Optimal distinguishing observables for quantum parameter estimation

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The quantum Cramér-Rao bound (QCRB) gives the ultimate limit on the precision of estimating an unknown parameter via quantum measurements followed by data-processing. A typical kind of data-processing in realistic cases is inferring the parameter from the expectation value of a physical observable. We derive the necessary and sufficient condition for the optimal observable to saturate the QCRB by invoking the Schrödinger-Robertson uncertainty relation. We elucidate the distinction between the optimal observables and the optimal measurements followed by general data-processing for the global optimization in the whole range of the parameter. We also point out that the optimal separable observable does not exist for the cases of typical mixed states.

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I. INTRODUCTION

An essential task in quantum parameter estimation is to suppress the fundamental bound on measurement precision imposed by quantum mechanics. Various quantum strategies have been developed and implemented to enhance the accuracy of the parameter estimation, which are closely related to some practical applications, such as the Ramsey spectroscopies, atomic clocks, and the gravitational wave detection [1–4]. Two approaches in common use for high-precision measurements are the parallel protocol with correlated (entangled or squeezed) multi-probes [5] and multi-round protocol with a single probe [6, 7]. Most recently, some novel methods, like environment-assisted metrology [8] and quantum error correction for metrology [9–12], are raised to achieve high precision in realistic experiments.

From quantum estimation theory [13–15], the ultimate limit to the estimation precision of the parameter \( \phi \) in quantum states, measured by the unit-corrected mean-square-root error of the estimator \( \hat{\phi}_{\text{est}} \) as [13, 16],

\[
\delta \phi_{\text{est}} := \left( \frac{\phi_{\text{est}}}{d(\phi_{\text{est}})/d\phi} - \phi \right)^2 \sqrt{\frac{1}{\nu}},
\]

is bounded from below by

\[
\delta \phi_{\text{est}} \geq \frac{1}{\sqrt{\nu} \sqrt{F_\phi}},
\]

where \( \nu \) is the repetitions of the experiment, and \( F_\phi \) is the quantum Fisher information (QFI) (see Eq. 5 for definition), which measures the statistical distinguishability of the parameter in quantum states. This lower bound is known as the quantum Cramér-Rao bound (QCRB), and it is asymptotically achieved for large \( \nu \) by optimal measurements followed by the maximum-likelihood estimator [13, 16]. Among the general measurements and general estimators (representing the data-processing of the measurement outcomes), a particular approach, which is often used in realistic experiments, is estimating the values of the parameter through the average value of a physical observable represented by a Hermitian operator [3, 17]. This motives us to consider the effectiveness of this observable approach.

In this paper, we will address this problem by invoking the Schrödinger-Robertson uncertainty relation to derive the necessary and sufficient condition for the optimal distinguishing observable saturating the QCRB-based sensitivity. We show the intrinsic relation between the optimal distinguishing observables and the symmetric logarithmic derivative (SLD) operators. For local optimization in the range of the parameter, the observable approach, like the general estimator approach, can always saturate the QCRB. However, for global optimization in the whole range of the parameter, there exist cases where a global optimal measurement followed by maximum-likelihood estimator can saturate the QCRB, but a global optimal observable does not exist. Finally, we take a typical model of parameter estimation for example, and discuss the problem of whether the QCRB-based sensitivity can be achieved via the separable observable approach for the cases of the initial mixed states which are evolved from the Greenberger-Horne-Zeilinger (GHZ) state under three different decoherence channels.

This paper is structured as follows. In Sec. III we briefly review the quantum parameter estimation via general estimators and that via average value of observables. In Sec. III we show the necessary and sufficient condition for the optimal distinguishing observable and compare with that for the optimal distinguishing measurement. In Sec. IV we discuss the optimal separable observable for the cases of typical mixed states. At last, a conclusion is given in Sec. V.

II. ULTIMATE BOUND FOR PARAMETER ESTIMATION

In this section, we first briefly review quantum parameter estimation, then, by invoking the Schrödinger-Robertson uncertainty relation, we investigate the case...
where the parameter is inferred through the expectation value of an observable and elucidate the distinction between the optimal observable and the optimal measurement with general estimators.

A general procedure of the quantum parameter estimation can be abstractly modeled by three steps: preparing the probe state, then evolving the state under the parameter-dependent Hamiltonian after which the state evolving the state under the measurement-induced probability distribution

\[ \rho_{\text{out}} = M_{\varphi} \rho_{\text{in}} \]

is parameter-dependent (parametrization), finally estimating the probe state, then evolving the state under the measurement followed by an estimator mapping the outcomes to the estimation value of the parameter. A particular case in common use is estimating the parameter through the expectation value of a distinguishing observable \( A \). In such case, the estimation error can be given by the error-propagation formula as

\[
\Delta \varphi_{\text{est}}^2 := \frac{\langle \Delta A^2 \rangle}{\partial_{\varphi}(A)^2}
\]

with \( \Delta A := A - \langle A \rangle \). For this scenario, the achievable lower bound of the estimation error can be easily derived by the Robertson-Schrödinger uncertainty relation, or equivalently, the Schwarz inequality, as follows. Let us first recall the Schrödinger-Robertson uncertainty relation (SRUR) [19, 20], which states that the uncertainty of two non-commuting observables must obey the following inequality

\[
\langle \Delta A^2 \rangle\langle \Delta B^2 \rangle \geq \frac{1}{4} \langle [A, B] \rangle^2 + \frac{1}{4} \langle [\Delta A, \Delta B]_+ \rangle^2,
\]

where \( [\cdot, \cdot]_+ \) denotes the anti-commutator, see Ref. [13]. It is remarkable that \( L \) may not be uniquely determined by Eq. (3), when \( \rho_{\varphi} \) is not of full rank [18]. If \( K \) is a Hermitian operator satisfying \( K \rho_{\varphi} = 0 \) and \( L \) is a SLD operator for \( \rho_{\varphi} \), then \( L + K \) is also a SLD operator for \( \rho_{\varphi} \).

In general, the value of the parameter \( \varphi \) cannot be directly measured. The most general method for estimating the parameter encoded in quantum states involves a generalized quantum measurement followed by an estimator mapping the outcomes to the estimation value of the parameter. A particular case in common use is estimating the parameter through the expectation value of a distinguishing observable \( A \). In such case, the estimation error can be given by the error-propagation formula as

\[
\Delta \varphi_{\text{est}}^2 := \frac{\langle \Delta A^2 \rangle}{\partial_{\varphi}(A)^2}
\]

where \( v \) is the number of the experiments and

\[
F_{\varphi|M} := \sum_x p_{\varphi}(x)[\partial_{\varphi} \ln p_{\varphi}(x)]^2
\]

is the Fisher information of the measurement-induced probability distribution \( p_{\varphi}(x) = \text{Tr}(\rho_{\varphi} M_x) \). The maximal Fisher information over all the POVMs is given by the so-called quantum Fisher information (QFI) defined by

\[
F_{\varphi} := \text{Tr}(\rho_{\varphi} L^2)
\]

with \( L \) being the symmetric logarithmic derivative (SLD) operator, which is the Hermitian operator determined by

\[
\partial_{\varphi} \rho_{\varphi} = \frac{1}{2}[\rho_{\varphi}, L]_+
\]
When the experiment is repeated $v$ times, the right-hand side of Eq. (12) is reduced by a factor of $v^{-1}$. Here the bound in Eq. (13) is the so-called quantum Cramér-Rao bound (QCRB) described in Eq. (4), and the bound in Eq. (12) is called as the generalized quantum Cramér-Rao bound (see Fig. 1).

III. OPTIMAL DISTINGUISHING OBSERVABLES

Now, we consider the achievability of the above bounds and the necessary and sufficient condition for optimal distinguishing observables. From the necessary and sufficient condition for equality in Schwarz inequality, the equality in Eq. (12) holds if and only if

$$\Delta A \sqrt{\rho_{\varphi}} = \alpha L \sqrt{\rho_{\varphi}}$$

(15)

is satisfied with some nonzero complex number $\alpha$. Furthermore, the equality in Eq. (13) holds if and only if

$$\text{Im} \langle AL \rangle = 0.$$  (16)

This condition Eq. (16) can be combined into the condition Eq. (15) by restricting $\alpha$ to be a real number, that is,

$$\Delta A \sqrt{\rho_{\varphi}} = \alpha L \sqrt{\rho_{\varphi}} \quad \text{with } \alpha \in \mathbb{R} \setminus \{0\}$$

(17)

is the necessary and sufficient condition for the optimal distinguishing observable $A$ that makes $\Delta \varphi_{\text{est}}$ achieve the QCRB. For pure states $\rho_{\varphi} = |\psi_{\varphi}\rangle \langle \psi_{\varphi}|$, the condition Eq. (17) is equivalent to

$$\Delta A |\psi_{\varphi}\rangle = \alpha L |\psi_{\varphi}\rangle \quad \text{with } \alpha \in \mathbb{R} \setminus \{0\}.$$  (18)

The necessary and sufficient condition Eq. (17) implies that for any optimal distinguishing observable $A$, the deviation operator $\Delta A$ must be proportional (with a real factor $\alpha$) to the SLD operator, since $L \sqrt{\rho_{\varphi}}$ is uniquely determined and any Hermitian operator $O$ satisfying $O \sqrt{\rho_{\varphi}} = L \sqrt{\rho_{\varphi}}$ is also a SLD operator for $\rho_{\varphi}$. In other words, Eq. (17) is equivalent to

$$[\Delta A, \rho_{\varphi}]_+ = 2\alpha \partial_{\varphi} \rho_{\varphi} \quad \text{with } \alpha \in \mathbb{R} \setminus \{0\},$$

(19)

which shows the intrinsic relation between the optimal distinguishing observables and the SLD operators.

Moreover, the theoretic optimal distinguishing observables are in general dependent of the value of the parameter, meaning that a pre-estimation giving a rough value of the parameter is required, according to which the optimal observable is chosen. In other words, the global optimal observable should be independent on the value of the parameter. Meanwhile, a measurement is said to be globally optimal, if it attains the maximal Fisher information at every point in the whole range of the parameter. We will show that even a global optimal measurement exists for a parameter estimation model, a
global optimal distinguishing observable may not exist, as the latter implicitly means that the estimator (data-processing) is restricted to a very special kind—infering the value of the parameter inversely from the expectation value of the observable.

Remind that the necessary and sufficient conditions for the optimal POVM, with which Fisher information of the measurement-induced probability distribution achieves the maximum, i.e., QFI of the quantum states, are given in Ref. [13] as follows:

$$\sqrt{M_x} \sqrt{\rho_{\varphi}} = u_x \sqrt{M_x L} \sqrt{\rho_{\varphi}}$$  (20)

for all the outcomes $x$, where $u_x$ are real numbers. Consider a parametric family of states

$$\rho_{\varphi} = \sum_j \lambda_{\varphi,j} |j\rangle \langle j|.$$  (21)

where $\lambda_{\varphi,j}$ are nonnegative numbers dependent of $\varphi$, and $\{|j\rangle\}$ is a basis independent of $\varphi$. Obviously, $\{|j\rangle\langle j|\}$ is a global optimal measurement, i.e., satisfying Eq. (20). To see that the global optimal distinguishing observables do not exist, let us assume that $\rho_{\varphi}$ is of full rank so that the SLD operator is uniquely determined and given by

$$L = \sum_j \partial_{\varphi} \ln \lambda_{\varphi,j} |j\rangle \langle j|.$$  (22)

Since the SLD operator is unique, the optimal observable must satisfy

$$A = \langle A \rangle + \alpha L$$

$$= \sum_j (\langle A \rangle + \alpha \partial_{\varphi} \ln \lambda_{\varphi,j}) |j\rangle \langle j|.$$  (23)

Even though we consider $\langle A \rangle$ and $\alpha$ as totally free variable, we cannot make $A$ be $\varphi$-independent for a general non-degenerate $L$. That is to say, there are some cases that a global optimal measurement exists but a global optimal observable does not.

A. Separable observable and local measurement

In realistic setups, it is often of interest to estimate the value of the unknown parameter by calculating the expectation of the separable observables, which are described by the tensor products of Hermitian matrices $I$. In Ref. [5], Giovannetti et al. have demonstrated that the separable observable does saturate the Heisenberg limit (HL) and they also gave an explicit example of an $N$-qubit parametric pure state

$$|\psi_{\text{GHZ}}(\varphi)\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes N + e^{iN\varphi} |1\rangle \otimes N),$$  (24)

for which the optimal distinguishing observable is given by $\sigma^x_\varphi$ with $\sigma_x$ the Pauli matrix. Therefore, for the
state in Eq. (24), there must exist the local measurement to attain the Fisher information in the corresponding HL. Here, the state of Eq. (24) is the \( N \)-qubit GHZ state of which each qubit has been rotated by \( \varphi \) along \( z \)-axis. A further discussion of the general expression of the optimal separable observable for this state will be given in Sec. IV A.

However, in a recent work [21], it was pointed out that the local measurement (the restricted readout procedure) might not be possible to go beyond the shot-noise limit (SNL) even for arbitrary entangled states. It seems that this conclusion is inconsistent with ours in the above discussion. In what follows, we clarify this issue by revisiting the method in [21] and showing the causes for this inconsistency.

For simplicity, let us consider the two-qubit state
\[
|\psi^{(2)}_{GHZ}(\varphi)\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{2i\varphi}|11\rangle).
\] (25)

Following Ref. [21], we restrict the local measurement to be the projective measurements \(|+\rangle\langle+|, |−\rangle\langle−|\) for each qubit with
\[
|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle).
\] (26)

Whether the above restricted measurement is the optimal measurement saturating the QCRB can be tested by asking whether or not the operators of the form
\[
K = \lambda_{++}|++\rangle\langle++| + \lambda_{+-}|+−\rangle\langle−+| + \lambda_{−+}|−+\rangle\langle−+| + \lambda_{−−}|−−\rangle\langle−−|
\] (27)

can be the SLD operator for the state of Eq. (25). By demonstating that for the state in Eq. (20) with \( \varphi = 0 \), there is no solution of the SLD equation \( \Delta \) for the coefficients \( \{\lambda_{++}, \lambda_{+-}, \lambda_{−+}, \lambda_{−−}\} \) in Eq. (27), the authors in Ref. [21] claimed that the projective measurement about \( \{|+\rangle, |−\rangle\} \) is not the optimal measurement for the state of Eq. (25).

Below, we shall show that for any other point except for \( \varphi = 0 \) in the range of the parameter, there do exist the SLD operator in form of Eq. (27). First, note that the SLD operator for the non-full-rank density matrices is not uniquely determined, but \( L \rho_\varphi \) (or \( L|\psi_\varphi\rangle \)) for pure state) is uniquely determined. Second, for pure states \( \rho_\varphi = |\psi_\varphi\rangle\langle\psi_\varphi| \), we have
\[
L = 2\partial_\varphi \rho_\varphi,
\] (28)

which can be seen by taking the differential with respect to \( \varphi \) on both sides of the identity equation \( \rho_\varphi^2 = \rho_\varphi \) and comparing with the SLD equation \( \Delta \). We then can see that
\[
L = -2ie^{-2i\varphi}|00\rangle\langle11| + 2ie^{2i\varphi}|11\rangle\langle00|.
\] (29)

is a SLD operator for the state of Eq. (26). Third, since \( L|\psi_\varphi\rangle \) is uniquely determined, then if \( K \) is the SLD operator for \( |\psi_\varphi\rangle \) if and only if
\[
L|\psi_\varphi\rangle = K|\psi_\varphi\rangle
\] (30)
is satisfied. Thus, substituting Eqs (25), (27) and (29) into Eq. (30), we obtain the solutions for the coefficients as
\[
\lambda_{++} = \lambda_{−−} = -2\tan \varphi, \quad \lambda_{+-} = \lambda_{−+} = 2\cot \varphi.
\] (31)

Apparently, for \( \varphi = 0 \), which is the case considered in [21], the above solution is singular. Whilst, for a general value of the parameter, the restricted local measurement considered here indeed saturate the HL-scaling sensitivity for the parametric state of Eq. (25).

\[A. \text{ Unitary parametrization}\]

In general, the parametrization processes of consideration are mainly divided into two groups: unitary [5] and non-unitary [22, 23]. In what follows, we consider that the parameter \( \varphi \) of interest is imprinted into the probe state via a unitary operation
\[
U_\varphi = \exp(-iG\varphi)
\] (32)

with \( G \) the Hermitian generator (see Fig. 1). Then the probe state becomes dependent on the parameter \( \varphi \) as
\[
\rho_\varphi = U_\varphi \rho_0 U_\varphi^\dagger.
\] (33)

Differentiating the above equation with respect to \( \varphi \) yields \( \partial_\varphi \rho_\varphi = -i[G, \rho_\varphi] \).

For pure states, the SLD operator is given by Eq. (28). By submitting Eq. (28) into Eq. (18), then the condition Eq. (18) reduces to
\[
\Delta A|\psi_\varphi\rangle = -2i\alpha \Delta G|\psi_\varphi\rangle \quad \text{with } \alpha \in \mathbb{R}
\] (34)

and \( \Delta G := G - \langle G \rangle \). The above equation was alternatively obtained by Hofmann [24], with which the author demonstrated that the measurement of photon number difference operator in two-path lossless optical interferometry achieves ultimate sensitivity given by the QCRB for all path-symmetric states [24]. It is remarkable that the condition Eq. (34) is the special case of the general condition Eq. (17).

\[IV. \text{OPTIMAL SEPARABLE OBSERVABLES FOR GHZ STATE WITH AND WITHOUT DECOHERENCE}\]

As mentioned in Sec. III A, it was shown that the separable observable \( \sigma^N \) can saturate the HL-scaling sensitivity for the parametric maximally entangled (PME) state given in Eq. (24) [3]. We now proceed to a discussion of whether the QCRB-based sensitivity can be achieved by the separable observable for the cases of typical mixed states which are evolved from the GHZ state under three different decoherence channels.

Let us consider a typical model: a quantum system consisting of \( N \) samplings (photons or atoms), which is
initially prepared in a known quantum state, and then evolves under the \( \varphi \)-dependent linear unitary operation \( U_\varphi \) (see Eq. (32) [2, 23]. The samplings are described as qubits, with standard basis states: \( |0\rangle \) and \( |1\rangle \), which may be used to represent the two modes of the photon or the internal states of the two-level atom in realistic setups. Here, we assume the generator \( G \) of unitary operator \( U_\varphi \) is

\[
G = J_z = \sum_{i=1}^{N} \frac{\sigma_z^{(i)}}{2}, \tag{35}
\]

where \( \sigma_z^{(i)} \) denotes the Pauli matrix acting on the \( i \)th qubit. This model has been widely investigated both for Mach-Zehnder interferometry [20, 27] and Ramsey interferometry [1, 28, 29].

After the unitary evolution \( U_\varphi \), the final state becomes dependent of the parameter \( \varphi \), and then it is followed by measurement of an observable \( A \). Here, we restrict \( A \) to be a separable observable which can be written in the following form

\[
A = A_q^{\otimes N}, \tag{36}
\]

where each local observable \( A_q \), acting on each qubit, may depend on four real coefficients \( \{a_0, a_1, a_2, a_3\} \), i.e.,

\[
A_q = a_0 I + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z. \tag{37}
\]

Then the task is to investigate whether there exist optimal separable observable of the above form saturating the QCRB-based sensitivity for the mixed probe states which are generated from the GHZ state under decoherence processes. Prior to this task, we first derive the general expression of the optimal separable observable for the case of the initial state to the GHZ state.

### A. GHZ state

We assume the system is initially prepared in an \( N \)-qubit GHZ state which is known as the maximally entangled state

\[
|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^\otimes N + |1\rangle^\otimes N). \tag{38}
\]

It was demonstrated that this state can saturate the ultimate HL 1/\( N \) on precision of measurement, which is a quadratic improvement over the SNL 1/\( \sqrt{N} \) [3]. Under the parametrization process generated by Eq. (35), the state of Eq. (38) accumulates a phase \( \varphi \) and then becomes the PME state given in Eq. (24) up to a global phase factor \( e^{-iN\varphi/2} \).

According to Eq. (28), we find a SLD operator for the PME state as

\[
L = -iNe^{-iN\varphi}(|0\rangle\langle 1|)^{\otimes N} + iNe^{iN\varphi}(|1\rangle\langle 0|)^{\otimes N} \tag{39}
\]

To determine the general optimal separable observable in form of Eq. (36) saturating the HL, we need to find the solutions of the four coefficients in Eq. (37) so that the necessary and sufficient condition Eq. (18) is satisfied. We find that the condition Eq. (18) is always satisfied for \( a_0 = a_3 = 0 \) and arbitrary real number \( a_1, a_2 \) that are not vanished simultaneously. Thus the general expression of the optimal separable observable for the PME state reads

\[
A_{\text{opt}} = (a_1 \sigma_x + a_2 \sigma_y)^{\otimes N}, \tag{40}
\]

which is independent of the parameter \( \varphi \), i.e., globally optimal in the whole range of the parameter. It is easy to be checked as follows. According to the error-propagation formula Eq. (7), we have

\[
\Delta \varphi_{\text{GHZ}} = \frac{\sqrt{\langle A_{\text{opt}}^2 \rangle - \langle A_{\text{opt}} \rangle^2}}{|\partial_{\varphi} \langle A_{\text{opt}} \rangle|} = \frac{1}{N}, \tag{41}
\]

as a result of

\[
\langle A_{\text{opt}} \rangle = \text{Re}[e^{-iN\varphi}(a_1 + ia_2)^N], \tag{42}
\]

\[
\langle A_{\text{opt}}^2 \rangle = (a_1^2 + a_2^2)^N. \tag{43}
\]

From Eq. (41), we see that the separable observable of Eq. (40) is indeed the optimal distinguishing observable for the PME state of Eq. (24).

### B. Decohered GHZ states

Below, we consider the system is initially prepared in the mixed states which are evolved from the GHZ state under decoherence processes. A quantum noisy dynamical process can be generally described by a map \( \mathcal{E} \) using the Kraus representation

\[
\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger, \tag{44}
\]

where \( K_i \) are the Kraus operators satisfying \( \sum_i K_i^\dagger K_i = \mathbb{1} \), which leads to the map being a completely positive and trace-preserving map [30].

We take account of three different decoherence channels: phase-damping channel (PDC), depolarizing channel (DPC), and amplitude-damping channel (ADC), and the corresponding Kraus operators are given by

\[
\mathcal{E}_{\text{PDC}} : \left\{ K_i = \sqrt{1-\eta} |i\rangle\langle i| \right\}_{i=0,1}, K_2 = \sqrt{\eta} \mathbb{1}_2;
\]

\[
\mathcal{E}_{\text{DPC}} : K_0 = \frac{\sqrt{1+3\eta}}{2} \mathbb{1}_2, \left\{ K_i = \frac{\sqrt{1-\eta}}{2} \sigma_i \right\}_{i=1,2,3};
\]

\[
\mathcal{E}_{\text{ADC}} : K_0 = \left( \begin{array}{cc} \sqrt{\eta} & 0 \\ 0 & 1 \end{array} \right), K_1 = \left( \begin{array}{cc} 0 & 0 \\ \sqrt{1-\eta} & 0 \end{array} \right),
\]

where \( \eta \in [0,1] \) may be time-dependent [30]. All these decoherence channels have been widely investigated in the modern science of the quantum information [31, 32].
and each one may correspond to a prototype model in realistic setups. For instance, the ADC is a schematic model of the decay of an excited state of a two-level atom due to spontaneous emission of a photon \[50\].

We suppose that the decoherence processes act locally on the qubits system which is initially prepared in the GHZ state. Then the time evolution of the density matrices of the system can be represented as

\[
\mathcal{E}_{\text{total}}(\rho_{\text{GHZ}}) = \frac{1}{2} \left[ \mathcal{E}_x((0)(0))^\otimes N + \mathcal{E}_x((0)(1))^\otimes N + \mathcal{E}_x((1)(0))^\otimes N + \mathcal{E}_x((1)(1))^\otimes N \right],
\]

where \(x\) corresponds to PDC, DPC, and ADC, respectively. Subsequently, we encode the density matrices given in Eq. \[45\] via the unitary operation \(U_{\phi}\) generated by Eq. \[35\], and they become dependent on the parameter \(\phi\). All these parametric density matrices can be expressed in the block-diagonal form as \[34\]

(i) PDC: 
\[
\rho_{\text{PDC}}(\phi) = \begin{pmatrix} \theta_0 & \xi_0 \otimes 1 \\ \xi_0 \otimes 1 & \theta_0 \end{pmatrix},
\]

with \(\theta_0\), zero-matrix of dimension \(d_1 = 2N - 2\).

(ii) DPC: 
\[
\rho_{\text{DPC}}(\phi) = \begin{pmatrix} \theta_0 & \xi_0 \otimes 1 \\ \xi_0 \otimes 1 & \theta_0 \end{pmatrix},
\]

where \(\theta_0\) is the binomial coefficient.

(iii) ADC: 
\[
\rho_{\text{ADC}}(\phi) = \begin{pmatrix} \theta_0 & \xi_0 \otimes 1 \\ \xi_0 \otimes 1 & \theta_0 \end{pmatrix},
\]

where \(\theta_0\) is the binomial coefficient.

By applying the method described in appendix \[A\] the analytical expressions of the SLD operator and the QFI for the three parametric density matrices given above are derived as follows. We note that all these density matrices are written in the block-diagonal form. Then the SLD operators can also be written in the same form by

\[
L^{(x)} = \bigoplus_{\nu} L^{(x)}_{\nu}.
\]

With the help of Eq. \[A13\], we derive the SLD operators corresponding to the above three cases as

\[
L^{(x)} = \begin{pmatrix} 0 & -i\beta_x e^{-iN\phi} \\ i\beta_x e^{iN\phi} & 0 \end{pmatrix} \otimes 0_{d_1},
\]

where \(\beta_{\text{PDC}} = N\eta^N\), \(\beta_{\text{DPC}} = \frac{N\eta^N}{\eta^2 + 1 - \eta^N}\), and \(\beta_{\text{ADC}} = \frac{2N\eta^N/2}{\eta^2 + 1 - \eta^N}\). In the derivation of \(L^{(x)}\), we have considered that for the sub-block density matrices given by Eqs. \[17\], \[19\], and \[11\], the corresponding SLD operators are equal to zero matrices, i.e., \(L^{(x)} = 0\). This result is easily to be identified as follows. For an arbitrary \(\phi\)-independent density matrix, its generalized Bloch vector must be \(\phi\)-independent, with the equality \(\partial_\phi \omega = 0\). From Eqs. \[A4\] and \[A7\] (or Eqs. \[A13\] and \[A15\]), we see that the SLD operator and the QFI are dependent on the term \(\partial_\phi \omega\). Thus the SLD operator must be zero-matrix and the QFI is vanished for the \(\phi\)-independent density matrix. In addition, it is remarkable that \(\rho_{\text{PDC}}\) is not a full-rank matrix so that the SLD operator cannot be uniquely determined. By substituting Eq. \[53\] into Eq. \[A15\], we obtain

\[
F^{(x)} = \gamma_{\nu} N^2,
\]

where \(\gamma_{\text{PDC}} = \eta^2N\), \(\gamma_{\text{DPC}} = \frac{\eta^2N}{\eta^2 + 1 - \eta^N}\), and \(\gamma_{\text{ADC}} = \frac{2\eta^N}{\eta^2 + 1 - \eta^N}\). According to the quantum Cramér-Rao theorem described by Eq. \[2\], we obtain the ultimate sensitivity of the estimation for the above three cases as

\[
\Delta \phi_x = \frac{1}{\sqrt{\xi_{\nu} N}}.
\]

It shows that when local decoherence is taken into account, the maximal possible quantum enhancement amounts generically to a constant factor in the asymptotic limit of infinite probes \[22\].

We now study the problem of whether the separable observable in form of Eq. \[30\] can saturate the sensitivities given in Eq. \[55\]. To solve this problem, we substitute Eq. \[50\], each parametric density matrix \(\rho_{\nu}(\phi)\) for the three cases, and each SLD operator \(L^{(x)}\) in Eq. \[53\] into the necessary and sufficient condition Eq. \[17\], then we find that no solutions exist for the coefficients \(\{a_0, a_1, a_2, a_3\}\) for each case. It means that there do not exist the optimal distinguishing observables in form of Eq. \[30\] saturating the ultimate precisions of the measurement given in Eq. \[55\]. Besides, we find that the local restricted measurement described by Eq. \[26\] also cannot reach the sensitivities in Eq. \[55\].

V. CONCLUSION

In this paper, we have addressed the general problem of the necessary and sufficient condition for the optimal observable to achieve the ultimate precision limit of the parameter estimation. We show the difference between the general estimation via estimator and that via the average value of the observable, especially for the global optimization in the whole range of parameter. We also clarify the question of whether the local restricted measurements can saturate the HL-scaling sensitivity. As
is shown in Ref. [21], the authors claimed that the local projective measurement might not be possible to go beyond the SNL-scaling accuracy even for arbitrary entangled states. We exemplify that the local projective measurement can reach the HL for the parametric maximally entangled state with nonzero value of the parameter. Meanwhile, we give the general expression of the optimal separable observable for the parametric maximally entangled state.

Moreover, we consider the estimation of a parameter \( \varphi \) involving \( N \) parallel samplings of a linear unitary operation with the mixed probe states which are generated by the GHZ state under three different decoherence channels. For these cases, we find that the separable observable fails to saturate the achievable sensitivity limit given by the quantum Cramér-Rao bound. Our results may be directly applied in the investigation of the optical interferometry or the standard Ramsey interferometry.

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Appendix A: A geometric method to derive the SLD operator and the QFI

In this appendix, we geometrically derive the SLD operator and the QFI for block-diagonal form of parametric density matrices. We first turn to the generalized Bloch vector representation, in which density matrices and observables can be expressed by an operator basis, which consists of the identity operator \( \mathbb{1} \) and the generators \( \hat{\lambda} = \{ \hat{\lambda}_i \}^d_{i=1} \) of the Lie algebra \( \text{su}(d) \), where \( d \) denotes the dimension of Hilbert space [33, 36]. The generators \( \hat{\lambda} \) satisfy \( \hat{\lambda}_i^2 = \hat{\lambda}_i, \text{Tr} \hat{\lambda}_i = 0, \) and \( \text{Tr}(\hat{\lambda}_i \hat{\lambda}_j) = 2\delta_{ij} \). Each \( \hat{\lambda} \) is characterized by the structure constants \( f_{ijk} \) (completely antisymmetric tensor) and \( g_{ijk} \) (completely symmetric tensor) as

\[
\begin{align*}
[\hat{\lambda}_i, \hat{\lambda}_j] &= 2i \sum_k f_{ijk} \hat{\lambda}_k, \\
[\hat{\lambda}_i, \hat{\lambda}_j]_+ &= 4 \sum_k g_{ijk} \hat{\lambda}_k.
\end{align*}
\]

In the above generalized Bloch vector representation, the density matrix of a finite \( d \)-dimensional quantum system is expressed as

\[
\rho_\varphi = \frac{1}{d} \mathbb{1}_d + \frac{1}{2} \omega \cdot \hat{\lambda} \quad \text{(A3)}
\]

where \( \omega \in \mathbb{R}^{d^2-1} \) denotes the generalized Bloch vector. Due to the dependence of \( \rho_\varphi \) on \( \varphi \), the components of \( \omega \) are real functions with respect to \( \varphi \). In Ref. [37], the geometric expressions of the SLD operator and the QFI with respect to \( \rho_\varphi \) are obtained

\[
\begin{align*}
L &= -(\partial_\omega \omega)^T M^{-1}(\omega \mathbb{1}_d - \hat{\lambda}), \\
\mathcal{F}_\varphi &= (\partial_\varphi \omega)^T M^{-1} \partial_\varphi \omega. \tag{A4, A5}
\end{align*}
\]

where \( M^{-1} \) is the inverse of the real symmetric matrix \( M \) defined by

\[
M = \frac{2}{d} - \omega \omega^T + G \quad \text{(A6)}
\]

with the elements of \( G \) being \( G_{ij} := \sum_k g_{ijk} \omega_k \) [37]. In general, \( M^{-1} \) may or may not exist as a result of \( M \) may having some zero eigenvalues. In this case, we define \( M^{-1} \) on the support of \( M \), i.e., \( \text{supp}(M) \), which is defined as a space spanned by those eigenvectors with nonzero eigenvalues [37, 38].

Furthermore, we consider a type of density matrices which can be written in the block-diagonal form

\[
\rho_\varphi = \bigoplus_{\nu=0}^n \rho_\nu(\varphi), \quad \text{(A7)}
\]

where \( \rho_\nu(\varphi) \) are Hermitian and positive semidefinite on the corresponding sub-block of dimension \( d_\nu \). Meanwhile, the SLD operator can also be written in the block-diagonal form

\[
L = \bigoplus_{\nu=0}^n L_\nu. \quad \text{(A8)}
\]

For this scenario, the SLD equation [6] can be divided into \( n+1 \) equations expressed as

\[
\partial_\nu \rho_\nu(\varphi) = \frac{1}{2} [\rho_\nu(\varphi), L_\nu]_+, \quad \text{(A9)}
\]

for \( \nu = 0, 1, ..., n \). With Eqs. (A7) and (A8), then the QFI of Eq. (6) becomes

\[
\mathcal{F}_\varphi = \sum_{\nu=0}^n \mathcal{F}_{\varphi,\nu} = \sum_{\nu=0}^n \text{Tr}[\rho_\nu(\varphi) L_\nu^2]. \tag{A10}
\]

Equation (A10) shows that the QFI \( \mathcal{F}_\varphi \) for the diagonal-block-form density matrix of Eq. (A7) is equivalent to sum up all sub-QFIs \( \mathcal{F}_{\varphi,\nu} \) in terms of each sub-block matrix \( \rho_\nu(\varphi) \).

To determine each \( L_\nu \), we represent \( \rho_\nu(\varphi) \) and \( L_\nu \) in the form of generalized Bloch vectors as

\[
\begin{align*}
\rho_\nu(\varphi) &= \frac{1}{d_\nu} \omega_{\nu,0} \mathbb{1}_{d_\nu} + \frac{1}{2} \omega_\nu \cdot \hat{\lambda}_\nu, \quad \text{(A11)}
L_\nu &= b_{\nu,0} \mathbb{1}_{d_\nu} + b_\nu \cdot \hat{\lambda}_\nu, \quad \text{(A12)}
\end{align*}
\]

where \( \omega_{\nu,0}, b_{\nu,0} \in \mathbb{R}, \omega_\nu, b_\nu \in \mathbb{R}^{d_\nu-1}, \) and \( \hat{\lambda}_\nu \) denote the generators of \( \text{su}(d_\nu) \) on the \( \nu \)th sub-block Hilbert space.
By substituting Eqs. (A11) and (A12) into Eq. (A9), we obtain
\[
L_{\nu} = - (\partial_\omega \omega_{\nu})^T M_{\nu}^{-1}(\omega_{\nu} \mathbb{1} - \hat{\lambda}^{(\nu)}).
\]  
(A13)

where \( M_{\nu} \) is still a real symmetric matrix defined by
\[
M_{\nu} = \frac{2}{d_\nu} \omega_{\nu,0} - \omega_{\nu,0}^{\nu} + G_{\nu}
\]  
(A14)

with \( G_{\nu,ij} := \sum_k g_{ijk} \omega_{\nu,k} \). Similarly, the inverse matrix \( M_{\nu}^{-1} \) above is defined on the support of \( M_{\nu} \), i.e., \( \text{supp}(M_{\nu}) \). With Eqs. (A10), (A11) and (A13), the geometric expression of the QFI for the block-diagonal form of density matrices in Eq. (A7) is obtained
\[
\mathcal{F}_\varphi = \sum_{\nu=0}^n (\partial_\omega \omega_{\nu})^T M_{\nu}^{-1} \partial_\omega \omega_{\nu}.
\]  
(A15)