Quantum theory for mesoscopic electric circuits

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A quantum theory for mesoscopic electric circuits in accord with the discreteness of electric charges is proposed. On the basis of the theory, the Schrödinger equation for the quantum LC design and L design is solved exactly. The uncertainty relation for electric charge and current is obtained and a minimum uncertainty state is solved. By introducing a gauge field, a formula for persistent current arising from magnetic flux is obtained.

I. INTRODUCTION

Along with the dramatic achievement in nanotechnology, such as molecular-beam epitaxy, atomic-scale fabrication or advanced lithography, mesoscopic physics and nanoelectronics are undergoing a rapid development.[1,2] It has been a strong and definite trend in the miniaturization of integrated circuits and components towards atomic-scale dimensions[3] for the electronic device community. When the transport dimension reaches a characteristic dimension, namely, the charge carrier inelastic coherence length, one must address not only the quantum mechanical property but also the discreteness of electron charge. Thus a correct quantum theory is indispensable for the device physics in integrated circuits of nanoelectronics. The classical equation of motion for an electric circuit of LC design is the same as that for a harmonic oscillator, whereas the ''coordinate'' means electric charge. The quantization of the circuit was carried out[4] in the same way as that of a harmonic oscillator. This only results in energy quantization. In fact, a different kind of fluctuation in mesoscopic systems, which inherently has nothing to do with energy quantization and interference of wave functions, is due to the quantization of electronic charge. Recently we studied the quantization of electric circuit of LC design under consideration of the discreteness of electric charge.[5]

In the present paper we extend the main idea of our previous Letter[5] and present a quantum mechanical theory for electric circuits based on the fact that electronic charge takes discrete values. In Sec. II, a finite-difference Schrödinger equation for the mesoscopic electric circuit is obtained. In Sec. III, the Schrödinger equation for a mesoscopic circuit of LC design is turned to the Mathieu equation in p-representation and solved exactly. The average value of electric current for the ground state is calculated. In Sec. IV, the uncertainty relation for charge and current is discussed. A minimum uncertainty state, which recovers the usual Gaussian wave packet in the limit of vanishing discreteness, is solved. In Sec. V, the Schrödinger equation for a quantum L design both in the presence of an adiabatic power source and in the absence of source are solved exactly. A gauge field is introduced and a formula for persistent current that is a periodic function of the magnetic flux is obtained. It provides a formulation of the persistent current in the mesoscopic ring from a different point of view. Finally, some discussions and conclusions are made in Sec. VI.

II. QUANTIZATION OF ELECTRIC CIRCUIT IN ACCORD WITH THE DISCRETENESS OF ELECTRIC CHARGE

We recall that for a classical nondissipative electric circuit of LC design in the presence of a source $\varepsilon(t)$, the equation of motion, as a consequence of Kirchoff's law, reads $d^2q/dt^2+(1/LC)q-(1/L)\varepsilon(t)=0$, where $q(t)$ stands for electric charge, L for inductance, and C for the capacity of the circuit. This equation of motion can be formulated in terms of Hamiltonian mechanics, namely,

$$\dot{q}=\frac{\partial H}{\partial p},\quad \dot{p}=-\frac{\partial H}{\partial q},$$

with $H(t)=(1/2L)p^2+(1/2C)q^2+\varepsilon(t)q$. Here the variable q stands for the electric charge instead of the conventional ''coordinate,'' while its conjugation variable $p(t)=Ldq/dt$ represents (apart from a factor L) the electric current instead of the conventional ''momentum.'' Analogous to the forced harmonic oscillator, the electric circuit was quantized by many authors,[4] where the electric charge was treated as a continuous variable. As a matter of fact, the electronic charge is discrete and it must play an important role in the theory for mesoscopic circuits. Taking account of the discreteness of electric charge, we must reconsider the quantization of a mesoscopic circuit. According to the standard quantization principle, one associates with each of the two observable quantities q and p a linear Hermitian operator, namely, $\hat{q}$ and $\hat{p}$. The Hamiltonian, also an observable quantity, corresponds to a Hermitian operator $H=(1/2L)\hat{p}^2+V(\hat{q})$, which is a function of the operators $\hat{p}$ and $\hat{q}$. The commutation relation for the conjugation variables are

$$[\hat{q},\hat{p}]=i\hbar. \tag{2.1}$$

Up to now, the discreteness of electronic charge was not taken into account. Regarding the discreteness, we must impose that the eigenvalues of the self-adjoint operator $\hat{q}$ take discrete values,[5] i.e.,

$$\hat{q}|q\rangle = nq_e|q\rangle, \tag{2.2}$$

where $n \in \mathsf{Z}$ (set of integers) and $q_e = 1.602\times 10^{-19}$ C, the elementary electric charge. Obviously, any eigenstate of $\hat{q}$ can be specified by an integer. This allows us to introduce a minimum "shift operator" $\hat{Q} := e^{iq_e\hat{p}/\hbar}$, which is shown to satisfy the following commutation relations[5]

$$[\hat{q},\hat{Q}] = -q_e\hat{Q},$$

$$[\hat{q},\hat{Q}^+] = q_e\hat{Q}^+,$$

$$\hat{Q}^+\hat{Q} = \hat{Q}\hat{Q}^+ = 1. \tag{2.3}$$

These relations can determine the structure of the whole Fock space. For $\hat{q}|n\rangle = nq_e|n\rangle$, the algebraic relations (2.3) enable us to derive the following

$$\hat{Q}^+|n\rangle = e^{i\alpha_{n+1}}|n+1\rangle,$$

$$\hat{Q}|n\rangle = e^{-i\alpha_n}|n-1\rangle, \tag{2.4}$$

where α_n's are undetermined phases. Obviously $\hat{Q}^+$ and $\hat{Q}$ are ladder operators, respectively, for charge increasing and decreasing in the diagonal representation of charge operator. The Fock space for our present algebra differs from the well-known Fock space for the Heisenberg-Weyl algebra, because the spectrum of the former is isomorphic to the set of integers Z but that of the latter is isomorphic to the set of non-negative integers $\mathsf{Z}^+ + \{0\}$. Since $\{|n\rangle | n \in \mathsf{Z}\}$ constitutes a Hilbert space, we have the completeness $\Sigma_{n\in \mathsf{Z}}|n\rangle\langle n| = 1$. We also have the orthogonality $\langle n|m\rangle = \delta_{nm}$ due to the self-adjointness of $\hat{q}$. As a result, the inner product in charge representation takes the form

$$\langle \phi|\psi\rangle = \sum_{n\in \mathsf{Z}} \langle \phi|n\rangle\langle n|\psi\rangle = \sum_{n\in \mathsf{Z}} \phi^*(n)\psi(n). \tag{2.5}$$

One can now study the eigenstates and eigenvalues of the operator $\hat{p}$. Obviously, if $\hat{p}|p\rangle = p|p\rangle$ then $f(\hat{p})|p\rangle = f(p)|p\rangle$ for any analytical function f. Supposing $|p\rangle = \Sigma_{n\in \mathsf{Z}} c_n(p)|\rangle$, and using $\hat{Q}|p\rangle = e^{iq_e p/\hbar}|p\rangle$ we can find that $c_{n+1}/c_n = \exp(iq_e p/\hbar + i\alpha_{n+1})$, which yields the following solution

$$|p\rangle = \sum_{n\in \mathsf{Z}} \kappa_n e^{inq_e p/\hbar}|n\rangle, \tag{2.6}$$

where $\kappa_n = e^{i\Sigma_{j=1}^{n}\alpha_j}$, $\kappa_{-n} = e^{-i\Sigma_{j=0}^{n-1}\alpha_{-j}}$ for $n>0$. Obviously $|p+\hbar(2\pi/q_e)\rangle = |p\rangle$, the eigenvalues of the operator $\hat{p}$ is a periodic parameter. Topologically, the parameter space of the spectrum is isotopic to the S^1.

Since the spectrum of charge is discrete and the inner product in charge representation is a sum instead of the usual integral, one may define a right and left discrete derivative operators ∇_{q_e} and $\bar{\nabla}_{q_e}$ by

$$\nabla_{q_e} f(n) = \frac{f(n+1)-f(n)}{q_e},$$

$$\bar{\nabla}_{q_e} f(n) = \frac{f(n)-f(n-1)}{q_e}. \tag{2.7}$$

They can be understood as the inverse of a discrete definite integral, which is in accord with the inner product (2.5), i.e.,

$$\int_{x_i}^{x_f} f(x)dx := \sum_{n=n_i}^{n_f} q_e f(nq_e)$$

$$= \begin{cases} \hat{Q}F(x_f) - F(x_i) & \text{if } \nabla_{q_e}F = f, \\ F(x_f) - \hat{Q}^+F(x_i) & \text{if } \bar{\nabla}_{q_e}F = f. \end{cases}$$

Clearly, it recovers the conventional differential-integral calculus as long as the minimum interval q_e goes to zero. The discrete derivative operators defined by (2.7) can be expressed explicitly by the minimum shift operators

$$\nabla_{q_e} = (\hat{Q}-1)/q_e,$$

$$\bar{\nabla}_{q_e} = (1-\hat{Q}^+)/q_e. \tag{2.8}$$

It is easy to check[6] that $\nabla_{q_e}^+ = -\bar{\nabla}_{q_e}$. Then we can write down two important self-adjoint operators: "momentum" operator,

$$\hat{P} = \frac{\hbar}{2i}(\nabla_{q_e} + \bar{\nabla}_{q_e}) = \frac{\hbar}{2iq_e}(\hat{Q} - \hat{Q}^+) \tag{2.9}$$

and free Hamiltonian operator

$$\hat{H}_0 = -\frac{\hbar^2}{2}\nabla_{q_e}\bar{\nabla}_{q_e} = -\frac{\hbar^2}{2q_e}(\nabla_{q_e} - \bar{\nabla}_{q_e})$$

$$= -\frac{\hbar^2}{2q_e^2}(\hat{Q} + \hat{Q}^+ - 2), \tag{2.10}$$

which we call as momentum and free Hamiltonian operators, respectively, because they are really those when $q_e \to 0$. Now we have finished the quantization of mesoscopic electric circuits and obtained the following finite-difference Schrödinger equation,

$$\left[-\frac{\hbar^2}{2q_e L}(\nabla_{q_e} - \bar{\nabla}_{q_e}) + V(\hat{q})\right]|\psi\rangle = E|\psi\rangle. \tag{2.11}$$

III. THE QUANTUM *LC* DESIGN

As an application of our quantization strategy of the mesoscopic circuit, we discuss a mesoscopic LC design in this section. We only considered the adiabatic approximation so that $\varepsilon(t)$ is considered as a constant ε. Then the Schrödinger equation (2.11) for a LC design is written as

$$\left[-\frac{\hbar^2}{2q_e L}(\nabla_{q_e} - \bar{\nabla}_{q_e}) + \frac{1}{2C}\hat{q}^2 + \varepsilon\hat{q}\right]|\psi\rangle = E|\psi\rangle. \tag{3.1}$$

We consider a representation in which the operator $\hat{p}$ is diagonal and called it the p representation. We must address that the $\hat{p}$ is the conjugation of the charge variable $\hat{q}$ within the meaning of the usual canonical commutator (2.1), and it is the "current" operator only if the charge is treated as a continuous variable. However, the operator $\hat{P}$ associated with the physical quantity, electric current (apart from a factor $1/L$), differs from the operator $\hat{p}$ as long as the discreteness of charge is taken into account. Clearly, $\hat{P}$ will become the usual $\hat{p}$ when q_e goes to zero. The orthogonality of eigenstates of $\hat{p}$ is an immediate consequence of (2.6) and the orthogonality of the charge eigenstates, i.e.,

$$\langle p|p'\rangle=\frac{2\pi}{q_e\hbar}\sum_{n\in Z}\delta\left[p-p'+n\left(\frac{2\pi}{q_e}\right)\hbar\right].$$

The completeness is also verified:

$$\frac{q_e}{2\pi}\int_{-\hbar(\pi/q_e)}^{\hbar(\pi/q_e)}\frac{dp}{\hbar}|p\rangle\langle p|=\sum_{n\in Z}|n\rangle\langle n|=1. \tag{3.2}$$

The transformation of wave functions between charge representation and p representation is given by

$$\langle n|\psi\rangle=\left(\frac{q_e}{2\pi\hbar}\right)\int_{-\hbar(\pi/q_e)}^{\hbar(\pi/q_e)}dp\langle p|\psi\rangle e^{-inq_ep/\hbar}. \tag{3.3}$$

Using (2.6), we can obtain the following relations

$$\langle p'|\nabla_{q_e}-\bar{\nabla}_{q_e}|p\rangle=\frac{4\pi\hbar}{q_e^2}\left[\cos\left(\frac{q_e}{\hbar}p\right)-1\right]\delta(p-p'),$$

$$\langle p'|\hat{q}^2|p\rangle=-\frac{2\pi\hbar^3}{q_e}\frac{\partial^2}{\partial p^2}\delta(p-p'). \tag{3.4}$$

In the p representation, the finite-difference Schrödinger equation (3.1) becomes a differential equation for $\tilde{\psi}(p):=\langle p|\psi\rangle$

$$\left\{-\frac{\hbar^2}{2C}\frac{\partial^2}{\partial p^2}-\frac{\hbar^2}{q_e^2L}\left[\cos\left(\frac{q_e}{\hbar}p\right)-1\right]\right\}\tilde{\psi}(p)=E\tilde{\psi}(p), \tag{3.5}$$

which is the well-known Mathieu equation.[7,8] This equation appeared in Ref. 9 on the discussion of Padé approximates. In deriving (3.5), we have adopted $\varepsilon=0$ for simplicity. Actually, the linear term in (3.1) can be moved by a translation in the "coordinate" (charge) space. Apart from a redefinition of $\hat{q}$ and a shift of the energy E, the same equation as (3.5) would be derived.

In terms of the conventional notations,[7,8] the wave functions in p representation can be solved as follows

$$\tilde{\psi}_l^+(p)=\mathrm{ce}_l\left(\frac{\pi}{2}-\frac{q_e}{2\hbar}p,\xi\right)$$

or

$$\tilde{\psi}_{l+1}^-(p)=\mathrm{se}_{l+1}\left(\frac{\pi}{2}-\frac{q_e}{2\hbar}p,\xi\right), \tag{3.6}$$

where the superscripts + and − specify the even and odd parity solutions, respectively; $l=0,1,2,\ldots$; $\xi=(2\hbar/q_e^2)^2C/L$; $\mathrm{ce}(z,\xi)$ and $\mathrm{se}(z,\xi)$ are periodic Mathieu functions. In this case, there exist infinitely many eigenvalues $\{a_l\}$ and $\{b_{l+1}\}$ that are not identically equal to zero. Then the energy spectrum is expressed in terms of the eigenvalues a_l, b_l of the Mathieu equation,

$$E_l^+=\frac{q_e^2}{8C}a_l(\xi)+\frac{\hbar^2}{q_e^2L},$$

$$E_{l+1}^-=\frac{q_e^2}{8C}b_{l+1}(\xi)+\frac{\hbar^2}{q_e^2L}. \tag{3.7}$$

As an exercise, one may calculate the fluctuation of electric current for the ground state. It is known that the explicit results of eigenvalues and eigenfunctions of the Mathieu equation are complicated. They are related to continued fractions and trigonometric series, respectively. For the concrete values of the Plank constant and the elementary electric charge, the WKB method is valid. From the series solution of Mathieu equation for ground state, we obtained the fluctuation of electric current $\hat{P}$ (apart from a factor $1/L$) for the ground state

$$\langle\hat{P}^2\rangle_{\text{ground}}=\frac{1}{2}\left(\frac{h}{q_e}\right)^2\left[1-\frac{3}{2}\left(\frac{\hbar^2C}{q_e^4L}\right)^2+\cdots\right]. \tag{3.8}$$

This result is valid for the case $C/L\ll(q_e^2/\hbar)^2$.

IV. UNCERTAINTY RELATION AND THE MINIMUM UNCERTAINTY STATE

In order to understand the main conclusions in this section, we begin with a brief view of the derivation of the uncertainty relation in standard quantum mechanics. If $\hat{A}$ and $\hat{B}$ are two Hermitian (self-adjoint) operators that do not commute, the physical quantities A and B cannot both be sharply defined simultaneously. The variances of A and B are defined as $(\Delta\hat{A})^2=\langle(\hat{A}-\langle\hat{A}\rangle)^2\rangle$ and $(\Delta\hat{B})^2=\langle(\hat{B}-\langle\hat{B}\rangle)^2\rangle$. Their positive square roots, ΔA and ΔB, are called the uncertainties in A and B. In terms of the properties of self-adjoint operators and the knowledge of Schwarz inequality, one can prove that

$$(\Delta\hat{A})^2(\Delta\hat{B})^2\geqslant|\langle(1/2)(\{\hat{A},\hat{B}\}-\langle A\rangle\langle B\rangle)\rangle|^2+|\langle(1/2)[\hat{A},\hat{B}]\rangle|^2, \tag{4.1}$$

where $\{,\}$ denotes the anticommutator, and the equality sign holds if and only if $\hat{B}|\psi\rangle\propto\hat{A}|\psi\rangle$. In deriving (4.1), the fact that the expectation value of a Hermitian (or anti-Hermitian) operator is a real number (or purely imaginary number) has been used. As a direct consequence of (4.1), the uncertainty relation is conventionally written as

$$(\Delta\hat{A})^2(\Delta\hat{B})^2\geqslant|\langle(1/2)[\hat{A},\hat{B}]\rangle|^2. \tag{4.2}$$

Clearly. the equality sign in (4.2) holds if and only if both the equality sign in (4.1) holds and the first term of the right-hand side in (4.1) vanishes. These conditions imply that

$$(\hat{B}-\langle\hat{B}\rangle)|\psi\rangle=\lambda(\hat{A}-\langle\hat{A}\rangle)|\psi\rangle,$$

$$\lambda=\frac{\langle\psi|[\hat{A},\hat{B}]|\psi\rangle}{2(\Delta\hat{A})^2}. \tag{4.3}$$

Now we go directly to our main purpose. After some calculations, we obtain the following commutation relations for the charge $\hat{q}$, the current $\hat{P}$, and the free Hamiltonian $\hat{H}_0$,

$$[\hat{H}_0,\hat{P}]=0,\ [\hat{H}_0,\hat{q}]=i\hbar\hat{P},\ [\hat{q},\hat{P}]=i\hbar\left(1+\frac{q_e^2}{\hbar^2}\hat{H}_0\right), \tag{4.4}$$

where the operators $\hat{P}$ and $\hat{H}_0$ have been defined respectively by (2.9) and (2.10). The term $(q_e^2/\hbar^2)\hat{H}_0$ in the third equation of (4.4) occurs due to the discreteness of electric charge. Now we are ready to write out the uncertainty relation for electric charge and electric current, namely,

$$\Delta\hat{q}\cdot\Delta\hat{P}\geqslant\frac{\hbar}{2}\left(1+\frac{q_e^2}{\hbar^2}\langle\hat{H}_0\rangle\right). \tag{4.5}$$

This uncertainty relation recovers the usual Heisenberg uncertainty relation if q_e goes to zero, i.e., in the case that the discreteness of electric charge vanishes. Moreover, the uncertainty relation (4.5) has shown us further knowledge than the traditional Heisenberg uncertainty relation.

It is of interest to study the particular state $|\psi\rangle$, for which (4.5) becomes an equality. This is the state in which the product of the uncertainties in electric charge and current is as small as the noncommutivity allows: $\Delta\hat{q}\cdot\Delta\hat{P}=(\hbar/2)(1+q_e^2/\hbar^2\langle\hat{H}_0\rangle)$. Such a minimum uncertainty state must obey the condition (4.3) for $\hat{A}=\hat{P}$ and $\hat{B}=\hat{q}$

$$(\hat{q}-\langle\hat{q}\rangle)|\psi\rangle=-\frac{i\hbar(1+q_e^2/\hbar^2\langle\hat{H}_0\rangle)}{2(\Delta\hat{P})^2}(\hat{P}-\langle\hat{P}\rangle)|\psi\rangle. \tag{4.6}$$

Using (2.6) and (2.4), one can find that

$$\langle p'|\hat{q}|p\rangle=\frac{h}{q_e}\frac{\hbar}{i}\frac{\partial}{\partial p}\delta(p-p')$$

$$\langle p'|\hat{P}|p\rangle=\frac{h}{q_e}\frac{\hbar}{q_e}\sin\left(\frac{q_e p}{\hbar}\right)\delta(p-p'). \tag{4.7}$$

Then (4.6) becomes the following differential equation in p representation

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial p}+\langle\hat{q}\rangle\right)\tilde{\psi}(p)=\frac{i\hbar(1+q_e^2/\hbar^2\langle\hat{H}_0\rangle)}{2(\Delta\hat{P})^2}\left(\frac{\hbar}{q_e}\sin(q_e p/\hbar)-\langle\hat{P}\rangle\right)\tilde{\psi}(p). \tag{4.8}$$

This differential equation is solved by a plane wave with modulated amplitude:

$$\tilde{\psi}(p)=N\exp\left\{\frac{1+q_e^2/\hbar^2\langle\hat{H}_0\rangle}{2(\Delta\hat{P})^2}\left[\frac{\hbar^2}{q_e^2}\cos\left(\frac{q_e}{\hbar}p\right)+\langle\hat{P}\rangle p\right]-\frac{i\langle\hat{q}\rangle p}{\hbar}\right\}. \tag{4.9}$$

where N is the normalization constant. Equation (4.9) is obviously a deformation of the usual Gaussian wave packet and recovers the Gaussian wave-packet if the discreteness vanishes.

V. QUANTUM *L* DESIGN, GAUGE FIELD, AND PERSISTENT CURRENTS

In this section, we will solve the Schrödinger equation for an L design in the presence of an adiabatic source and in the absence of source. Introducing a gauge field and gauge transformation, we derive a formula for persistent current in a pure L design, i.e., a mesoscopic metal ring.

A. The *L* design in the presence of an adiabatic source

The Schrödinger equation for an L design in the presence of an adiabatic source reads

$$\left[-\frac{\hbar^2}{2q_eL}(\nabla_{q_e}-\bar{\nabla}_{q_e})+\varepsilon\hat{q}\right]|\psi\rangle=E|\psi\rangle. \tag{5.1}$$

In order that the quantization of a mesoscopic circuit be valid, the size of the circuit must be restricted, while the voltage source can come from an infinite reservoir to keep the chemical potential constant. We consider the present problem in charge representation and expand the eigenstate of (5.1) in terms of the orthonormal set of charge eigenstates, namely, $|\psi\rangle=\Sigma_{n=-\infty}^{\infty}u_n|n\rangle$. Substituting it into (5.1), we obtain the following recursion relations:

$$2\left(\frac{\hbar^2}{q_eL}+nq_e\varepsilon-E\right)u_l-\frac{\hbar^2}{2q_eL}(u_{l-1}+u_{l+1})=0. \tag{5.2}$$

The knowledge of the recursion formula of Bessel functions, $z[J_{\nu+1}(z)+J_{\nu-1}(z)]=2\nu J_\nu(z)$ enables us to write down a solution of (5.2)

$$u_n=J_{nq_e\varepsilon+z_0-E}(z_0), \tag{5.3}$$

where $z_0=\hbar^2/q_eL$. In terms of (5.3) the eigenstates of the Schrödinger equation (5.1) are written out

$$|\psi_E\rangle=\sum_{n=-\infty}^{\infty}J_{nq_e\varepsilon+z_0-E}(z_0)|n\rangle, \tag{5.4}$$

which is the solution of eigenstates for quantum L design in the presence of an adiabatic source.

B. Pure *L* design

Now we consider a pure L design, $\hat{H}_L=-(\hbar^2/2q_eL)(\nabla_{q_e}-\bar{\nabla}_{q_e})$. The Hamiltonian operator of the pure L design $\hat{H}_L$, the current operator $\hat{P}$, and the operator

$\hat{p}$ commute each other, so they can have simultaneous eigenstates. Actually, (2.6) is the simultaneous eigenstate of those operators, i.e.,

$$\hat{P}|p\rangle = \frac{\hbar}{q_e}\sin\left(\frac{q_e p}{\hbar}\right)|p\rangle,$$

$$\hat{H}_L|p\rangle = \frac{\hbar^2}{q_e^2 L}\left[1-\cos\left(\frac{q_e p}{\hbar}\right)\right]|p\rangle. \tag{5.5}$$

This result tells us that the magnitude of electric current in a mesoscopic electric circuit of pure L design are bounded taking values between $-\hbar/q_e L$ and $\hbar/q_e L$. It also indicates that the maximum quantum noise in a pure L design (a mesoscopic ring is an example) takes a finite value if the elementary charge q_e should not be considered as the infinitesimal (particularly for the mesoscopic circuit). It is also worthwhile to notice that both the current and energy of a pure L design become null when $p=2\pi\hbar/q_e$ as long as q_e is not zero. Clearly, the lowest-energy states correspond to $p=nh/q_e$ for any integer n. Thus the energy spectrum is infinitely degenerated.

C. Gauge field and persistent current

In the previous discussion, we have used the terminology p representation and solved the eigenstates of $\hat{p}$. Now let us find out what the eigenvalues of $\hat{p}$ means. If introducing a operator $\hat{G}:=e^{-i\beta\hat{q}/\hbar}$, we can find that $\hat{G}|p\rangle=|p-\beta\rangle$ and $\hat{G}^+|p\rangle=|p+\beta\rangle$. Considering a unitary transformation to the eigenstates of the Schrödinger operator given by

$$|\psi\rangle \to |\psi'\rangle = \hat{G}|\psi\rangle,$$

we find that the Schrödinger equation (2.11) is not covariant. This requires that we introduce a gauge field and define a reasonable covariant discrete derivative. By making the following definitions,

$$D_{q_e} := e^{-i(q_e/\hbar)\phi}\frac{\hat{Q}-e^{i(q_e/\hbar)\phi}}{q_e},$$

$$\bar{D}_{q_e} := e^{i(q_e/\hbar)\phi}\frac{e^{-i(q_e/\hbar)\phi}-\hat{Q}^+}{q_e}, \tag{5.6}$$

we can verify that they are covariant under a gauge transformation. The gauge transformations are expressed as

$$\hat{G}D_{q_e}\hat{G}^{-1} = D'_{q_e},$$

$$\hat{G}\bar{D}_{q_e}\hat{G}^{-1} = \bar{D}'_{q_e}, \tag{5.7}$$

as long as the gauge field ϕ transforms in such a way that

$$\phi \to \phi' = \phi - \beta.$$

From either the transformation law or the dimension of the field ϕ, we may realize that ϕ plays the role of the magnetic flux threading the circuit.

In terms of those covariant discrete derivatives (5.6), one can write down the Schrödinger equation in the presence of the gauge field (magnetic flux). Here we write out the Schrödinger equation for a pure L design in the presence of magnetic flux,

$$-\frac{\hbar^2}{2q_e L}(D_{q_e}-\bar{D}_{q_e})|\psi\rangle = E|\psi\rangle \tag{5.8}$$

because its eigenstates can be simultaneous eigenstates of $\hat{p}$. Equation (5.8) is solved by the same eigenstate $|p\rangle$ in (2.6). The energy spectrum is easily calculated as

$$E(p,\phi) = \frac{2\hbar}{q_e^2}\sin^2\left[\frac{q_e}{2\hbar}(p-\phi)\right], \tag{5.9}$$

which has an oscillatory property with respect to ϕ or p. Differing from the usual classical pure L design, the energy of a mesoscopic quantum pure L design cannot be larger than $2\hbar/q_e^2$. Clearly, the lowest-energy states are those states where $p=\phi+n(h/q_e)$. Thus the eigenvalues of the electric current [i.e., $(1/L)\hat{P}$] of the ground state are calculated

$$I(\phi) = \frac{\hbar}{q_e L}\sin\left(\frac{q_e}{\hbar}\phi\right). \tag{5.10}$$

Obviously, the electric current on a mesoscopic circuit of pure L design is not null in the presence of a magnetic flux except $\phi=n(h/q_e)$. Clearly, this is a pure quantum characteristic. Equation (5.10) exhibits that the persistent current in a mesoscopic L design is an observable quantity periodically depending on the flux ϕ. Because a mesoscopic metal ring is a natural pure L design, the formula (5.10) is valid for persistent current on a single mesoscopic ring.[10] Differing from the conventional formulation of the persistent current on the basis of quantum dynamics for electrons, our formulation presented a method from a new point of view. Formally, the $I(\phi)$ we obtained here is a sine function with periodicity of $\phi_0=h/q_e$. But either the model where the electrons move freely in an ideal ring,[11] or the model where the electrons have hard-core interactions between them[12] can only give the sawtooth-type periodicity. Obviously, the sawtooth-type function is only the limit case for $q_e/\hbar \to 0$.

Certainly the experiment[13] should be considered as the case of persistent current in a LC design because the junction of semiconductors will contribute a capacitance to the "circuit."

VI. CONCLUSIONS AND DISCUSSIONS

In the above, we studied the quantization of a mesoscopic electric circuit. Differing from the literature, in which it is simply treated as the quantization of a harmonic oscillator, we addressed the importance of the discreteness of electric charge. Taking the discreteness into account, we proposed a quantum theory for mesoscopic electric circuit and give a finite-difference Schrödinger equation for the mesoscopic electric circuit. As the Schrödinger equation for LC design in p representation becomes the well-known Mathieu equation, it is exactly solved. We obtain the wave functions in terms of Mathieu functions and the energy spectrum in terms of the eigenvalues of Mathieu equation. The discussion on the uncertainty relation for the charge and current shed some light on the knowledge of the transitional Heisenberg uncertainty relation. The discreteness of electric charge increased the uncertainty, which is related to the expectation value of the ‘‘free’’ Hamiltonian. The minimum uncertainty state we obtained is a deformation of the standard Gaussian wave packet. As further applications of our theory, the eigenstates of L design in the presence and in the absence of source were solved respectively. Introducing a gauge field and gauge transformation, we successfully obtained a formula for the persistent current on the mesoscopic pure L design in the presence of the magnetic flux. As the mesoscopic metal ring is a natural pure L design, the formula is certainly valid for the persistent current on mesoscopic rings. In our formula, the mass of electrons, the carriers for electric current, is not involved. This is worthwhile to check by experiment. Our present theory is believed to explain the Coulomb blockade on which research is in progress.

In addition, all the results in the present paper will recover the standard knowledge if one takes the continuous limit $q_e \rightarrow 0$, e.g., (5.10) becomes $\phi = LI$ in the limit of $q_e \rightarrow 0$, the well-known formula in electromagnetism. So the whole theory and their results are believed to be consistent and reasonable. One may notice that we used the charge representation and the so-called p representation. Because of the discreteness of electric charge, $\hat{p}$ is no longer a current operator, but should be understood as the usual Dirac conjugation of the charge operator satisfying (2.1) only. The operator $\hat{P}$, which is associated with a physical observable, electric current, obeys the commutation relation (4.4). Clearly, the current operator $\hat{P}$ is not a Dirac conjugation of the charge operator $\hat{q}$. So we need a new definition about such conjugation defined by (4.4).

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