Spectrum of a quantum parity non-linear Schrödinger model

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Abstract

A quantum parity non-linear Schrödinger model, a variant of the well known one, is introduced and shown to be exactly solvable. Its quantum spectrum is obtained exactly by means of the Bethe ansatz approach for both periodic or rapidly decreasing boundary conditions.

1. Over the past two decades, modern mathematical physics has witnessed a vast development of the research on integrable models. Integrable models are quite interesting from both the mathematical as well as the physical point of view. Mathematically, we encounter new concepts such as the infinite dimensional algebra, quantum group etc. From the physical point of view, these models describe physical phenomena in such diverse areas as nonlinear optics, condensed matter, plasma and high energy physics etc. The non-linear Schrödinger equation [1] is undoubtedly an important one among those models [2], and has been studied by many authors. In the present Letter, we introduce a parity non-linear Schrödinger equation which is a variant of the well known one. The equation is shown to possess a Hamiltonian structure. The quantum spectrum problem turns out to be equivalent to a quantum mechanical many-body problem with $\delta$-function interactions of both particle–particle and particle–“image” types. The spectra are solved exactly for both periodic boundary conditions and rapidly decreasing boundary conditions. The spectrum we obtained is different from the well known one [3] in which there is only a $\delta$-function interaction of particle–particle type. On the other hand, the consistency of complete integrability involves a variant of the usual Yang–Baxter equation besides the usual one.

2. The quantum parity non-linear Schrödinger equation, which we will discuss, reads

$$i\hbar \frac{\partial \hat{\Psi}}{\partial t} = -\hbar^2 \frac{\partial^2 \hat{\Psi}}{\partial x^2} + 2c \left( \hat{\Psi} \hat{\Psi}^\dagger + \hat{\rho} \hat{\rho}^\dagger \cdot \hat{\rho} \hat{\rho} \right) \hat{\Psi},$$  

(1)

1 Mailing address.
where \( \hat{P} \) stands for the parity operator defined by \((\hat{P} \hat{\Psi})(x) := \hat{\Psi}(-x)\). The ordered quantum Hamiltonian for the system takes the form
\[
H = \int_{-\infty}^{\infty} dx \left[ \hat{P} \hat{\Psi}^* \hat{P} \hat{\Psi} + c \hat{\Psi}^* \left( \hat{P} \hat{\Psi} + \mathcal{F} \hat{\Psi}^* \right) \hat{\Psi} \right].
\]
(2)

It follows that Eq. (1) is given by the Heisenberg equation
\[
\frac{\partial}{\partial t} \hat{\Psi} = \frac{i}{\hbar} \left[ \hat{\Psi}, H \right]
\]
(3)
under the following canonical commutation relations,
\[
[\hat{\Psi}(x, t), \hat{\Psi}^*(y, t)] = \delta(x - y), \quad [\hat{\Psi}(x, t), \hat{\Psi}(y, t)] = [\hat{\Psi}^*(x, t), \hat{\Psi}^*(y, t)] = 0.
\]
(4)

Since \( \hat{\Psi} \) and \( \hat{\Psi}^* \) satisfy the canonical commutation relations even in the presence of a non-linear interaction, one can build up the Hilbert space for the system. If one defines a vacuum state \( |0\rangle \) which satisfies
\[
\hat{\Psi}(x, t) |0\rangle = 0,
\]
(5)
\( \hat{\Psi}^* \) can be regarded as the creation operator. Then the \( N \)-particle state with momentum \( k_1, k_2, \ldots, k_N \) can be expressed as
\[
|k_1, \ldots, k_N\rangle = \frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} dx_1 \ldots dx_N \phi(x_1, \ldots, x_N; k_1, \ldots, k_N) \hat{\Psi}^*(x_1) \ldots \hat{\Psi}^*(x_N) |0\rangle,
\]
(6)
where \( \phi(x_1, \ldots, x_N; k_1, \ldots, k_N) = \langle 0 | \hat{\Psi}(x_1) \ldots \hat{\Psi}(x_N) |k_1, \ldots, k_N\rangle \) is the \( N \)-particle wave function of the system. Let the Hamiltonian (2) act on Eq. (6), and using the commutation relations given in Eq. (4) repeatedly and taking account of the condition \( \hat{\Psi}^*(\infty) = 0 \), we obtain that
\[
H |k_1, \ldots, k_N\rangle = \frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} dx_1 \ldots dx_N \left( -\sum_{j=1}^{N} \delta_{x_j}^2 + 2c \sum_{i<j}^{N} \left[ \delta(x_i - x_j) + \delta(x_i + x_j) \right] \right) \Phi(x_1, \ldots, x_N; k_1, \ldots, k_N) \hat{\Psi}^*(x_1) \ldots \hat{\Psi}^*(x_N) |0\rangle.
\]
(7)

Obviously, \( |k_1, \ldots, k_N\rangle \) is an eigenstate of the total Hamiltonian (2) if the wave function \( \Phi \) is an eigenfunction of the \( N \)-particle Schrödinger equation with \( \delta \)-function interactions. Thus the discussion of the spectrum of the parity non-linear Schrödinger equation is equivalent to that of the following \( N \)-body quantum mechanical problem,
\[
\left( -\sum_{j=1}^{N} \delta_{x_j}^2 + 2c \sum_{i<j}^{N} \left[ \delta(x_i - x_j) + \delta(x_i + x_j) \right] \right) \Phi(x_1, \ldots, x_N) = E \Phi(x_1, \ldots, x_N).
\]
(8)

For the non-interaction case, \( c = 0 \), the wave function would simply be a plane wave one. However, the wave function here would have more structure due to the \( \delta \)-function interaction. Clearly, it is different from the problem in Ref. [3]. Eq. (8) can be considered to describe an \( N \)-particle system with point interactions between both particle and particle, and particle and "image".

3. Now we solve the spectrum for periodic boundary conditions. In Euclidean space \( \mathbb{R}^N \) with Cartesian coordinates \( x := (x_1, \ldots, x_N) \), the set of hyperplanes \( \{ \{ x \mid x_i \mp x_j = 0 \} \mid i, j = 1, \ldots, N \} \) partition \( \mathbb{R}^N \) into finitely many regions. We use the convention that the scalar product of two \( N \)-dimensional vectors is written as \( (x \mid y) = \sum_{i=1}^{N} x_i y_i(x) \). The hyperplanes mentioned above are Weyl reflection hyperplanes \((x \mid \alpha) = 0\) where \( \alpha = e_i - e_j \) or \( e_i + e_j \) are roots of the Lie algebra \( D_N \). We denote a Weyl reflection hyperplane \( \mathbb{P}_{\alpha} := \{ x \mid (x \mid \alpha) = 0 \} \). The connected components of \( \mathbb{R}^N \setminus \{ \mathbb{P}_{\alpha} \} \) are called the Weyl chambers [4] of the Lie algebra.
We denote the Weyl group of $D_N$ by $\mathcal{W}_N$. Its basic elements are $\sigma_i$ ($i = 1, \ldots, N - 1$) and $\tilde{\sigma}_N$. They are defined by: $\sigma_i: (x_1, \ldots, x_i, x_{i+1}, \ldots, x_N) \rightarrow (x_1, \ldots, x_{i+1}, x_i, \ldots, x_N)$; $\tilde{\sigma}_N: (x_1, \ldots, x_i, x_{i+1}, \ldots, x_N) \rightarrow (x_1, \ldots, x_{i+1}, -x_i, \ldots, x_N)$. As Weyl chambers can be specified by the elements of the Weyl group, we denote it by $\mathcal{E}(\tau)$ for $\tau \in \mathcal{W}_N$.

We can find that the Schrödinger equation (8) has plane wave solutions on the domain $\mathbb{R}^N \setminus \{P_\alpha\}$. Thus we may look for solutions of the following Bethe [5] ansatz form,

$$
\Phi_i(x) = \sum_{\sigma \in \mathcal{W}_N} A(\sigma, \tau) e^{i(\lambda k \mid \sigma)} , \quad x \in \mathcal{E}(\tau),
$$

(9)

where $\mathcal{W}_N$ stands for the Weyl group of the Lie algebra $A_{N-1}$, which has $\sigma_i$ ($i = 1, \ldots, N - 1$) as its basic elements and is isomorphic to the permutation group $S_N$: $\lambda k$ stands for the image of a given $k := (k_1, \ldots, k_N)$ by the mapping $\sigma \in \mathcal{W}_N$. Eq. (9) denotes that the wave function is a piece-wise continuous function defined on a separate Weyl chamber $\mathcal{E}(\tau)$.

Substituting Eq. (9) into the discontinuity relations [6] of the derivatives of the wave function along the normal of the Weyl hyperplane $P_\alpha$, we obtain the following relations,

$$
i[(\sigma k)_i - (\sigma k)_{i+1}] [A(\sigma, \sigma_i \tau) - A(\sigma \sigma_i, \sigma_i \tau) - A(\sigma, \tau) + A(\sigma \sigma_i, \tau)]
= 2c [A(\sigma, \tau) + A(\sigma \sigma_i, \tau)],
$$

(10)

$$
i[(\sigma k)_i + (\sigma k)_{i+1}] [A(\sigma, \tilde{\sigma}_{i+1} \tau) - A(\sigma, \tau)] = c [A(\sigma, \tau) + A(\sigma, \tilde{\sigma}_{i+1} \tau)].
$$

(11)

Eq. (10) is due to the relation for $\alpha = \epsilon_i - \epsilon_{i+1}$ and Eq. (11) to that for $\alpha = \epsilon_i + \epsilon_{i+1}$.

For a bosonic system, the wave-function is supposed to be symmetric under any permutation of the coordinates. Because any element of the permutation group can be expressed as a product of the neighboring interchanges $\sigma_i$ ($i = 1, 2, \ldots, N - 1$), it requires ($\sigma_i \Phi(x) = \Phi(x)$. Since $\Phi$ is a scalar function, $(\sigma_i \Phi)$ is well defined by $\Phi(\sigma_i^{-1} x)$. Thus both sides can be written out by using Eq. (9). In terms of the evident identity $(\sigma k \mid \sigma^{-1}_i x) = (\sigma \sigma_k \mid x)$ and the rearrangement theorem of group theory, we obtain

$$
A(\sigma, \sigma_i \tau) = A(\sigma_i \sigma, \tau), \quad \text{for } i = 1, 2, \ldots, N - 1.
$$

(12)

With this relation, Eq. (10) gives

$$
A(\sigma \sigma_i, \tau) = -Y_i(\sigma k) A(\sigma, \tau),
$$

(13)

where

$$
Y_i(\sigma k) = \frac{c + i[(\sigma k)_i - (\sigma k)_{i+1}]}{c - i[(\sigma k)_i - (\sigma k)_{i+1}]}.
$$

(14)

As a result, all the coefficients $A$ defined on the same Weyl chamber are determined up to an overall scalar factor by using Eq. (13) repeatedly. Eq. (11) can be written as

$$
A(\sigma, \tilde{\sigma}_{i+1} \tau) = -Z_{i+1}(\sigma k) A(\sigma, \tau),
$$

(15)

where

$$
Z_{i+1}(\sigma k) = \frac{c + i[(\sigma k)_i + (\sigma k)_{i+1}]}{c - i[(\sigma k)_i + (\sigma k)_{i+1}]}.
$$

(16)

Eqs. (15) and (12) provide relations between the coefficients $A$ defined on different Weyl chambers. Thus we obtained an explicit result for the wave function in the form of Eq. (9).

Now we determine the secular equation of the spectrum $\{k_i\}$ for the periodic boundary conditions $\Phi(x_1, \ldots, x_j + L, \ldots, x_N) = \Phi(x_1, \ldots, x_j, \ldots, x_N)$. Suppose $x = (x_1, \ldots, x_j, \ldots, x_N)$ is a point on the Weyl chamber $\mathcal{E}(\tau)$, as a result of periodicity, $x' = (x_1, \ldots, x_j + L, \ldots, x_N)$ must be a point on another Weyl
chamber $\mathcal{C}(\gamma \tau)$, where $\gamma := \sigma_1 \sigma_2 \ldots \sigma_{N-1} \bar{\sigma}_N \ldots \bar{\sigma}_2$. Substituting Eq. (9) into the periodic boundary condition, we find that it requires

$$A(\sigma, \gamma \tau) e^{i(\sigma k) L} = A(\sigma, \tau).$$

(17)

Using the procedure

$$A(\sigma, \sigma_1 \ldots \sigma_{N-1} \bar{\sigma}_N \ldots \bar{\sigma}_2 \tau)$$

$$= A(\sigma_{N-1} \ldots \sigma_1 \sigma, \bar{\sigma}_N \ldots \bar{\sigma}_2 \tau)$$

$$= \frac{A(\sigma_{N-1} \ldots \sigma_1 \sigma, \bar{\sigma}_N \ldots \bar{\sigma}_2 \tau)}{A(\sigma_{N-2} \ldots \sigma_1 \sigma, \bar{\sigma}_N \ldots \bar{\sigma}_2 \tau)} \cdot \frac{A(\sigma_{N-2} \ldots \sigma_1 \sigma, \bar{\sigma}_N \ldots \bar{\sigma}_2 \tau)}{A(\sigma_{N-3} \ldots \sigma_1 \sigma, \bar{\sigma}_N \ldots \bar{\sigma}_2 \tau)} \ldots \frac{A(\sigma_1 \sigma, \bar{\sigma}_2 \tau)}{A(\sigma, \tau)} \cdot A(\sigma, \tau)$$

such that each fraction can be calculated in terms of Eq. (14) or Eq. (16), we obtain the following transcendental equations from Eq. (17).

$$e^{-i k L} = \prod_{j=1}^{N} \frac{c + i(k_j - k_i)}{c - i(k_j - k_i)} \frac{c + i(k_j + k_i)}{c - i(k_j + k_i)},$$

(18)

which are the secular equations for the spectrum $\{k_j\}$. Therefore, we obtained the exact eigenstates of the full Hamiltonian (2),

$$H | k_1, \ldots, k_N \rangle = \left( \sum_{j=1}^{N} k_j^2 \right) | k_1, \ldots, k_N \rangle.$$

(19)

4. Now we discuss the spectrum for rapidly decreasing boundary conditions. Above, we discussed periodic boundary conditions, where the particles can be thought of as situated on a circle of length $L$. In that case, the system is invariant under a translation $x_i \rightarrow x_i + L$ for all $i = 1, 2, \ldots, N$. This implies that

$$\exp \left( -i L \sum_{j=1}^{N} \hat{\partial}_{j} \right) \Phi( x_1, \ldots, x_N ) = \Phi( x_1, \ldots, x_N ),$$

consequently, $\sum_{j=1}^{N} k_j = n(2 \pi / L)$. This is checked by taking the logarithm of Eq. (18). For the cases beyond periodic boundary conditions, however, the system has no invariance of translation of any scale due to the $\delta(x_i + x_j)$ term. Since $\sum_{j=1}^{N} (\sigma k_j)$ equals the same value for any $\sigma \in \mathcal{W}_{\mathcal{G}}$, which is a characteristic of a system with translational invariance, the Bethe ansatz form given in Eq. (9) must be modified for rapidly decreasing boundary conditions. Thus we adopt the following form,

$$\Phi_{\tau}( x ) = \sum_{\sigma \in \mathcal{W}_{\mathcal{G}}} A(\sigma, \tau) e^{i(\sigma k \cdot x)}, \quad \tau \in \mathcal{W}_{\mathcal{G}}.$$

(20)

Different from Eq. (9), the summation here is taken over the Weyl group $\mathcal{W}_{\mathcal{G}}$ instead of $\mathcal{W}_{\mathcal{S}}$ so that the conservation of the total momentum is violated.

In the present case, the discontinuity relation along the direction $\alpha - e_i - e_{i+1}$ gives the same relation as Eq. (10), but that along $\alpha - e_i + e_{i+1}$ gives a relation which is different from its counterpart, Eq. (11), namely,

$$i[(\sigma k)_i + (\sigma k)_{i+1}][A(\sigma, \bar{\sigma}_{i+1} \tau) - A(\bar{\sigma}_{i+1} \sigma, \bar{\sigma}_{i+1} \tau) - A(\sigma, \tau) + A(\bar{\sigma}_{i+1} \sigma, \tau)]$$

$$= 2c \left[ A(\sigma, \tau) + A(\bar{\sigma}_{i+1} \sigma, \tau) \right],$$

(21)

where $\sigma \in \mathcal{W}_{\mathcal{G}}$. Noticing the invariance of Eq. (8) under $x_i \rightarrow -x_i$ for any one $i$ and using Eq. (12), we obtain that $A(\sigma, \bar{\sigma}_{i+1} \tau) = A(\bar{\sigma}_{i+1} \sigma, \tau)$ regardless of the parity for a single particle. Then Eq. (21) gives

$$A(\bar{\sigma}_{i+1} \sigma, \tau) = -\bar{Y}_{i+1}(\sigma k) A(\sigma, \tau),$$

(22)
where

\[
\tilde{Y}_{i+1}((\sigma k)) = \frac{c + i\left[(\sigma k)_{i+1} + (\sigma k)_{i+1}^*\right]}{c - i\left[(\sigma k)_{i+1} - (\sigma k)_{i+1}^*\right]}.
\]

(23)

Because the whole Weyl group $W_{\mathbb{Z}}$ can be generated by the basic elements $\sigma_1, \sigma_2, \ldots, \sigma_{N-1}, \tilde{\sigma}_N$ all coefficients $A$ on any Weyl chamber $\mathcal{C}(\tau)$ are determined up to an overall factor by $Y_j(\sigma k), i = 1, 2, \ldots, N - 1$ and $\tilde{Y}_N(\sigma k)$. Thus the wave function given in Eq. (20) is solved exactly.

If the coupling constant $c$ is negative, then there exists a bound state solution. That is the case for the rapidly decreasing boundary condition

\[
\Phi(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]

(24)

Obviously the solution given in Eq. (20) will satisfy this condition if only the $k = (k_1, \ldots, k_N)$ is purely imaginary, $k = i\kappa$, while only such exponentials occur that $(\sigma k | x) > 0$. Within each Weyl chamber there is only one of the vectors $(\sigma k)$ which satisfies this inequality [7]. In a given Weyl chamber $\mathcal{C}(\tau)$, we suppose $(\sigma k | x) > 0$ for a certain $\sigma' \in W_{\mathbb{Z}}$. In order to follow our present boundary condition (24), the other coefficients $A(\sigma, \tau) (\sigma \neq \sigma')$ must be zero so that the additional exponentials which will increase at infinity do not occur. It is guaranteed as long as

\[
Y_j(i\sigma k) = 0, \quad j = 1, 2, \ldots, N - 1, \quad \tilde{Y}_N(i\sigma k) = 0.
\]

(25)

This requires us to solve the following equations,

\[
(\sigma' k)_j - (\sigma' k)_{j+1} = c, \quad j = 1, 2, \ldots, N - 1, \quad (\sigma' k)_{N-1} + (\sigma' k)_N = c.
\]

(26)

The solutions of these equations are easily found,

\[
(\sigma' k)_N = 0, \quad (\sigma' k)_{N-1} = c, \quad \ldots, \quad (\sigma' k)_1 = (N - 1)c.
\]

(27)

Then the bound state energy is calculated as

\[
E = -\sum_{j=1}^{N} (\sigma' k)_j^2 = -\frac{1}{2}c^2N(N - 1)(2N - 1).
\]

(28)

which agrees with Ref. [7] where the Coxeter matrix was adopted.

5. We have discussed the eigenstates and spectra of a quantum parity non-linear Schrödinger equation. They are solved exactly by means of the Bethe ansatz approach for both periodic boundary conditions and rapidly decreasing boundary conditions. We have shown that Eqs. (13), (15) and (22) provide relations between the coefficients $A$ that are related via the basic elements of the Weyl group. Because any element of the Weyl group must be a product of some of its basic elements, we concluded that every coefficient is determined up to an overall factor in principle. One may wonder if those relations are all consistent. In order to answer this question, we have to examine some consistency conditions. As a matter of fact, the Weyl group $W_{\mathbb{Z}}$ has $\sigma_i^2 = 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ as identities. These identities must involve some relations $Y_j(\sigma k)Y'_j(\sigma k) = 1$ and $Y_j(\sigma_{i+1}\sigma_{i+1}^*\sigma_{i+1}\sigma_{i+1}^*\sigma_{i+1}\sigma_{i+1}^*) = Y_j(\sigma_{i+1}\sigma_{i+1}^*\sigma_{i+1}\sigma_{i+1}^*\sigma_{i+1}\sigma_{i+1}^*)Y_j(\sigma_{i+1}\sigma_{i+1}^*\sigma_{i+1}\sigma_{i+1}^*\sigma_{i+1}\sigma_{i+1}^*)Y_j(\sigma k)$ which are obtained by using Eq. (13) repeatedly. The former is the unitarity relation; the latter is the well known Yang–Baxter equation [3]. It is easy to check that the explicit expression given in Eq. (14) verifies those consistency conditions. For the Weyl group $W_{\mathbb{Z}}$, apart from the ones we mentioned we have the additional identity $\sigma_{N-2}\sigma_N\sigma_{N-2} = \sigma_N\sigma_{N-2}\sigma_N$. This leads to the relation [8]

\[
Y_{N-2}(\sigma_{N-2}\sigma k)\tilde{Y}_N(\sigma_{N-2}\sigma k)Y_{N-2}(\sigma k) = \tilde{Y}_N(\sigma_{N-2}\sigma_N\sigma k)Y_{N-2}(\sigma_N\sigma k)\tilde{Y}_N(\sigma k),
\]

which is obtained by using Eqs. (13) and (22) repeatedly. This relation is a variant of the standard Yang–Baxter equation and is verified by the explicit results given in Eqs. (14) and (23).
On the other hand, we did not solve the discontinuity relations at all the hyperplanes which are determined by all the $\delta$-function terms (in other words, the hyperplanes related to all elements of the Weyl group), but only at the hyperplanes which are related to the basic elements of the Weyl group. However, they exhausted all the independent relations due to the group characteristics and the abovementioned consistency.

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