \), for the transverse field Ising model. \( N=2M+1 \) is the number of spins.

**FIG. 3.** (Color online). Time average of the second term of the OFS as a function of \( \lambda \) and \( \beta \) for a given \( M=1000 \) (2001 spins).
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