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Dynamical properties of partially coherent Bose-Einstein condensates in double wells

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Dedicated to Ulrich Eckern on the occasion of his 60th birthday.

Some dynamical features of partially coherent Bose-Einstein condensate systems confined to double-well potentials are investigated in mean field approximation. For a one-component system, the phenomenon of self-trapping is studied and it is shown that the partially coherent system, like the completely coherent one, can exhibit self-trapping phenomena which are affected by both the degree of coherence and the initial relative phase. The adiabatic properties are studied by introducing the fidelity for mixed states. It is shown that the fidelity can characterize the adiabatic condition of a partially coherent system. For a two-component system, the time evolution of the degree of coherence for a time-dependent tunneling strength is investigated. The corresponding results imply that the degree of coherence for such a system can exhibit decoherence phenomena even without a condensate-environment coupling.

1 Introduction

Since the realization of Bose-Einstein condensates (BECs) in experiment [1–3], the study of cold atoms has become a remarkable research area which is now entangled with many other research areas, such as optics, laser, superfluids and so on. In particular, the Bose-Einstein condensate in double wells offers a versatile tool to explore the underlying physics of various nonlinear phenomena because almost all of the parameters, such as the inter-well tunneling strength, the atomic interaction and the energy bias between the two wells, can be tuned experimentally. So far many fascinating features, like Rabi oscillations [4–6], Josephson oscillations [6–8], and self trapping [8–11], have been extensively studied in terms of BECs in double wells. In those works, the system were mostly assumed to be in completely coherent states at the initial time. Thus most of the authors only paid attention to the time evolution of the particle distribution but neglected that of the relative phase difference between the two wells. But we know that both the degree of coherence and the relative phase difference can affect the dynamical properties of the system, which is worthwhile to be studied.

Additionally, the Bose-Einstein condensate system in double wells is essentially a two-level system, so it is expected either to be employed as a possible qubit or to simulate certain issues [12] in quantum computation and information. Recently, there have been many interdisciplinary studies on BECs and quantum information. For example, the effect of decoherence of BECs in double wells was investigated experimentally by means of interference between BECs [13, 14] and studied theoretically in terms of the single-particle density matrix [15]. In order to exhibit the phenomena of decoherence, those authors [13–15] had to introduce the condensate-environment coupling because one-species BEC were considered there. Whereas, for two-component BECs in double wells the degree of coherence can change with time even without the condensate-environment coupling, different from the one-component case. So it is also worthwhile to investigate the coherent dynamics of two-component BEC systems.

In this paper, we study some dynamical properties of one-component and two-component BEC systems confined in double-well potentials, respectively. For a one-component system, we show that the system can exhibit self-trapping phenomena even if the system is not completely coherent. We also study the adiabatic properties of the system by introducing the fidelity for a mixed

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state. For the two-component system, with the help of the reduced single-particle density matrix, we show that such a system can exhibit decoherence phenomena without condensate-environment coupling. In the next section, we study the coherent dynamics and the time evolution of the fidelity for one-component partially coherent BECs in double wells. In Sect. 3, we investigate the time evolution of the degree of coherence for a time-dependent tunneling strength. Then a brief summary is given in Sect. 4.

2 Dynamics and fidelity for one-component BECs

We consider a one-component Bose-Einstein condensate system confined in a double well, where the atoms can tunnel between the two wells. The Hamiltonian describing such a system can be written as

$$\hat{H} = -J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{4}(\hat{n}_1 - \hat{n}_2)^2 + \frac{\gamma}{2}(\hat{n}_1 - \hat{n}_2), \quad (1)$$

where the bosonic operators $\hat{a}_\mu^\dagger$ and $\hat{a}_\mu$ ($\mu = 1, 2$) create and annihilate an atomic state in the μth well, respectively. Here the parameter γ is the energy bias between the two wells, J denotes the interwell tunneling strength, and U the interaction strength between atoms. Although this model was widely studied in mean-field approximation [16, 17], the authors assumed the system to be completely coherent. Whereas, we know that a realistic system of condensates may not always be in completely coherent states for various situations.

In order to study the partially coherent system, we need to introduce the single-particle density matrix ρ with entities $\rho_{\mu\nu}(t) = \langle \hat{a}_\mu^\dagger(t)\hat{a}_\nu(t)\rangle/N$. Clearly ρ_{11} and ρ_{22} represent the population in the 1st and in the 2nd well, respectively. The conservation of particle number requires that $\rho_{11} + \rho_{22} = 1$. In mean-field approximation, one can write the dynamical equations for the elements of $\hat{\rho}$ with the help of Heisenberg motion equation for operators,

$$\begin{aligned}
i\frac{d\rho_{11}}{dt} &= -J(\rho_{12} - \rho_{21}),\\
i\frac{d\rho_{22}}{dt} &= J(\rho_{12} - \rho_{21}),\\
i\frac{d\rho_{12}}{dt} &= -\gamma\rho_{12} + J(\rho_{22} - \rho_{11}) - UN(\rho_{11} - \rho_{22})\rho_{12},\\
i\frac{d\rho_{21}}{dt} &= \gamma\rho_{21} - J(\rho_{22} - \rho_{11}) + UN(\rho_{11} - \rho_{22})\rho_{21}, \quad (2)
\end{aligned}$$

where $\hbar$ is set to unit. As the density matrix is 2×2, the degree of coherence corresponding to $\hat{\rho}$ is given by [18]

$$\eta = 2\mathrm{Tr}\rho^2 - 1, \quad (3)$$

which is an important quantity that affects the dynamical properties of the system. With the help of the definition of the degree of coherence, we will investigate the dynamical properties of the partially coherent system based on Eqs. (2) in this section. The mean-field approach is valid in the semiclassical limit. We know that the magnitude of quantum fluctuations around the condensate state is on the scale of $1/\sqrt{N}$ at zero temperature. Since the particle number N in a BEC experiment is in the range of $10^4 - 10^7$, quantum fluctuations are characteristically small. So the mean-field approximation is an excellent one. For the system consisting of such a large number of particles, the exact diagonalization method is not useful. Note that in [19], the authors compared the results of mean-field approximation with that of the exact diagonalization method, which showed that the results obtained by the two methods are almost the same for a system consisting of 800 particles.

2.1 The phenomenon of self-trapping

We know that the phenomenon of self-trapping is the most prominent nonlinear feature of atomic tunneling between two wells. In order to study such a nonlinear feature in the partially coherent BEC system, one needs to solve Eqs. (2). Note that the mean-field equations for the completely coherent BEC system have been solved analytically by Jacobi elliptic functions in [4]. Although the equations (2) that describe the partially coherent system are different from that in [4], they can also be solved analytically by means of Jacobi elliptic functions if the parameters of the system are not time-dependent. However, since we consider a time-dependent energy bias γ in the calculation, the equations (2) can not be solved analytically. So we solve them numerically. In the calculation, we adopt a linearly time-dependent energy bias $\gamma = \alpha t$ where α is a constant characterizing the rate of the change of the energy bias γ. The initial values are taken as $\rho_{11} = \rho_{22} = \frac{1}{2}$ and $\rho_{12} = \frac{1}{2}\sqrt{\eta}\exp i\phi$, where ϕ is the phase difference between the condensates in the two wells at initial time. Our results are summarized in Fig. 1 and Fig. 2.

In Fig. 1, we plot the time-evolution of the population difference $\Delta\rho = \rho_{11} - \rho_{22}$ between the two wells for different initial relative phases and degrees of coherence. From this figure, we find that the system can exhibit the

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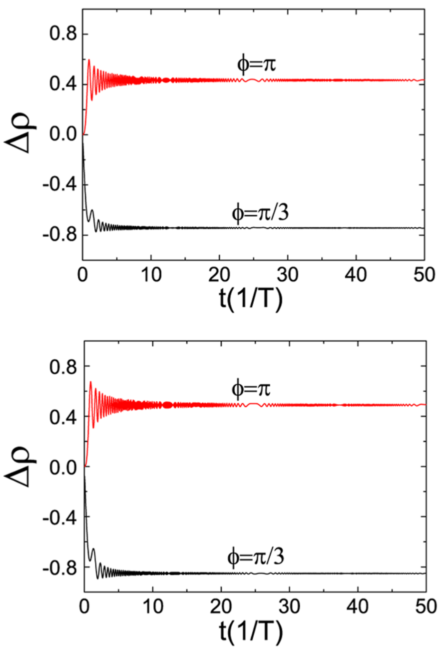

Figure 1 (online color at: www.ann-phys.org) Time-evolution of the population difference $\Delta\rho$ between the two wells for different initial relative phases and degrees of coherence ($\eta = 0.6$ for the upper panel and $\eta = 0.8$ for the lower panel). The other parameters are $UN/T = 4$ and $\alpha/T^2 = 5$.

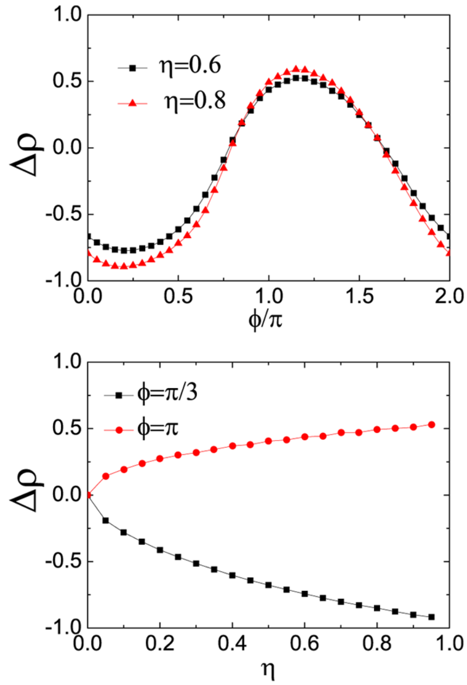

Figure 2 (online color at: www.ann-phys.org) The final population difference $\Delta\rho$ between the two wells versus the initial relative phase (upper panel) and the degree of coherence (lower panel). The parameters are the same as in Fig. 1.

phenomenon of self-trapping even if the system is not completely coherent, i.e., $0 < \eta < 1$. Comparing the two lines in the same panels of Fig. 1, we see that the number of particles in the lower well is always larger than that in the upper well (i.e., $\rho_{22} > \rho_{11}$) for $\phi = \pi/3$, which is in contrast to the case $\phi = \pi$. This implies that the initial relative phase can affect the particle distribution between the two wells. In Fig. 2, we plot the final population difference $\Delta\rho$ versus the initial relative phase and the degree of coherence in the upper and lower panel, respectively. The upper panel of Fig. 2 confirms that the initial relative phase can affect the distribution of atoms for fixed degrees of coherence. Whereas, the lower panel of Fig. 2 shows that when the initial relative phase difference is fixed, the larger the degree of coherence is, the larger the population difference between the two wells at the final time will be. Note that in this subsection, because we consider an isolated system, i.e., the system does not couple to an environment, the degree of coherence does not change with time in the calculation. Although the self-trapping phenomena have been studied in [4, 8–11], the authors mainly considered a completely coherent system. Here we consider a partially coherent system which can exhibit more fascinating self-trapping phenomena.

2.2 Adiabatic fidelity for partially coherent system

Now we are in the position to study the adiabatic properties of the partially coherent system with the help of fidelity for mixed states. As a prominent property of the quantum system, the adiabatic evolution of linear systems were widely investigated and the adiabatic condition law has been given in many textbooks of quantum mechanics. Although the adiabatic features of nonlinear system were studied in [20], the authors assumed the system to be in a completely coherent state. It is therefore worthwhile to study the adiabatic properties of the partially coherent BEC system in double wells. We introduced the definition of fidelity for an ultracold atom-to-molecule conversion system to study the relation between the atom-to-molecule conversion efficiency and the adiabatic property of the system in [21], where the system is in a completely coherent state. Now in order to study the adiabatic property for the partially coherent BEC system, we introduce the definition of fidelity for a mixed state:

$$F(\rho_1\rho_2) = \mathrm{Tr}(\rho_1\rho_2) + 2(\det\rho_1\det\rho_2)^{\frac{1}{2}}. \qquad (4)$$

The above definition requires that the matrix $\sqrt{\rho_1}\rho_2\sqrt{\rho_1}$ is 2×2 and its eigenvalues are not negative. Note that such a condition can be always satisfied in the following calculations.

We express the elements of the density matrix $\hat{\rho}$ as

$$\rho_{11} = \frac{1+\sqrt{\eta}Z}{2},$$
$$\rho_{22} = \frac{1-\sqrt{\eta}Z}{2}$$

and

$$\rho_{12} = \frac{\sqrt{\eta}}{2}(R+iI),$$

where η is the degree of coherence. Substituting the above expressions of $\rho_{\mu\nu}$ into the dynamical Eqs. (2), we get the dynamical equations for Z, R and I:

$$\frac{dZ}{dt} = -2TI,$$
$$\frac{dR}{dt} = -(\gamma+UN\sqrt{\eta}Z)I,$$
$$\frac{dI}{dt} = (\gamma+UN\sqrt{\eta}Z)R+2TZ. \tag{5}$$

From Eq. (3), we can see that the parameters Z, R and I satisfy the relation $Z^2+R^2+I^2=1$. It is easy to find that Eqs. (5) are equivalent to the nonlinear dynamical equations for a BEC confined in triple wells under the following identifications: $Z=P_l$, $I=iP_m$ and $R=P_r$, where P_l, P_m and P_r denote the population distributions in the left, middle and right well, respectively [22]. The atoms can tunnel between the left-middle and right-middle wells, and the corresponding tunneling strengths are $2T$ and $\gamma+UN\sqrt{\eta}P_L$, respectively. Note that for the BEC system in double wells the variables Z, R and I are real quantities, while for the BEC system in triple wells the variables P_l, P_m and P_r are complex numbers.

Recently, there have been several proposals to coherently manipulate single BECs [23–25] in triple-well potentials by the stimulated Raman adiabatic passage (STIRAP) technique. The success of this STIRAP technique requires that the system can adiabatically follow a dark state. The dark state involves only two states, i.e., $P_m=0$. In the extreme weak interaction limit, i.e., $UN\sqrt{\eta}/T \ll 1$, the dark state can be written as

$$|d(\theta)\rangle = \begin{pmatrix} P_r \\ 0 \\ P_l \end{pmatrix} = \begin{pmatrix} \cos\theta(t) \\ 0 \\ -\sin\theta(t) \end{pmatrix}, \tag{6}$$

where

$$\theta(t) = \arctan\left(\frac{\gamma+UN\sqrt{\eta}P_l}{2T}\right).$$

Note that $\theta(t)$ is not a parameter of the system but a dynamical variable since it contains the population difference $Z(P_l)$ in its definition. According to the above relations, the density matrix corresponding to the dark state can be written as

$$\hat{\rho}_0(t) = \begin{pmatrix} \frac{1-\sqrt{\eta}\sin\theta(t)}{2} & \frac{\sqrt{\eta}\cos\theta(t)}{2} \\ \frac{\sqrt{\eta}\cos\theta(t)}{2} & \frac{1+\sqrt{\eta}\sin\theta(t)}{2} \end{pmatrix}. \tag{7}$$

Now we calculate numerically the fidelity for the dark state $\hat{\rho}_0(t)$ to characterize whether the dark state $\hat{\rho}_0$ can be followed adiabatically. In the calculation, we assume that the system is in the dark state at the initial time and take the parameters $\gamma=\alpha t$ and $U=0.005T$. For these parameters, the initial values are $\rho_{11}(0)=\rho_{22}(0)=1/2$ and $\rho_{12}(0)=\sqrt{\eta}/2$. Then we can determine the time evolution of $\hat{\rho}(t)$ with the help of Eqs. (2). According to Eq. (4), it is easy to get the time evolution of the fidelity $F(\rho_0,\rho)$. In addition, we know that for the linear system, the adiabatic condition requires that $|\dot{\theta}|$ should be much smaller than the energy separation ΔE between the dark state and the energetically closest one. For the dark state Eq. (7), $\Delta E=\sqrt{4T^2+(\gamma+UN\sqrt{\eta}P_l)^2}$. So in the extreme weak interaction limit, the adiabatic condition of the BEC system reads $|\dot{\theta}| \ll \Delta E$. Our results are summarized in Fig. 3 and Fig. 4, which show that both F ~ 1 and $|\dot{\theta}| \ll \Delta E$ can be regarded as the adiabatic condition.

In Fig. 3, we plot the time evolution of the fidelity for the dark state and the ratio of $\dot{\theta}$ and ΔE in the case of $\eta=0.8$. From Fig. 3(a), we can see that the value of the fidelity is equal to 1 when $\alpha=0.0025(T^2)$, which implies that the dark state $\hat{\rho}_0$ can be adiabatically followed. Such a case is in contrast to that of $\alpha=0.25(T^2)$ and $\alpha=2.5(T^2)$. The results plotted in Fig. 3(a) show that the adiabatic fidelity for mixed states can characterize the adiabatic condition. Meanwhile, from the textbook of quantum mechanics, we know that if the dark state $\hat{\rho}_0$ can be adiabatically followed, the ratio of $|\dot{\theta}|$ and ΔE must be much smaller than 1, i.e., $g=|\dot{\theta}|/\Delta E \ll 1$. From Fig. 3(b), we can find that the adiabatic condition is satisfied when $\alpha=0.0025(T^2)$ but it not be satisfied when $\alpha=0.25(T^2)$ and $\alpha=2.5(T^2)$, which agrees with the results shown in Fig. 3(a). In Fig. 4, we plot the dependence of the minimum value of the fidelity and the maximum value of g on the rate of the change of the energy bias γ. From this figure, we can see that when $g \ll 1$, the value of fidelity approaches 1, which confirms that both the

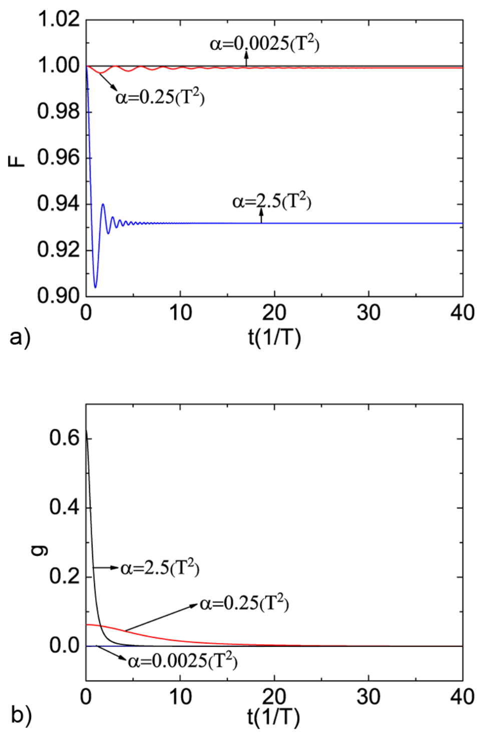

Figure 3 (online color at: www.ann-phys.org) The time evolution of the fidelity for the dark state (a), and the time evolution of the ratio of $|\dot{\theta}|$ and ΔE (b). The parameters are $U = 0.005T$ and $\eta = 0.8$.

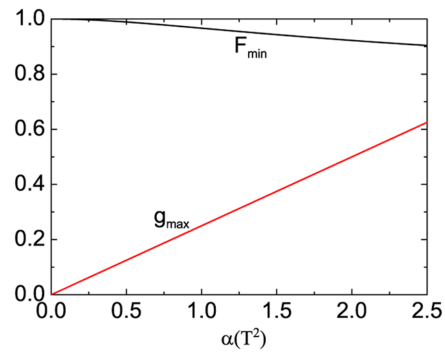

Figure 4 (online color at: www.ann-phys.org) The dependence of the minimum value of the fidelity and the maximum value of the ratio of $|\dot{\theta}|$ and ΔE on the rate of change of the energy bias γ. The parameters are the same as in Fig. 3.

fidelity for mixed state and g can characterize the adiabatic condition.

3 Time-evolution of the degree of coherence in two-component BECs

In the above section, we considered the one-component BEC system whose degree of coherence is time-independent without a condensate-environment coupling. In this section, we will study the time evolution of the degree of coherence for the two-component BEC system in double wells and show that the degree of coherence for such a two-component system can evolve with time even without the condensate-environment coupling. The Hamiltonian of the system is given by

$$\hat{H} = -J_a(\hat{a}_1^\dagger \hat{a}_2 + \text{H.c.}) - J_b(\hat{b}_1^\dagger \hat{b}_2 + \text{H.c.}) + \sum_i \frac{1}{2}(\tilde{U}_{aa}\hat{n}_{ai}\hat{n}_{ai} + \tilde{U}_{bb}\hat{n}_{bi}\hat{n}_{bi}) + \sum_i \tilde{U}_{ab}\hat{n}_{ai}\hat{n}_{bi}, \tag{8}$$

where $\hat{a}_i^\dagger$ ($\hat{b}_i^\dagger$) and $\hat{a}_i$ ($\hat{b}_i$) create and annihilate a bosonic atom of component a (b) in the ith well, respectively; $\hat{n}_{ai} = \hat{a}_i^\dagger \hat{a}_i$ ($\hat{n}_{bi} = \hat{b}_i^\dagger \hat{b}_i$) denotes the particle number operator of component a (b).

In mean-field approximation, the expectation values of the annihilation operators can be replaced by complex numbers, i.e., $a_i = \langle \hat{a}_i \rangle / \sqrt{N}$ and $b_i = \langle \hat{b}_i \rangle / \sqrt{N}$. For simplicity of writing, we introduce the wave function $|\psi\rangle$,

$$|\psi\rangle = \left(a_1,\, a_2,\, b_1,\, b_2\right)^{\mathrm{T}} = \sum_{\sigma,i} |\sigma\rangle \otimes |i\rangle, \tag{9}$$

where the state $|\sigma\rangle = |a\rangle$ or $|b\rangle$ specifies the two different species while $|i\rangle = |1\rangle$ or $|2\rangle$ specifies the two wells. Then we can write the dynamical equations for $|\psi\rangle$ with the help of Heisenberg equation for operators, namely,

$$i\frac{d}{dt}|\psi\rangle = H_{\text{eff}}|\psi\rangle, \tag{10}$$

where

$$H_{\text{eff}} = \begin{pmatrix} U_a|a_1|^2 + U_{ab}|b_1|^2 & -J_a & 0 & 0 \\ -J_a & U_a|a_2|^2 + U_{ab}|b_2|^2 & 0 & 0 \\ 0 & 0 & U_b|b_1|^2 + U_{ab}|a_1|^2 & -J_b \\ 0 & 0 & -J_b & U_b|b_2|^2 + U_{ab}|a_2|^2 \end{pmatrix}, \tag{11}$$

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with $U_a = \tilde{U}_{aa}N$, $U_b = \tilde{U}_{bb}N$, and $U_{ab} = \tilde{U}_{ab}N$. In order to study the time evolution of the system, we introduce the density matrix $\hat{\rho}' = |\psi\rangle\langle\psi|$ which is a 4×4 matrix. Its diagonal elements ρ'_{11} (ρ'_{22}) and ρ'_{33} (ρ'_{44}) represent the population of component a (b) in the first and second well, respectively. From the definition of $\hat{\rho}'$, we can see that the density matrix $\hat{\rho}'$ describes a pure state. So the time evolution of the elements of $\hat{\rho}'$ can be obtained by solving Eqs. (10). Here we only focus on the total particle distribution between the two wells but not distinguish the two components, so the system can be described by the reduced density matrix $\hat{\rho}^r = \sum_\sigma \langle\sigma|\hat{\rho}'|\sigma\rangle$ whose elements are $\rho^r_{11} = \rho'_{11} + \rho'_{33}$, $\rho^r_{12} = \rho'_{12} + \rho'_{34}$, $\rho^r_{21} = \rho'_{21} + \rho'_{43}$, and $\rho^r_{22} = \rho'_{22} + \rho'_{44}$. Clearly, the diagonal elements of $\hat{\rho}^r$ denote the total population distribution in the first and second well, respectively. The density $\hat{\rho}^r$, as a 2×2 matrix, describes a mixed state and its corresponding degree of coherence is determined by Eq. (3).

Now we are in the position to calculate the time evolution of the system described by the reduced density matrix $\hat{\rho}^r$. In the numerical calculation, we take $J_a = J_b = J$ for simplicity and choose the initial state $a_1 = \sqrt{0.3}$, $a_2 = \sqrt{0.2}\exp(i\pi)$, $b_1 = \sqrt{0.5}$, and $b_2 = 0$. Here we consider a time-dependent J,

$$J = J_0 \sin\omega t \qquad (0 \le t \le \pi/\omega), \tag{12}$$

i.e., J increases from zero to its maximum value J_0 and then decreases to zero again in the end of the calculation. We first obtain the time evolution of the elements of $\hat{\rho}'$ by solving Eqs. (10). Then we get the time evolution of the degree of coherence with the help of Eq. (3) and the definition of $\rho^r_{\mu\nu}$ ($\mu, \nu = 1, 2$).

In Fig. 5, we plot the time evolution of the degree of coherence for the form of inter-well tunneling strength given in Eq. (12). This figure shows that the degree of coherence for the two-component system can change with time without the condensate-environment coupling, which is different from the case of a one-component system. For the form of J given in Eq. (12), the degree of coherence oscillates with time and the tendency of oscillation depends on the initial states and the parameters of the system. So one can control the time evolution of the degrees of coherence by changing the initial states or tuning the parameters of the system. Note that the time evolution of the degree of coherence is not periodic for the time-dependent J we considered here. If the degree of coherence oscillates periodically with time, one can control the time evolution of degree of coherence more easily. Such a case has been discussed in [26].

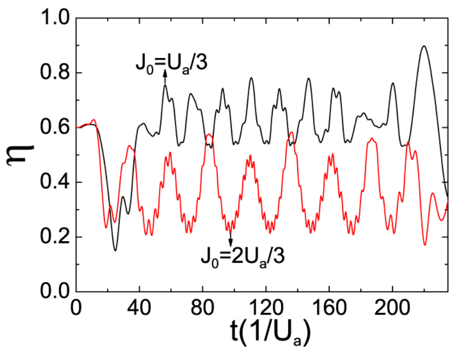

Figure 5 (online color at: www.ann-phys.org) Time evolution of the degree of coherence for a time-dependent inter-well tunneling strength. The parameters are $U_b = 2U_a/3$, $U_{ab} = U_a/3$, and $\omega = 0.04U_a/3$.

4 Summary

In the above, we studied some dynamical properties of a Bose-Einstein condensate system in double wells. For one-component BECs, we firstly studied the phenomenon of self-trapping for partially coherent systems and showed that the system can also exhibit the self-trapping phenomenon even if it is not completely coherent. Our results imply that both the degree of coherence and the initial relative phase difference can affect the final particle population distribution between the two wells. Secondly, we investigated the adiabatic property of the one-component partially coherent system by introducing the definition of fidelity for mixed states. We showed that the adiabatic fidelity can characterize whether a state can be followed adiabatically in the extreme weak interaction limit. In order to confirm our results, we compared the adiabatic condition described by the adiabatic fidelity with that given in textbooks of quantum mechanics. For a two-component BEC system, we studied the time evolution of the degree of coherence with the help of the reduced density matrix and showed that unlike the one-component system, the degree of coherence for the two-component system can change with time even without a condensate-environment coupling. This is helpful in order to control the degree of coherence of a two-component BEC system in double wells.

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Key words. Bose-Einstein condensate, degree of coherence, self-trapping, fidelity, decoherence.

References

[1] M. H. Anderson et al., Science **269**, 198 (1995).
[2] K. B. Davis et al., Phys. Rev. Lett. **75**, 3969 (1995).
[3] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).
[4] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A **55**, 4318 (1997).
[5] J. M. Choi, G. N. Kim, and D. Cho, Phys. Rev. A **77**, 010501 (2008).
[6] B. Sun and M. S. Pindzola, Phys. Rev. A **80**, 033616 (2009).
[7] R. Gati and M. K. Oberthaler, J. Phys. B **40**, R61 (2007).
[8] S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A **59**, 620 (1999).
[9] M. Albiez et al., Phys. Rev. Lett. **95**, 010402 (2005).
[10] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).
[11] G. F. Wang, L. B. Fu, and J. Liu, Phys. Rev. A **73**, 013619 (2006).
[12] D. Sokolovski and S. A. Gurvitz, Phys. Rev. A **79**, 032106 (2009).
[13] T. Schumm et al., Nature Phys. **1**, 57 (2005).
[14] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Nature (London) **449**, 324 (2007).
[15] W. Wang, L. B. Fu, and X. X. Yi, Phys. Rev. A **75**, 045601 (2007).
[16] B. Wu and J. Liu, Phys. Rev. Lett. **96**, 020405 (2006).
[17] C. Lee, Phys. Rev. Lett. **102**, 070401 (2009).
[18] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).
[19] Y. Khodorkovsky, G. Kurizki, and A. Vardi, Phys. Rev. A **80**, 02369 (2009).
[20] B. Wu and J. Liu, Phys. Rev. Lett. **96**, 020405 (2006).
[21] L. H. Lu and Y. Q. Li, Phys. Rev. A **77**, 033619 (2009).
[22] C. Ottaviani, V. Ahufinger, R. Corbalan, and J. Mompart, Phys. Rev. A **81**, 043621 (2010).
[23] E. M. Graefe, H. J. Korsch, and D. Witthaut, Phys. Rev. A **73**, 013617 (2006).
[24] M. Rab et al., Phys. Rev. A **77**, 061602(R) (2008).
[25] V. O. Nesterenko, A. N. Nikonov, F. F. de Souza Cruz, and E. L. Lapolli, Laser Phys. **19**, 616 (2009).
[26] L. H. Lu, X. Q. Xu, and Y. Q. Li, J. Phys. B **44**, 145301 (2011).