Constructing solvable models of the quantum measurement process

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Received 22 December 1994; revised manuscript received 10 May 1995; accepted for publication 10 May 1995

Communicated by P.R. Holland

Abstract

The quantum measurement process is discussed by considering a quantum object and the measuring detector as a closed system. A way to construct explicitly solvable models is suggested. Several models of an ultrarelativistic particle coupled with various measuring detectors are given and discussed.

In order to understand quantum mechanics completely, the measurement problem [1] is a difficult question, in which the von Neumann or the Lüder projection postulate plays a crucial role. The postulate says that if a physical quantity of a quantum object is measured successively twice, then the same value is obtained each time. This means that once a determined value about the observable $A$ is measured, the state vector of the quantum object $Q$ (usually a certain superposition of some kind of basis in Hilbert space) must collapse into the corresponding eigenstate of the operator $A$. Simplified solvable models in quantum mechanics are undoubtedly very helpful to comprehend the so-called measurement problem [1,2]. In Ref. [3], the author multiplies the evolution operator of his model only by one of the components of the initial state of the quantum object and measuring device system. Thus the discussion is only about a quantum phase shifter instead of wave packet collapse. In the present Letter, we will analyse the measure process formally in terms of quantum mechanics and will formulate a condition of explicit solvability. From this condition we construct several explicitly solvable models, such as an ultrarelativistic particle interacting with fermionic or bosonic oscillators in one dimension. We also formulate models interacting with a spin array and an angular momentum array. The energy fluctuations of the measuring device around the average and the relative fluctuations for each case are calculated. In the limit of weak coupling between the quantum object and the measuring device, the results tend to the same one for each case.

As quantum mechanics is a fundamental theory governing the whole universe [4,5], we consider the quantum object $Q$ to be measured and the measuring device $M$ as a closed quantum system $Q + M$. Let $\mathcal{H}_Q$ and $\mathcal{H}_M$ stand for the Hilbert spaces of $Q$ and $M$, respectively. We assume that the operator $A$ on $\mathcal{H}_Q$ representing the observable quantity under consideration has a spectrum $\{\lambda\}$. Let $|\psi(\lambda)\rangle \in \mathcal{H}_Q$ be the normalized eigenvector corresponding to the eigenvalue $\lambda$. Obviously if there is no interaction between $Q$ and $M$, the evolutions of the states in the Hilbert space $\mathcal{H}_Q$ are independent of that in $\mathcal{H}_M$. Thus the evolution of the state of the closed
system \(Q + M\) is simply a tensor product of the independent evolution of the states in the Hilbert spaces \(\mathcal{H}_Q\) and \(\mathcal{H}_M\),

\[
\sum_{\lambda\mu} C_{\lambda}(t) D_{\mu}(t) |\psi_\lambda\rangle \otimes |\phi_\mu\rangle.
\]  

(1)

The von Neumann postulate requires for such a final state of \(Q+M\) after the measure process that \(\lambda\) and \(\mu\) are correlated, i.e., \(\mu := \mu(\lambda)\). Of course the correlation does not happen unless the mutual interaction between \(Q\) and \(M\) in the measure process is taken into account. The models which we will construct in the present Letter exactly lead to such correlations.

In order to find dynamical models which can fulfill the von Neumann postulate, we consider a total Hamiltonian for the \(Q + M\) system,

\[
H = H_Q + H_M + H',
\]  

(2)

independent of time, where \(H_Q\) and \(H_M\) are the free Hamiltonian of \(Q\) and \(M\), respectively, and \(H'\) the interaction Hamiltonian. Here "free" means that the Hamiltonians \(H_Q\) and \(H_M\) contain the operators of their own system only, i.e., they commute with each other. In this case, it is convenient to work in the interaction picture and it is easy to obtain the time evolution of the state in the Hilbert space \(\mathcal{H}_{Q+M}\),

\[
|\Psi(t)\rangle = \exp\left(-i\frac{1}{\hbar}(H_Q + H_M)t\right) U(t, t') \exp\left(i\frac{1}{\hbar}(H_Q + H_M)t'\right) |\Psi(t')\rangle,
\]  

(3)

where \(|\Psi\rangle \in \mathcal{H}_{Q+M}\) and the evolution operator fulfills

\[
i\hbar \frac{\partial U(t, t')}{\partial t} H'_I(t) U(t, t'),
\]  

(4)

with

\[
H'_I(t) = \exp\left(i\frac{1}{\hbar}(H_Q + H_M)t\right) H'(t) \exp\left(-i\frac{1}{\hbar}(H_Q + H_M)t\right),
\]  

(5)

and \(U(t, t) = 1, t'\) is a parameter due to the infinitely many interaction pictures.

From Eq. (4) we can write the evolution operator as an integral-integral,

\[
U(t, t') = \exp\left(i\frac{1}{\hbar} \int_{t'}^{t} H'_I(t'') \, dt''\right).
\]  

(6)

Of course, if the commutator of the interaction Hamiltonians in the interaction picture at different times vanishes, i.e.

\[
[H'_I(t'), H'_I(t'')] = 0,
\]  

(7)

then the evolution operator has an explicit solution. Therefore Eq. (7) is a starting point to construct explicitly solvable models for the quantum measure process.

Let us first consider an ultrarelativistic particle as the quantum object and consider a one-dimensional free bosonic or fermionic oscillator array as the measure detector. In this case, the free Hamiltonians for \(Q\) and \(M\) are

\[
H_Q = c \hat{P}, \quad H_M = \hbar \omega \sum_{l=1}^{N} (a_l^+ a_l + \frac{1}{2}),
\]  

(8)
where \( \hat{P} \) is the momentum operator of the particle, \( a_i^+ (a_i) \) is the creation (annihilation) operator of the oscillator in the \( i \)th site on the array. The interaction Hamiltonian must contain some of the operators of both \( Q \) and \( M \). Considering the changes from \( H' \) to \( H'_1 \) brought out by the free Hamiltonian (Eq. (8)), we can write the interaction Hamiltonian as

\[
H' = \sum_{i=1}^{N} V(x - x_i) f \left( a_i \exp \left( \frac{i \omega x}{c} \right), a_i^+ \exp \left( -\frac{i \omega x}{c} \right) \right),
\]

(9)

where \( x \) is the position operator of the particle, \( x_i \) is a parameter indicating the site of the \( i \)th oscillator, \( V \) is a real potential and \( f \) can be any analytical function. This interaction Hamiltonian is transformed into the one in the interaction picture via Eq. (5). The only change is that \( V(x - x_n) \) becomes \( V(x + ct - x_n) \), and it is easy to show that condition (7) is really fulfilled. Thus the evolution operator is explicitly solvable. Although an arbitrary analytical function was included in the interaction Hamiltonian (Eq. (9)), we are not sure whether this is the most general one such that commutator (7) vanishes.

Let us consider the one photon interaction case,

\[
H' = \sum_{i=1}^{N} V(x - x_i) \left[ a_i \exp \left( \frac{i \omega x}{c} \right) + a_i^+ \exp \left( -\frac{i \omega x}{c} \right) \right].
\]

(10)

This Hamiltonian has been first introduced in Ref. [9]. Of course this is also the general case for fermionic oscillators. Then the evolution operator is solved as the following product,

\[
S = \lim_{t' \to -\infty} U(t, t') = \prod_{i=1}^{N} S_{(i)},
\]

(11)

where

\[
S_{(i)} = \exp \left( \frac{\nu \delta^2}{\hbar^2 c^2} \right) \exp \left[ \frac{\nu \delta}{\hbar c} \exp \left( -\frac{i \omega x}{c} \right) a_i^+ \right] \exp \left[ \frac{\nu \delta}{\hbar c} \exp \left( \frac{i \omega x}{c} \right) a_i \right],
\]

(12)

where \( \nu \delta := \int_{-\infty}^{\infty} V(x) \, dx \). In deriving the above result, the known Baker–Compell–Hausdorff formula [6] has been used.

Suppose the initial state of \( Q \) is a plane wave \( |p\rangle \) (eigenstates of \( H_Q \)) and \( M \) is in its rest state (i.e. ground state) \( |0\rangle \) before the interaction. Notice that \( a_i|0\rangle = 0 \), the evolution of the initial state is obtained without much difficulty. For the case of \( M \) being bosonic oscillators, the result is

\[
|\Psi_{(i)}\rangle = S |\Psi_{(i)}\rangle = \exp \left[ N(\nu \delta /\hbar c)^2 \right] \sum_{n=0}^{\infty} \left( \frac{N^n}{n!} \right)^{1/2} \left( \frac{\nu \delta}{\hbar c} \right)^n |p - n\hbar \omega /c, n\rangle.
\]

(13)

Here and in the following \( |p', n\rangle \) stands for \( |p'\rangle \otimes |n\rangle \) and \( |n\rangle \) is the symmetrized state of \( M \). For the case of a fermionic oscillator, the creation and annihilation operators are Grassmannian \( (a^+)^2 = a^2 = 0 \). Taking this into account we obtained that the evolution operator is a product of the following operators,

\[
S^{(i)} = \exp \left( \frac{\nu \delta^2}{\hbar^2 c^2} \right) \left[ 1 + \frac{\nu \delta}{\hbar c} \exp \left( -\frac{i \omega x}{c} \right) a_i^+ \right] \left[ 1 + \frac{\nu \delta}{\hbar c} \exp \left( \frac{i \omega x}{c} \right) a_i \right].
\]

Similarly, the initial state of \( Q \) is chosen as a plane wave and that of \( M \) as a ground state \( |0\rangle \) before the interaction. Using the fact that \( a_i|0\rangle = 0 \), we obtain the evolution of the initial state,
\[ S|\Psi_i\rangle = \exp\left[N(v\delta/hc)^2\right] \sum_{n=0}^{N} \binom{N}{n}^{1/2} \left(\frac{v\delta}{hc}\right)^n |p - nh\omega/c, n\rangle. \] \hfill (14)

where \( \binom{N}{n} \) stands for the binomial coefficients and \( |n\rangle \) stands for the normalized state of M in which there are \( n \) fermionic oscillators in the excited state. If M is a spin array, one can also construct an explicit solvable interaction Hamiltonian. The total Hamiltonian of this case is

\[ H = c\hat{P} + \hbar\omega \sum_{l=1}^{N} (1 + \frac{1}{2}\sigma_z^{(l)}) + \sum_{l=1}^{N} V(x - x_l) \left[ \sigma_+^{(l)} \exp\left(-i\frac{\omega}{c}x\right) + \sigma_-^{(l)} \exp\left(i\frac{\omega}{c}x\right) \right]. \] \hfill (15)

This is nearly the Nakazato Hamiltonian as in Ref. [7], a development of the Coleman–Hepp model [8]. One can solve the evolution operator, which is also a product of individual operators,

\[ S^{(l)} = \cos(v\delta/hc) - i\sin(v\delta/hc) \left[ \sigma_+^{(l)} \exp\left(-i\frac{\omega}{c}x\right) + \sigma_-^{(l)} \exp\left(i\frac{\omega}{c}x\right) \right]. \] \hfill (16)

This leads to the following evolution of the state,

\[ S|p, 0\rangle = \sum_{j=0}^{N} \binom{N}{j}^{1/2} \left[ \cos(v\delta/hc) \right]^{N-j} \left[ -i\sin(v\delta/hc) \right]^{j} |p - jh\omega/c, j\rangle. \] \hfill (17)

where \( |j\rangle \) stands for the state of M in which there are \( j \) spin up and \( N - j \) spin down states.

Similarly, one can find the solvable Hamiltonian of the ultrarelativistic particle coupling with the angular momentum array. There are two possibilities:

\[ H = c\hat{P} + \hbar\omega \sum_{l=1}^{N} J_z^{(l)} + \sum_{l=1}^{N} V(x - x_l) \left[ J_+^{(l)} \exp\left(-i\frac{\omega}{c}x\right) + J_-^{(l)} \exp\left(i\frac{\omega}{c}x\right) + J_z \right] \]

or

\[ H = c\hat{P} + \sum_{l=1}^{N} \frac{J_l^2}{2l_l} + \sum_{l=1}^{N} V(x - x_l) \mathbf{b} \cdot \mathbf{J}. \] \hfill (18)

For the sake of space we omit further calculations of this case.

From the above explicit solvable models, we find that the evolution caused by the interaction Hamiltonian really leads to the correlation of the eigenstates in \( \mathcal{H}_Q \) and in \( \mathcal{H}_M \). Thus they do fulfill the von Neumann postulate. As we have obtained the evolution of the system Q + M for M as bosonic oscillators, fermionic oscillators and a spin array, respectively, we can calculate the energy fluctuation of M around their average. The results are

\[ \langle \delta H_M \rangle = \hbar\omega \sqrt{N} (v\delta/hc), \quad \langle \delta H_M \rangle = \hbar\omega \sqrt{N} \frac{v\delta/hc}{1 + (v\delta/hc)^2}, \]

\[ \langle \delta H_M \rangle = \hbar\omega \sqrt{N} \sin(v\delta/hc) \cos(v\delta/hc) \] \hfill (19)

for the bosonic, fermionic and spin array cases, respectively. The relative fluctuations are

\[ \frac{\langle \delta H_M \rangle}{\langle H_M \rangle} = \frac{2}{\sqrt{N} \left[ 1 + 2(v\delta/hc)^2 \right]} \quad \frac{\langle \delta H_M \rangle}{\langle H_M \rangle} = \frac{2}{\sqrt{N} \left[ 1 + 3(v\delta/hc)^2 \right]}, \]

\[ \frac{\langle \delta H_M \rangle}{\langle H_M \rangle} = \frac{2}{\sqrt{N} \left[ 2 \sin^2(v\delta/hc) + 1 \right]} \] \hfill (20)
for the bosonic, fermionic and spin array cases, respectively. From Eq. (19) we can see that for either the bosonic case, fermionic case or spin array, the energy fluctuations are proportional to $\hbar \omega \sqrt{N}$. Moreover, their relative fluctuation (Eq. (20)) is inversely proportional to the square root of $N$. Furthermore, we can find that both Eq. (19) and Eq. (20) are model independent in the limit of weak coupling, i.e. when $\nu \delta/\hbar c$ becomes very small.

Above we discussed the quantum measurement process. The key point is that $Q + M$ is considered as a closed system. Thus the total Hamiltonian is a sum of their free Hamiltonian $H_Q + H_M$ and the interaction Hamiltonian $H_I$ between $Q$ and $M$. In order that the time evolution operator will be explicitly solvable, the interaction Hamiltonians Eq. (9), Eq. (15) or Eq. (18) were constructed by considering that the commutator of the interaction Hamiltonian at different times vanishes in the interaction picture. It is also worthwhile to mention that the so-called free Hamiltonian in Ref. [3] is not really a free one, because the Hamiltonian for the spin array director includes functions of $x$, the position operator of the ultrarelativistic particle. In Ref. [3] a discussion about decoherence of the quantum phase shifter in the macroscopic limit was confused with the wave packet reduction. It is also worthwhile to mention that the relative fluctuation in Ref. [7] is divergent when the interaction between the quantum object and the measuring detector vanishes. This is due to the fact that the energy of the ground state of the detector was defined as zero in Ref. [7]. Although the divergence is avoided in the model of the present Letter, the free part of the Hamiltonian for the measuring device in Eq. (15) has no appealing physical meaning. It is explained in Ref. [9,10] that the divergence is a zero-temperature effect.

The authors acknowledge a referee for his constructive remarks and helpful suggestions. This work is supported by NSFC and NSF of Zhejiang province.

References