Yang Baxterization

You-Quan Li
CCAST (World laboratory), P. O. Box 8730, Beijing 100080, People's Republic of China

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On the basis of the Yang–Baxter equation and the initial condition of $\bar{R}$ matrix, Yang Baxterization of representations of braid group and that of Temperley–Lieb algebra are discussed systematically. It is shown that there exist many Yang–Baxterization formulas beyond the knowns. New developments of Yang–Baxterization approach are presented. As examples some new solutions of the Yang–Baxter equation are given.

I. INTRODUCTION

Since the Yang–Baxter equation plays a crucial role in the study of solvable models in two-dimensional statistical mechanics, quantum integrable systems, and low dimensional quantum field theory, etc., it is undoubtedly significant to study solutions of the Yang–Baxter equation. In principle, it is quite difficult to solve the Yang–Baxter equation directly, because they are functional equations.

On the basis of quantum group, or the $q$ analogs of the universal enveloping Lie algebra, Jimbo constructed a family of trigonometric solutions of the quantum Yang–Baxter equation from the classical $r$ matrix of Belavin and Drinfeld for the case of nonexceptional affine Lie algebra.

In order to understand the connection between knot theory and solvable models, Jones proposed a procedure, he called "Baxterization," to produce solutions of the Yang–Baxter equation from representations of braid group. He also gave explicit formulas to Baxterize to trigonometric solutions for the Hacke case and Birman–Murakami–Wenzl case. Later, Ge et al. succeeded in constructing Jimbo-type trigonometric $\tilde{R}(x)$ matrix via a developed approach, called Yang Baxterization, meanwhile some exotic solutions of the Yang–Baxter equation were found. Zhang et al. mainly considered representations of braid group arising from such finite-dimensional irreducible representations of quantum groups that any irreducible representation can be affinized and the tensor product of it with itself is multiplicity-free. By employing a tensor product graph and its maximal trees, an explicit formula for Jimbo-type trigonometric solutions of the Yang–Baxter equation was derived, and new solutions were found.

In this paper, further developments of the Yang–Baxterization approach are presented. In the next section, we give some general considerations on the Yang–Baxter equation. It is simple but very helpful for simplifying the discussion of the consistency condition. In Sec. III, we systematically study self Yang Baxterization of braid group representations. We find that, beyond the formulas in Refs. 9, 10 there may exist many other formulas of Yang Baxterization. As an example we derive a formula beyond the simplest case (the knowns). In Sec. IV, we derive the consistency condition for Yang Baxterization mainly for the trigonometric case. In Sec. V, we study Yang Baxterization of representations of Temperley–Lieb algebra, which includes both trigonometric and rational Yang Baxterization. In Sec. VI we present some applications of the above mentioned results, where some examples which are new solutions of the Yang–Baxter equation are given. Finally, in Sec. VII we give some discussions about the results of the paper.

"Mailing address: Department of Physics, Shanghai Jiaotong University, Shanghai 200030, People's Republic of China."
II. GENERAL CONSIDERATION

Quantum Yang–Baxter equations (in this paper we briefly say Yang–Baxter equation) are equations for a set of functions, which are generally taken as

\[
\tilde{R}_{gk}^{bc}(u)\tilde{R}_{kg}^{de}(u+v)\tilde{R}_{de}^{fg}(v) = \tilde{R}_{eg}^{bc}(v)\tilde{R}_{de}^{fg}(v+u)\tilde{R}_{ef}^{hg}(u),
\]

where \( u, v \in \mathbb{C} \) are additive spectral parameters. Convenitently Eq. (2.1) is written as the following matrix form:

\[
\tilde{R}_1(u)\tilde{R}_2(u+v)\tilde{R}_1(v) = \tilde{R}_2(v)\tilde{R}_1(v+u)\tilde{R}_2(u),
\]

where \( \tilde{R}_1 := \tilde{R} \otimes I, \tilde{R}_2 := I \otimes \tilde{R} \), \( \tilde{R} \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \) and \( I \) stands for an \( N \times N \) unit matrix. Owing to the physical significance, whether for the cases of solvable models in two-dimensional statistical mechanics and chain models in one-dimensional quantum mechanics, or in the case of two-dimensional factorized scattering theory, \( \tilde{R}(u) \) is subjected to satisfy the called initial condition.

\[
\tilde{R}(0) \propto I.
\]

Let us observe Eq. (2.2) now. If an \( \tilde{R} \) matrix under consideration satisfies condition (2.3), one can obtain from Eq. (2.2) by taking \( v = -u \) that

\[
\tilde{R}_1(u)\tilde{R}_1(-u) = \tilde{R}_2(-u)\tilde{R}_2(u).
\]

This leads to

\[
\tilde{R}(u)\tilde{R}(-u) = \tilde{R}(-u)\tilde{R}(u) = \rho(u)I,
\]

where \( \rho(u) \) must be an even scalar function. Furthermore one may notice that Eq. (2.2) is an identity either \( u = 0 \) or \( v = 0 \) as long as initial condition (2.3) holds. These characteristics will be useful for the discussion of consistency of Yang Baxterization in Sec. IV.

It is known that \( S \) matrix of a representation of braid group is a solution of the Yang–Baxter relation

\[
S_1S_2S_1 = S_2S_1S_2,
\]

which is the spectral parameter independent case of the Yang–Baxter equation.

If there exists two scalar functions \( \xi(u) \) and \( \zeta(u) \) [\( \zeta(u) \) is often a constant] such that

\[
\lim_{u \to u_0} \rho(u)\xi(u)\zeta(u) = \rho,
\]

\[
\lim_{u \to u_0} \tilde{R}(u)\xi(u)\zeta(u) = S,
\]

then

\[
\lim_{u \to -u_0} \xi(-u)\tilde{R}(u) = \rho S^{-1}.
\]

Obviously, when \( u_0 = 0 \) or \( \infty \), or \( \tilde{R}(u) \) is parametrized by periodic (or double periodic) functions, it is possible to take Eq. (2.6) as a limit case of Eq. (2.2).

Our task in the following is to explore some route or procedure [easier than to solve Eq. (2.2) directly] to obtain solutions \( \tilde{R}(u) \) of the Yang–Baxter equation. Since Yang–Baxter
relations (2.6) are algebraic equations and is certainly easier to solve than Eq. (2.2), it is significant to determine when it is possible and how to introduce a spectral parameter into a representation of braid group. On the other hand, it is known\(^{11}\) that many representations of braid group can give rise to representations of Birman–Murakami–Wenzl algebra,\(^{12}\) which contains Temperley–Lieb algebra as its subalgebra. So to Baxterize representations of Temperley–Lieb algebra to be solutions of Yang–Baxter Eq. (2.2) is another significant route.

**III. YANG BAXTERIZATION OF BRAID GROUP REPRESENTATION**

Yang Baxterization is a procedure originated by Jones\(^{8}\) to produce solutions of the Yang–Baxter equation from a known representation of braid group. The aim of the procedure is to insert a spectral parameter into a given representation of braid group so that the Yang–Baxter equation is satisfied and so that the original representation of braid group is the limit in the spectral parameter of Baxterized version.

In this section, we will derive a sequent of formulas starting from the initial condition, i.e., Eqs. (2.3) and (2.5), the latter is an immediate consequence of the former.

Given that the \(S\) matrix of a representation of braid group has \(n\) distinct eigenvalues \(\lambda_i\) \((i=1,2,...,n)\) and obeys an \(n\)th order reduction relation

\[
\prod_{i=1}^{n} (S-\lambda_i) = 0 \quad (3.1)
\]

then we have

\[
S = \sum_{i=1}^{n} \lambda_i P_i, \quad (3.2a)
\]

where \(P_i\) is the projector onto the subspace of \(\lambda_i\) and is solved uniquely to be

\[
P_i = \prod_{j \neq i} \frac{(S-\lambda_j)}{(\lambda_i-\lambda_j)}. \quad (3.2b)
\]

We restrict ourselves to consider the case that \(\tilde{R}(u)\)'s commute among themselves for different values of \(u\), i.e., \([\tilde{R}(u),\tilde{R}(u')] = 0\). This means that \(\tilde{R}(u)\)'s can be diagonalized (for real symmetric or complex without eigenvector of the null norm) "simultaneously." This reminds us to consider

\[
\tilde{R}(u) = \sum_{i=1}^{n} \Lambda_i(u) P_i, \quad (3.3)
\]

where \(\Lambda_i(u)\) are scalar functions to be determined so that

\[
\Lambda_i(u_0) - \lambda_i, \quad \Lambda_i(0) = \Lambda_j(0), \quad (3.4)
\]

\[
\Lambda_i(u) \Lambda_j(-u) = \Lambda_j(u) \Lambda_i(-u) \quad \text{for} \quad i \neq j.
\]

These conditions come from the requirements of \(\tilde{R}(u_0) = S\), Eqs. (2.3) and (2.5), respectively.

It is found without much difficulty that

\[
\Lambda_i(u) = \prod_{j=1}^{i-1} \left( \frac{\lambda_j}{\lambda_{j+1}} f_j(u) + 1 \right) \prod_{k=i}^{n-1} \left( f_k(u) + \frac{\lambda_k}{\lambda_{k+1}} \right) \cdot \lambda_n \quad (3.5)
\]
obeys Eq. (3.4), where \( f_j(u) (l=1,2,\ldots,n-1) \) are any such functions that
\[
\begin{align*}
  f_j(u_0) &= 0, \quad f_j(0) = 1, \\
  f_j(-u) &= f_l^{-1}(u).
\end{align*}
\]

(i) If we consider \( f_j(u) = v(u) \) for all \( l \), Eq. (3.5) becomes
\[
\Lambda_i(u) = \prod_{j=1}^{n-1} \left( \frac{\lambda_j}{\lambda_j+1} y(u) + 1 \right) \prod_{k=1}^{n-1} \left( y(u) + \frac{\lambda_k}{\lambda_k+1} \right) \lambda_n. \tag{3.7}
\]

It is worthwhile mentioning that the permutation of \( n \) known concrete eigenvalues of a given \( S \) matrix can be arbitrary, i.e., it can be any permutation from a given order via actions of permutation group \( S_n \). But one may notice that Eq. (3.7) is invariant under
\[
\pi': (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) \mapsto (\lambda_n, \lambda_{n-1}, \ldots, \lambda_2, \lambda_1). \tag{3.8}
\]

So we need not consider situations related to all elements of \( S_n \) but only need to consider those related to the coset representatives with respect to subgroup \( \{ e, \pi' \} \). From Eqs. (3.7), (3.3), and (3.2b), one can write out the following formulas.

(a) The case with two eigenvalues is
\[
\tilde{R}(u) = S + \lambda_1 \lambda_2 v(u) S^{-1}. \tag{3.9}
\]

An explicit formula for this case was carried out earlier in Ref. 8.

(b) The case with three eigenvalues is
\[
\tilde{R}(u) = L(u) S + M(u) I + N(u) S^{-1}, \tag{3.10a}
\]

where
\[
\begin{align*}
  L(u) &= 1 - y(u), \\
  M(u) &= (\lambda_1 + \lambda_2) (\lambda_2 + \lambda_3) \lambda_2^{-1} y(u), \tag{3.10b} \\
  N(u) &= \lambda_1 \lambda_3 v(u) [y(u) - 1].
\end{align*}
\]

(c) The case with four eigenvalues is
\[
\tilde{R}(u) = K(u) S^2 + L(u) S + M(u) I + N(u) S^{-1}, \tag{3.11a}
\]

where
\[
\begin{align*}
  K(u) &= y(u) [y(u) - 1] \alpha, \quad L(u) = [1 - y(u)] [1 + (\alpha \sigma_1 + \beta) y(u)], \\
  M(u) &= y(u) [(\alpha \sigma_2 + \lambda_1 \lambda_4 \gamma) y(u) + \beta (\lambda_2 + \lambda_3) y(u) + 1] + \lambda_2 \lambda_3 \beta, \\
  N(u) &= y(u) [y(u) - 1] [\lambda_1 \lambda_4 v(u) + \lambda_2 \lambda_3 \beta],
\end{align*}
\]

\[
\begin{align*}
  \alpha &= \frac{\lambda_1 \lambda_4 - \lambda_2 \lambda_3}{\lambda_2 \lambda_3 (\lambda_1 - \lambda_4)}, \\
  \beta &= \frac{(\lambda_2 - \lambda_3) \lambda_1}{\lambda_2 \lambda_3 (\lambda_1 - \lambda_4)}.
\end{align*}
\]
These formulas give rise to solutions of the Yang-Baxter equation and will be discussed in the next section.

(ii) As an example beyond the above simplest case, we consider \( f_1(u) = y(u) \), \( f_2(u) = y^2(u) \) or \( f_1(u) = y^2(u) \), \( f_2(u) = y(u) \) for the case with three eigenvalues. For the former case

\[
\begin{align*}
\Lambda_1 &= \left( y(u) + \frac{\lambda_1}{\lambda_2} \right) \left( y^2(u) + \frac{\lambda_2}{\lambda_3} \right) \lambda_3, \\
\Lambda_2 &= \left( \frac{\lambda_1}{\lambda_2} y(u) + 1 \right) \left( y^2(u) + \frac{\lambda_2}{\lambda_3} \right) \lambda_3, \\
\Lambda_3 &= \left( \frac{\lambda_1}{\lambda_2} y(u) + 1 \right) \left( \frac{\lambda_2}{\lambda_3} y^2(u) + 1 \right) \lambda_3.
\end{align*}
\] (3.12)

Here Eq. (3.12) does not have the symmetry properties like Eq. (3.7), so we should consider all the six permutations of eigenvalues related to \( S_3 \). Using Eqs. (3.12), (3.3), and (3.2b), we obtain that

\[
\tilde{K}(u) = L(u)S + M(u)I + N(u)S^{-1},
\] (3.13a)

where

\[
\begin{align*}
L(u) &= [y(u) - 1] [by(u) - a]a^{-1}, \\
M(u) &= (\lambda_1 + \lambda_2) (\lambda_2 + \lambda_3) y(u) [\lambda_1 - \lambda_3 y(u)] a^{-1}, \\
N(u) &= y(u) [y(u) - 1] [a\lambda_3 y(u) + b\lambda_2]a^{-1}, \\
a &= (\lambda_1 - \lambda_3)\lambda_2, \\
b &= (\lambda_1 + \lambda_2)\lambda_3.
\end{align*}
\] (3.13b)

For the latter case, the result is the same as that of Eq. (3.13) by turning \( \lambda_1, \lambda_2, \lambda_3 \), respectively, to be \( \lambda_3, \lambda_2, \lambda_1 \), which is apparently (because we will consider all six permutations) included in formula (3.13). Equation (3.13) can give rise to solutions of the Yang-Baxter equation and will be discussed in the next section.

IV. CONSISTENCY CONDITION

Now we discuss the consistency condition for the Yang-Baxterization formulas obtained in the last section to satisfy the Yang-Baxter equation. Let us first consider Eq. (3.9). Substituting it into Eq. (2.2) and using Eq. (2.6), one reaches the following equation:

\[
y(u + v) (S_1 S_2^{-1} S_1 - S_2 S_1^{-1} S_2) + y(u) y(v) \lambda_1 \lambda_2 (S_1^{-1} S_2 S_1^{-1} - S_2^{-1} S_1 S_2^{-1}) = 0.
\] (4.1)

For the \( S \) matrix with quadratic reduction relation, it can be shown to satisfy the following identity:

\[
(S_1 S_2^{-1} S_1 - S_2 S_1^{-1} S_2) + \lambda_1 \lambda_2 (S_1^{-1} S_2 S_1^{-1} - S_2^{-1} S_1 S_2^{-1}) = 0.
\] (4.2)
Then Eqs. (4.1) and (4.2) lead to
\[ y(u+v) = y(u)y(v), \tag{4.3} \]
which tells that \( y(u) = e^{-u} \), i.e., trigonometric parametrization. So Eq. (3.9) satisfies the Yang-Baxter equation (2.2) identically only for the trigonometric case.

In order to discuss the consistency condition of Yang Baxterization formulas for the case of three eigenvalues, we substitute either Eqs. (3.10a) or (3.13a) into (2.2). After using Yang-Baxter relation (2.6) and cubic reduction relation, one obtains the following equation:

\[ \theta_2^+(S_1S_2^{-1}S_1S_2^{-1}-S_2S_1^{-1}) + \theta_3^-(S_2^{-1}S_1S_2^{-1}-S_1S_2^{-1}) \]
\[ + \theta_1^+(S_1-S_2) + \theta_1^-(S_1^{-1}-S_2^{-1}) = 0, \tag{4.4a} \]

where

\[ \theta_2^+ = L(u)N(u+v)L(v), \quad \theta_3^- = N(u)\cdot L(u+v)N(v), \]
\[ \theta_1^+ = [N(u)L(u+v) - L(u)N(u+v)]\cdot M(v), \]
\[ \theta_1^- = M(u)[N(u+v)L(v) - L(u+v)N(v)], \]
\[ \theta_2^+ = [L(u)M(v) + M(u)L(v)]N(v) + \sigma_1 L(u)L(v) + \sigma_3^{-1}N(u)\cdot N(v)]M(u+v) \]
\[ - M(u)L(u+v)M(v), \]
\[ \theta_1^- = [N(u)M(v) + M(u)N(v)]N(v) + \sigma_3 L(u)L(v) + \sigma_2^{-1}N(u)\cdot N(v)]M(u+v) \]
\[ - M(u)N(u+v)M(v), \]
\[ \sigma_1 = \sum_{i} \lambda_i, \quad \sigma_2 = \sum_{i<j} \lambda_i \lambda_j, \quad \sigma_3 = \lambda_1 \lambda_2 \lambda_3. \tag{4.4b} \]

By the way, we have a few words to say about Eq. (3.6). It is easy to find that exponential function (e.g., \( y = e^{-u} \)) obeys Eq. (3.6) with \( u \to 0 \) at infinite, which gives a trigonometric parametrization of the \( \tilde{R}(u) \) matrix. Besides, Eq. (3.6) also permits pole-type parametrization, for example, \( y = \frac{x(u_0) - x(u)}{x(u_0) + x(u)} \), where \( x(u) \) is an odd function. General analysis of the latter is worth studying further, here we only discuss the trigonometric case.

If we adopt notations
\[ y := e^{-u}, \quad x := e^{-u} \tag{4.5} \]
then we know (from the discussion in Sec. II) that Yang-Baxter equation (2.2) holds for the Yang-Baxterization formulas under consideration at least when \( x = 0, y = 0, x = 1, y = 1 \), and \( x = y^{-1} \). This means that the \( \theta \)'s in Eq. (4.4b) must have \( xy(x-1)(y-1)(xy-1) \) as a common factor.

Substituting Eq. (3.10b) into Eq. (4.4b), factorizing them, and dropping out that common factor, we get one consistency condition
\[ \Phi_2 - \lambda_1 \lambda_3 \Phi_3^{-1} + (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)\lambda_2^{-1}[\Phi_2 - \Phi_2^{-1} + \lambda_2^{-1}\Phi_1 - \lambda_2 \Phi_1^{-1}] = 0, \tag{4.6a} \]

where...
\[ \Phi_3 := S_1 S_2^{-1} S_1 - S_2 S_1^{-1} S_2, \]
\[ \Phi_2 := S_1^{-1} S_2 - S_1 S_2^{-1}, \quad \Phi_1 := S_1 - S_2. \]  
(4.6b)

Therefore if Eq. (4.6) holds for some permutations of eigenvalues of \( S \) matrix, then relating to the permutation, the \( \tilde{R} \) matrix obtained from Eq. (3.10) is a solution of Yang–Baxter equation (2.2).

Similarly, using Eq. (3.13b) we obtain seven consistency conditions, which take the following forms:

\[ \varphi_3 \Phi_3 - \varphi_2 \Phi_2^{-1} + (\lambda_1 + \lambda_2) (\lambda_2 + \lambda_3) \left[ \varphi_2 \Phi_2 - \varphi_1 \Phi_1^{-1} + \varphi_1 \Phi_1 - \varphi_1 [\Phi_1^{-1}] \right] = 0. \]  
(4.7)

where \( \varphi \)'s are related to \( a, b, \) and \( \lambda_i \); \( \Phi \)'s are defined by Eq. (4.6b). When \( \lambda_1 + \lambda_2 = 0 \), they reduce to be one simple condition

\[ (S_1 S_2^{-1} S_1 - S_2 S_1^{-1} S_2) - \lambda_1 \lambda_3 (S_1^{-1} S_2 S_1^{-1} - S_2^{-1} S_1 S_2^{-1}) = 0. \]  
(4.8)

Especially if \( (S_1 S_2^{-1} S_1 - S_2 S_1^{-1} S_2) = 0 \) meanwhile \( \lambda_1 + \lambda_2 = 0 \) or \( \lambda_2 + \lambda_3 = 0 \), the consistency condition (4.7) holds identically.

For the case of four eigenvalues, substituting Eq. (3.11) into Eq. (2.2), we find that the consistency condition are seven complicated equations. It takes a long tedious careful calculation to obtain those results. For brevity, we omit them here. The concrete discussion will be presented in detail elsewhere.

V. YANG BAXTERIZATION OF REPRESENTATIONS OF TEMPERLEY–LIEB ALGEBRA

The relationship between the Temperley–Lieb algebra and the Yang–Baxter equation, i.e., Eq. (5.14), is well known.13 As in some cases11 the representation of braid group can give rise to representation of Birman–Murakami–Wenzl algebra,12 which contain Temperley–Lieb algebra as its subalgebra. For a given representation of braid group, one can check projectors (or their combinations) determined by \( S \) matrix and find those which are representations of Temperley–Lieb algebra (if there exists). As another route to Baxterize braid group representations we reformulate Yang Baxterization of Temperley–Lieb algebra. Now we are supposed to have a representation of Temperley–Lieb algebra \( T_i := T^{(1)} \otimes \cdots \otimes T^{(i-1)} \otimes T \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)} \), where \( T \) satisfies

\[ T^2 = \mu T \]  
(5.1)

and

\[ T_1 T_2 T_1 = T_1, \quad T_2 T_1 T_2 = T_2. \]  
(5.2)

As the \( \tilde{R}(u) \) matrix is only determined up to a scalar function multiplier by the Yang–Baxter equation, we can assume that

\[ \tilde{R}(u) = I + f(u) T. \]  
(5.3)

Substituting Eq. (5.3) into Eq. (2.5), we obtain that

\[ f(u) + f(-u) + \mu f(u) f(-u) = 0. \]  
(5.4)

The solution of Eq. (5.4) is found to be
where $y(u)$ is an odd function.

Owing to Eqs. (5.1), (5.2), and (5.3), Yang–Baxter equation (2.2) becomes

$$[ f(u) + f(v) - f(u + v) + f(u)f(v) + f(u + v)f(v) ](T_1 - T_2) = 0. \quad (5.6)$$

Obviously, the consistency of the Yang–Baxter equation requires the first factor of Eq. (5.6) to be null. Using Eq. (5.5) we obtain the following consistency condition:

$$y(u) + y(v) - y(u + v)\left[ 1 - \frac{4}{\mu^2} \right] y(u)y(v) + 1 = 0. \quad (5.7)$$

If $1 - 4/\mu^2 = 0$, i.e., $\mu = \pm 2$, we can immediately get a solution of Eq. (5.7)

$$y(u) = \frac{1}{\kappa} u. \quad (5.8)$$

Then we obtain a rational solution

$$f(u) = \pm \frac{u}{\kappa - u}, \quad (5.9)$$

where $\kappa$ is an arbitrary constant.

If $1 - 4/\mu^2 \neq 0$, recalling an addition formula for hyperbolic functions said

$$th(\alpha_1 + \alpha_2) = \frac{th\alpha_1 + th\alpha_2}{1 + th\alpha_1 th\alpha_2}$$

we can write out a solution of Eq. (5.7) at once

$$y(u) = \frac{1}{\nu} thu, \quad \nu^2 = 1 - \frac{4}{\mu^2}. \quad (5.10)$$

Then we have one more solution

$$f(u) = (1 - \nu^2)^{1/2} \frac{thu}{\nu - ithu}. \quad (5.11)$$

One can also introduce a new parameter $\eta$ defined by $\nu = ith\eta$ and write Eq. (5.11) again

$$f(u) = \frac{shu}{sh(\eta - u)}. \quad (5.12)$$

Finally, we conclude that if $\mu = \pm 2$ or if we can fix some parameters in $T$ such that $\mu = \pm 2$, we can obtain a rational solution of the Yang–Baxter equation, which is

$$\bar{R}(u) = I_{\pm \frac{u}{\kappa - u}} T. \quad (5.13)$$

Otherwise we can obtain a trigonometric solution of the Yang–Baxter equation for the case that $\mu$ is a parameter.
\[
\tilde{R}(u) = I + \frac{shu}{sh(\eta - u)} T,
\]

where \(\eta\) is determined by \(th^2\eta = 1 - 4/\mu^2\). We would like to give concrete examples in the next section.

VI. EXAMPLES

In this section, we will apply the previous formulations to give several concrete examples. First we consider the following braid group representation

\[
S = q \sum_a E_{aa} \otimes E_{aa} + \sum_{a < b} (t E_{ab} \otimes E_{ba} + t^{-1} E_{ba} \otimes E_{ab}).
\]

Its inverse is

\[
S^{-1} = q^{-1} \sum_a E_{aa} \otimes E_{aa} + \sum_{a < b} (t E_{ab} \otimes E_{ba} + t^{-1} E_{ba} \otimes E_{ab})
\]

of which the eigenvalues are \(q, 1,\) and \(-1\). It is easy to find that Eq. (6.1) satisfies

\[
S_1S_2^{-1}S_1S_2^{-1}S_2 = 0.
\]

Then Eq. (4.7) hold for the \(S\) matrix given by Eq. (6.1) when either \(\lambda_1 = \pm 1, \lambda_2 = \mp 1, \lambda_3 = q\) or \(\lambda_1 = q, \lambda_2 = \pm 1, \lambda_3 = \mp 1\). For the former case, we calculate from Eq. (3.13) a trigonometric \(\tilde{R}(u)\) matrix which are new solutions of the Yang–Baxter equation

\[
\tilde{R}(u) = (q \neq x^2) \sum_a E_{aa} \otimes E_{aa} + (1 \neq qx^2) \sum_{a < b} (t E_{ab} \otimes E_{ba} + t^{-1} E_{ba} \otimes E_{ab}),
\]

where \(x = e^{-u}\) and the overall scalar factor 1 – \(x\) has been omitted. For the latter case one can get the corresponding \(\tilde{R}(x)\) matrix similarly, which is just equivalent to Eq. (6.2).

As an example of Sec. V, we consider a braid group representation

\[
S = \begin{pmatrix}
q & 0 & 0 & 0 \\
t & 0 & p & 0 \\
t & p & 0 & 0 \\
2t^2 & -t & -t & q
\end{pmatrix}
\]

It has been shown\(^{10}\) to give rise to two representations of Temperley–Lieb algebra. One is

\[
T = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

which satisfies \(T^2 = -2T\). Then we obtain from Eq. (5.13) a rational solution of the Yang–Baxter equation.
\[
\tilde{R}(u) = I - \frac{u}{\kappa - u} T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \kappa & -u & 0 \\
0 & -u & \kappa & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]  

(6.4)

The other is

\[
T' = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-\frac{2t}{p-q} & -1 & -1 & 0 \\
-\frac{2t}{p-q} & -1 & -1 & 0 \\
\frac{4t^2}{(p-q)^2} & -\frac{2t}{p-q} & -\frac{2t}{p-q} & 0
\end{bmatrix},
\]

which satisfies \(T^2 = -2T\). Then we get another rational solution of the Yang–Baxter equation

\[
\tilde{R}(u) = \frac{1}{\kappa - u} \begin{bmatrix}
\kappa - u & 0 & 0 & 0 \\
\alpha u & \kappa & u & 0 \\
\alpha u & u & \kappa & 0 \\
-\alpha^2 u & -\alpha u & -\alpha u & \kappa - u
\end{bmatrix},
\]

(6.5)

where \(\alpha = 2t/(p-q)\). More explicit examples will appear in a forthcoming letter.

VII. DISCUSSIONS

Above we studied Yang Baxterization of representations of braid group and that of Temperley–Lieb algebra. When Baxterizing representations of braid group, the \(\tilde{R}\) matrix to be solved is desired to satisfy (i) initial condition, Eq. (2.3) and its immediate consequence, Eq. (2.5); (ii) \([\tilde{R}(u), \tilde{R}(v)] = 0\) for \(u \neq v\), as well as the Yang–Baxter equation. Although the formulation of Yang-Baxterization formulas is based on the \(S\) matrix having \(n\) distinct eigenvalues, the results with a slight change are also available to the case beyond that. This is because solutions of the Yang–Baxter equation permit an arbitrary overall scalar factor. The formulas (3.9) and (3.10) are valid without any change. One may notice that the denominators of Eqs. (3.11) and (3.13) would be zero if it were the case \(\lambda_1 = \lambda_4\) and \(\lambda_1 = \lambda_3\), respectively. However, we can drop out that (i.e., multiply Eq. (3.11) by \((\lambda_1 - \lambda_4)\), Eq. (3.13) by \((\lambda_1 - \lambda_3)\) as long as the result is consistent with the Yang–Baxter equation. At the time, the \(S\) matrix is no longer a limit of \(\tilde{R}(u)\).

When the \(\tilde{R}(u)\) matrices calculated from Yang-Baxterization formulas are solutions of the Yang–Baxter equation will be answered after checking the consistency condition of the Yang–Baxter equation. It has been shown that the formula (3.9) for the case that \(S\) obeys a quadratic reduction relation only presents trigonometric solutions and follows the Yang–Baxter equation identically. For \(S\) obeying a cubic reduction relation, we only discussed the trigonometric case, where the Jimbo’s type formula provides solutions of the Yang–Baxter equation as long as one consistency condition holds; another type formula provides solutions of the Yang–Baxter equation when seven consistency conditions hold.
For a representation of Temperley–Lieb algebra, it has been shown that either a rational or trigonometric solution of the Yang–Baxter equation can always be obtained. Therefore, if a representation of braid group can give rise to representations of Birman–Murakami–Wenzl algebra, one can obtain at least one solution of the Yang–Baxter equation by Baxterizing its subalgebra (Temperley–Lieb algebra).

Evidently, to develop a Yang-Baxterization procedure to reach elliptical solutions of the Yang–Baxter equation is worthwhile as well.

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