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RESEARCH ARTICLE

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Concurrence vectors for entanglement of high-dimensional systems

Abstract The concurrence vectors are proposed by employing the fundamental representation of A_n Lie algebra, which provides a clear criterion to evaluate the entanglement of bipartite systems of arbitrary dimension. Accordingly, a state is separable if the norm of its concurrence vector vanishes. The state vectors related to $SU(3)$ states and $SO(3)$ states are discussed in detail. The sign situation of nonzero components of concurrence vectors of entangled bases presents a simple criterion to judge whether the whole Hilbert subspace spanned by those bases is entangled, or there exists an entanglement edge. This is illustrated in terms of the concurrence surfaces of several concrete examples.

Keywords entanglement, concurrence, high-dimensional Hilbert space

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1 Introduction

Entanglement, as one of the most intriguing features of quantum systems, has been a subject of much study in recent years. It is regarded as a valuable resource in quantum computation and communication processions, which allows quantum physics to perform tasks that are classically impossible [1, 2]. The qubit used to be considered as the building blocks of quantum computers, and the entanglement of bipartite systems of qubits has been well studied. Most recently, moreover, an arbitrary polarization state of a single-mode biphoton was considered [3] to generate a qutrit, which is a system whose states constitute a three-dimensional Hilbert space. Experimentally, a new technique has been introduced to generate and control entangled qutrits [4], which reveals a source capable of generating maximally entangled states with a net state fidelity. The qutrit system was also suggested to be realized by nuclear magnetic resonance (NMR) utilizing deuterium nuclei partially oriented in liquid crystalline phase [5] or by trapped ions [6]. The use of qutrits instead of qubits was shown [7] to be more secure against symmetric attacks on a quantum key distribution protocol while the violation of local realism for two maximally entangled N-dimensional state is stronger than for qubits [8, 9]. Thus, an effective measurement of entanglement for high dimensional Hilbert space becomes useful.

It is known that Peres [10] proposed the partial transposition criterion as a necessary condition for separability (independent of the dimension of the state), which was later shown to be sufficient for the 2 by 2 and 2 by 3 cases by Horodecki *et al.* [11, 12]. More authors carried out further discussions and proposed various procedures [13, 14] for the similar purpose. Since the concurrence introduced earlier by Hill and Wootters [15] is an important evaluation of entanglement for qubits, a systematical extension for qutrits as well as to high-dimensional systems should be helpful.

In this paper, we present an extension of Hill-Wootters' concurrence on the basis of roots of A_n Lie algebra. In next section, we define a concurrence vector whose norm can be used to evaluate the entanglement of pure states, i.e., a separable state has a vanishing norm. Its relation to other entanglement measurement is also

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discussed. In Section 3, we extend the concurrence vector to mixed states. In Section 4, we discuss the concurrence surface for several concrete cases, which is expected to be helpful for the study of entanglement evolution. The advantage of our proposal is that one can easily judge whether a Hilbert subspace spanned by some entangled states is fully entangled, or there exists entanglement edge in the subspace. In the last section, a brief summary is given.

2 Concurrence vector for pure states: from qubits to qudits

2.1 General consideration

Let us recall the "concurrence" introduced by Hill and Wootters:

$$C(\psi) = |\langle\psi \mid \tilde{\psi}\rangle|$$

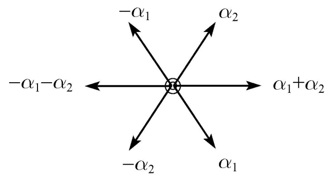
$$\mid \tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y) \mid \psi^*\rangle \tag{1}$$

which provides an easy evaluation of entanglement for a pair of qubits. The qubit is a category of two-dimensional Hilbert space well described by Pauli matrices that carry out the fundamental representation of group $SU(2)$. It is convenient to choose the $SU(2)$ generators as J_z, J_+ and J_- satisfying

$$[J_z,\, J_\pm] = \pm J_\pm, \quad [J_+,\, J_-] = 2J_z \tag{2}$$

In terms of these operators, Hill-Wootters' definition (1) can be equivalently replaced by $|\tilde{\psi}\rangle = (J_+ - J_-) \otimes (J_+ - J_-)|\psi^*\rangle$. This expression can be conveniently extended to the case of high-dimensional Hilbert space because $\{J_x,\, J_y\, J_z\}$ does not have a counterpart for high rank groups but $\{J_+,\, J_-\, J_z\}$ has. We need to describe a qutrit by the generators of $SU(3)$ group. Furthermore, the $SU(N)$ group is required for the case of N-dimensional Hilbert space. Hereafter, we will adopt the standard terminology in group theory in order to avoid possible ambiguities, while keeping physics perspective as much as possible.

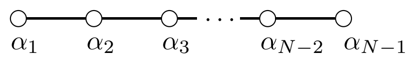
According to Cartan-Weyl analysis, the generators can be divided into two sets: the Cartan subalgebra which is the maximal Abelian subalgebra, and the remaining generators which play the similar role as the above $J_\pm$. The structure constants for the commutation relations of those operators can be described by the called root space diagram. The root space diagram for A_2 Lie algebra has a hexagonal shape:



The double circle at the center indicates the existence of two generators H_1, H_2 in the Cartan subalgebra. Unlike the spin systems whose states are labeled by the eigenvalue of J_z, which is the called magnetic quantum number, the states of $SU(3)$ system are labeled by the eigenvalues of $(H_1,\, H_2)$ which is therefore a vector called weight vector. We adopt nonorthogonal bases to expend the root vectors choosing the *simple roots* α_1 and α_2 as coordinate bases, which is clear and convenient for physicists. Placing contravariant components, $(\alpha)^i$ in conventional parentheses, we have the positive roots $\alpha_1 = (1,\, 0)$, $\alpha_2 = (0,\, 1)$, $\alpha_1 + \alpha_2 = (1,\, 1)$, and the negative roots $-\alpha_1 = (-1,\, 0)$, $-\alpha_2 = (0,\, -1)$, $-\alpha_1 - \alpha_2 = (-1,\, -1)$. When placing covariant components, $(\alpha)_j$, in square parentheses, we easily obtain from the above root space diagram that $\alpha_1 = [1,\, -1/2\,]$, $\alpha_2 = [-1/2,\, 1\,]$, $\alpha_1 + \alpha_2 = [1/2,\, 1/2\,]$, $-\alpha_1 = [-1,\, 1/2\,]$, $-\alpha_2 = [1/2,\, -1\,]$, and $-\alpha_1 - \alpha_2 = [-1/2,\, -1\,]$. Then, one can easily write out the following commutation relations:

$$\begin{aligned}
&\big[H_i,\, H_j\big] = 0 \\
&\big[H_j,\, E_\alpha\big] = (\alpha)_j E_\alpha \\
&\big[E_\alpha,\, E_{-\alpha}\big] = 2(\alpha)^i H_i \\
&\big[E_\alpha,\, E_\beta\big] = E_{\alpha+\beta}, \quad \text{if } \alpha + \beta \in \varDelta
\end{aligned} \tag{3}$$

where $\varDelta$ denotes the set of nonzero roots of nonexceptional Lie algebra. The above commutation relations imply that $E_{\pm\alpha_j}$ play the roles of raising/lowering operators like $J_\pm$ of the angular momentum operator. Moreover, there are more than one operators, H_j's, that commute to each other. For the $SU(N)$ case which corresponds to A_{N-1} Lie algebra, there are $N-1$ generators in the Cartan subalgebra, hence $i, j = 1, \cdots, N-1$ and Eq. (3) also fulfil. Because $N-1$ dimensional vectors are difficult to depict, the root space diagram is represented graphically by a two-dimensional diagram, called Dynkin diagram:



where each open dot "○" denotes a simple root, the angle between a pair of simple roots is 120° if a line connects them; that is 90° if no line connects them. Their covariant components of simple roots are easily calculated from the Dynkin diagram:

$$\begin{aligned}
\alpha_1 &= [1, -1/2, 0, \cdots, 0] \\
\alpha_2 &= [-1/2,\ 1,\ -1/2,\ 0, \cdots, 0] \\
&\cdots \\
\alpha_{N-1} &= [0,\ 0, \cdots, 0,\ -1/2,\ 1\,]
\end{aligned} \tag{4}$$

The simple roots are normalized to unity so that the structure constants in Eq. (3) differ from the Cartan matrix in textbook of group theory by a factor 1/2. The advantage of our convention is that when returning to the $SU(2)$, whose root space diagram has a simple line shape:

$$-\alpha \longleftarrow\!\!\!-\!\!\!-\!\!\!\bullet\!\!\!-\!\!\!-\!\!\!\longrightarrow \alpha$$

the eigenvalues of S_z are $1/2$ and $-1/2$, respectively for spin up and down, otherwise, they would be 1 and -1.

We consider an N-dimensional Hilbert space which carries out the fundamental representation of Lie algebra A_{N-1}, the whole weight vectors that label the state bases can be easily produced from the highest weight vector $[1/2, 0, \cdots]$ by Weyl reflection which is easily realized by means of the covariant component of simple roots, i.e.,

$$\left[\frac{1}{2},\ 0, \cdots\right] \xrightarrow{-\alpha_1} \left[-\frac{1}{2},\ \frac{1}{2}, 0, \cdots\right] \xrightarrow{-\alpha_2} \cdots \xrightarrow{-\alpha_{N-1}} \left[0, \cdots, -\frac{1}{2}\right].$$

Consequently, the N state vectors are generated by the lowering operators:

$$|1\rangle \xrightarrow{E_{-\alpha_1}} |2\rangle \xrightarrow{E_{-\alpha_2}} \cdots |N-1\rangle \xrightarrow{E_{-\alpha_{N-1}}} |N\rangle \tag{5}$$

The positive simple roots, on the other hand, just give the reverse of the above relations.

2.2 Formal extension

Now we are in the position to analyze the separability of the bipartite state $|\,\psi\rangle$. Supposing it is separable, we will have $|\,\psi\rangle = |\,\phi_A\rangle \otimes |\,\phi_B\rangle$ and generally $|\,\phi_A\rangle = \sum_{m=1}^{N} a_m\ |\,m\rangle$; $|\,\phi_B\rangle = \sum_{m=1}^{N} b_m\ |\,m\rangle$. The action of E_α on the state $|\,\phi_A\rangle$ (similar for $|\,\phi_B\rangle$) makes one term in the summation not vanish merely. This is because E_α maps some state saying $|\,m'\rangle$ to another one saying $|\,m''\rangle$ but the others to null. As a result, we have $\langle\phi_1\ |\ E_\alpha\ |\,\phi_A^*\rangle = a_{m''}^* a_{m'}^*$. Employing the operator related to the corresponding negative root, we have $\langle\phi_1\ |\ E_{-\alpha}\ |\ \phi_A^*\rangle = a_{m'}^* a_{m''}^*$. Referring to Hill and Wootters' strategy for the two-qubit case, one can extend the concept of concurrence to a *concurrence vector* defined by

$$\boldsymbol{C} = \{\langle\psi\ |\ (E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta})\ |\,\psi^*\rangle\ |\ \alpha, \beta \in \Delta^+\} \tag{6}$$

where Δ^+ denotes the set of positive roots of A_{N-1} Lie algebra. As there are totaly $N(N-1)/2$ positive roots, the concurrence vector is an $N^2(N-1)^2/4$ dimensional vector. The criterion for the separability of a joint pure state of bipartite system of arbitrary dimension is that the norm of the concurrence vector is zero, otherwise the state is entangled.

2.3 The relation to other entanglement measurements

In the above, we proposed the concurrence vector on the basis of mathematical analogy. It is worthwhile to observe the relationship between the aforementioned concurrence vector and other entanglement measurements.

For a pair of qubit and qutrit which can be regarded as a pair of spin-1/2 and spin-1, the concurrence vector is a three-dimensional vector given by

$$\boldsymbol{C} = \{\langle\psi\ |\ (\sigma_+ - \sigma_-) \otimes (E_\alpha - E_{-\alpha})\ |\,\psi^*\rangle\ \ |\ \alpha \in \Delta^+\} \tag{7}$$

where $\sigma_\pm = (\sigma_x \pm \mathrm{i}\sigma_y)/2$, and Δ^+ for A_2 contains three positive roots. Thus, the concurrence vector here is of three-dimensional. As we know, any state of a bipartite system can be expanded as

$$|\psi\rangle = \sum_{\mu,j} a_{\mu j} |\mu\rangle \otimes |j\rangle \tag{8}$$

where $a_{\mu j}$ is complex coefficients, and in our present case, $\mu = 1, 2$ and $j = 1, 2, 3$. It is easy to obtain the norm of concurrence, $|\boldsymbol{C}|^2 = C_1^2 + C_2^2 + C_3^2 = 4(a_{11}a_{22} - a_{12}a_{21})^2 + 4(a_{12}a_{23} - a_{13}a_{22})^2 + 4(a_{11}a_{23} - a_{13}a_{21})^2$.

In order to show the reliability of the concurrence vector, we consider the von Neumann entropy. The reduced density matrix ρ_A and ρ_B can be easily obtained:

$$\rho_A = aa^\dagger = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \\ a_{13}^* & a_{23}^* \end{pmatrix}$$

It is a 2×2 matrix, thus there are two eigenvalues κ_1^2 and κ_2^2, that are squares of the coefficients of Schmidt decomposition $|\psi\rangle = \kappa_1 |x_1\rangle_A |y_1\rangle_B + \kappa_2 |x_2\rangle_A |y_2\rangle_B$. Here, the κ_1^2 and κ_2^2 are the roots of the following secular equation

$$\lambda^2 - \lambda + |\boldsymbol{C}|^2/4 = 0 \tag{9}$$

where $|\boldsymbol{C}|$ is precisely the norm of the concurrence vector we proposed.

From Eq. (9), we obtain

$$\kappa_{1,2}^2 = \frac{1 \pm \sqrt{1 - |\boldsymbol{C}|^2}}{2} \tag{10}$$

So the von Neumann entropy is given by

$$E_N(|\psi\rangle) = h[(1-\sqrt{1-|\boldsymbol{C}|^2})/2] \tag{11}$$

where

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

On the other hand, one obtains ρ_B by tracing out the degree of freedom of part A, i.e.,

$$\rho_B = a^\dagger a \tag{12}$$

This is a 3×3 matrix whose eigenvalues are denoted by $\tilde{\kappa}_1^2$, $\tilde{\kappa}_2^2$, $\tilde{\kappa}_3^2$ are roots of the algebraic equation:

$$\lambda^3 - \lambda^2 + \frac{|\boldsymbol{C}|^2}{4}\lambda - \det\ \rho_B = 0 \tag{13}$$

The reduced density matrix ρ_B is of rank 2, i.e., $\det\rho_B = 0$, then there are only two non-zero eigenvalues. The von Neumann entropy takes the same form as in Eq. (11). Just like the case of Wootters [16], the von Newmann entropy here is also a monotonous function of the norm of concurrence vectors: $|\boldsymbol{C}|^2$. Therefore, the concurrence vector is a reliable measurement of entanglement of the states of qubit-qutrit system.

For bipartite qutrit system, the general state can be written as

$$|\psi\rangle = \sum_{i,j} g_{ij}|i\rangle\otimes|j\rangle \tag{14}$$

where $i,\ j = 1,2,3$. The reduced density matrix $\rho_B = \mathrm{tr}_A|\psi\rangle\langle\psi| \equiv g^\dagger g$, is clearly a positive-definite matrix, in which $g = \mathrm{mat}(g_{ij})$. Let κ_1^2, κ_2^2, κ_3^2 be the eigenvalues of ρ_B, which solve the following algebraic equation:

$$\lambda^3 - \lambda^2 + \frac{|\boldsymbol{C}|^2}{4}\lambda - \det\rho_B = 0 \tag{15}$$

where $|\boldsymbol{C}|^2$ is just the square of the norm of the concurrence vector we proposed, namely,

$$|\boldsymbol{C}|^2 = 4(\kappa_1^2\kappa_2^2 + \kappa_2^2\kappa_3^2 + \kappa_3^2\kappa_1^2)$$

The calculation of the von Neumann entropy exhibits an explicit relation to the norm of the concurrence vector:

$$E_N(|\psi\rangle) = h\big(x^+, x^-\big)$$

where

$$h(x^+,x^-) = -x^+\log_2 x^+ - x^-\log_2 x^- \\ -(1-x^+-x^-)\log_2(1-x^+-x^-) \tag{16}$$

and

$$x^\pm = \frac{1}{3} + \mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{3}}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \\ +\mathrm{e}^{\mp\mathrm{i}\frac{2\pi}{3}}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

$$p = \frac{|\boldsymbol{C}|^2}{4} - \frac{1}{3}$$

$$q = \det\rho_B + \frac{2}{27} - \frac{|\boldsymbol{C}|^2}{12}$$

Clearly, the von Neumann entropy depends on the norm of the concurrence vector monotonically. Unlike the von Neumann entropy for qubit which depends only on the concurrence [15], it also depends on the determinate of the reduced density matrix. The supremum (dashed line) and infimum for the von Neumann entropy versus $|\boldsymbol{C}|$ was plotted in Fig. 1, where both points of the entanglement maximum and non-entanglement coincide. It is clearly a convex function.

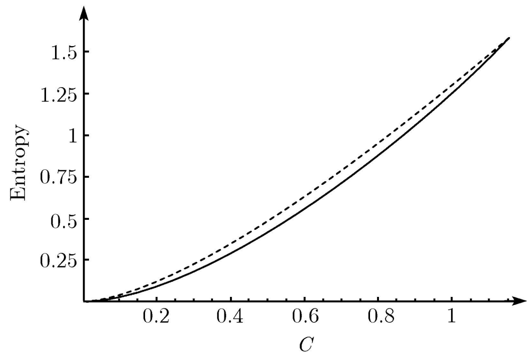


Fig. 1 The von Neumann entropy versus the norm of concurrence vector. Dashed line is the supremum and solid line the infimum.

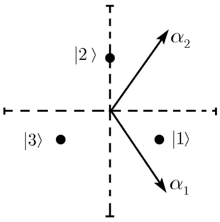
The linear entropy, also a measurement of entanglement, is given by

$$\begin{aligned} E_L(|\psi\rangle) &= 1 - \mathrm{tr}\ \rho_B^2 \\ &= (\kappa_1^2+\kappa_2^2+\kappa_3^2)^2 - \big(\kappa_1^4+\kappa_2^4+\kappa_3^4\big) \\ &= 2(\kappa_1^2\kappa_2^2+\kappa_2^2\kappa_3^2+\kappa_3^2\kappa_1^2) = |\boldsymbol{C}|^2/2 \end{aligned}$$

which indicates a direct relation to the norm of the concurrence vector. The magnitude of the linear entropy arranges from 0 to $1-1/d$ (here $d=3$). It is clearly a monotonically increasing function versus the norm of concurrence vector. Hence, the concurrence vector we proposed is a reasonable measurement of entanglement for qutrits.

2.4 Qutrit via $SU(3)$ states

Now we consider the qutrit as a concrete example. The states $|1\rangle$, $|2\rangle$ and $|3\rangle$ and the corresponding weights (denoted by black dots) $[1/2, 0]$, $[-1/2, 1/2]$ and $[0, -1/2]$ are plotted in the following:



This represents the fundamental representation of $SU(3)$. It is well known that the direct product representation is reduced to two irreducible representations, i.e., $\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^* \oplus \mathbf{6}$. The bases for the $\mathbf{3}^*$ dimensional representation are all entangled states, $|\psi_1^-\rangle$, $|\psi_2^-\rangle$, $|\psi_3^-\rangle$, but they are not the maximally entangled states. Additionally, it is not possible to produce maximally entangled state by superposition in this subspace.

Among the bases for the $\mathbf{6}$ dimensional representation (hexad), $|\psi_1^+\rangle$, $|\psi_2^+\rangle$ and $|\psi_3^+\rangle$ are entangled (not maximally entangled) states, but the other three $|11\rangle$, $|22\rangle$ and $|33\rangle$ are not entangled states. Whereas, three maximally entangled states, $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\varphi_3\rangle$ can be obtained by superposition of those three states. All the mentioned states are given in the following:

$$
\begin{aligned}
|\varphi_1\rangle &= \frac{1}{\sqrt{3}}\big(|11\rangle + |22\rangle + |33\rangle\big)\\
|\varphi_2\rangle &= \frac{1}{\sqrt{3}}\big(|11\rangle + \mathrm{e}^{\mathrm{i}2\pi/3}\,|22\rangle + \mathrm{e}^{-\mathrm{i}2\pi/3}\,|33\rangle\big)\\
|\varphi_3\rangle &= \frac{1}{\sqrt{3}}\big(|11\rangle + \mathrm{e}^{-\mathrm{i}2\pi/3}\,|22\rangle + \mathrm{e}^{\mathrm{i}2\pi/3}\,|33\rangle\big)\\
|\psi_1^\pm\rangle &= \frac{1}{\sqrt{2}}\big(|12\rangle \pm |21\rangle\big)\\
|\psi_2^\pm\rangle &= \frac{1}{\sqrt{2}}\big(|23\rangle \pm |32\rangle\big)\\
|\psi_3^\pm\rangle &= \frac{1}{\sqrt{2}}\big(|13\rangle \pm |31\rangle\big)
\end{aligned} \tag{17}
$$

The above nine states are orthonormal to each other. Their concurrence vectors are calculated to be

$$
\begin{aligned}
\boldsymbol{C}_{|\varphi_1\rangle} &= \left(\frac{2}{3},\, 0,\, 0,\, 0,\, \frac{2}{3},\, 0,\, 0,\, 0,\, \frac{2}{3}\right)\\
\boldsymbol{C}_{|\varphi_2\rangle} &= \left(\frac{2}{3}\mathrm{e}^{-\mathrm{i}2\pi/3}, 0, 0, 0, \frac{2}{3}, 0, 0, 0, \frac{2}{3}\mathrm{e}^{\mathrm{i}2\pi/3}\right)\\
\boldsymbol{C}_{|\varphi_3\rangle} &= \left(\frac{2}{3}\mathrm{e}^{\mathrm{i}2\pi/3}, 0, 0, 0, \frac{2}{3}, 0, 0, 0, \frac{2}{3}\mathrm{e}^{-\mathrm{i}2\pi/3}\right)\\
\boldsymbol{C}_{|\psi_1^\pm\rangle} &= (\mp 1,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0)\\
\boldsymbol{C}_{|\psi_2^\pm\rangle} &= (0,\, 0,\, 0,\, 0,\, \mp 1,\, 0,\, 0,\, 0,\, 0)\\
\boldsymbol{C}_{|\psi_3^\pm\rangle} &= (0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, \mp 1)
\end{aligned} \tag{18}
$$

In addition to $|\varphi_1\rangle, \cdots, |\varphi_3\rangle$, there are six more orthonormal bases,

$$
\begin{aligned}
|\varphi_4\rangle &= (|12\rangle + |23\rangle + |31\rangle)/\sqrt{3}\\
|\varphi_5\rangle &= (|12\rangle + \mathrm{e}^{\mathrm{i}2\pi/3}\,|23\rangle + \mathrm{e}^{-\mathrm{i}2\pi/3}\,|31\rangle)/\sqrt{3}\\
&\cdots\\
|\varphi_9\rangle &= (|21\rangle + \mathrm{e}^{-\mathrm{i}2\pi/3}\,|32\rangle + \mathrm{e}^{\mathrm{i}2\pi/3}\,|13\rangle)/\sqrt{3}
\end{aligned}
$$

which were adopted as a generalization of EPR pairs when discussing teleportation [17]. The concurrence vectors for those states are easily calculated, for example, $\boldsymbol{C} = (0, 2/3, 0, 0, 0, -2/3, -2/3, 0, 0)$ for $|\varphi_4\rangle$. Actually, all the states $|\varphi_l\rangle$, $(l = 4, \cdots, 9)$ are shown to be maximally entangled by calculating the norm of their concurrence vectors $|\boldsymbol{C}|^2 = 4/3$.

2.5 Qutrit via $SO(3)$ states

The fundamental representation of $SU(2)$ is carried out by spin-1/2 system which refers to the qubit. We known the Bell bases, $|\psi_\pm^B\rangle = \big(|\!\uparrow\downarrow\rangle \pm |\!\downarrow\uparrow\rangle\big)/\sqrt{2}$, $|\varphi_\pm^B\rangle = \big(|\!\uparrow\uparrow\rangle \pm |\!\downarrow\downarrow\rangle\big)/\sqrt{2}$ are simultaneous eigenstates of $S_x \otimes S_x$ and $S_z \otimes S_z$, which is not valid for spin-1 [i.e., $SO(3)$ representation] or high spin systems. This is because $S_x \otimes S_x$, $S_y \otimes S_y$ and $S_z \otimes S_z$ commute to each other only for spin-1/2 case. We consider such a pair of qutrits that their direct product representation decomposes into $SO(3)$ irreducible representations, i.e., $\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$. Some of their states read,

$$
\begin{aligned}
|\chi_1^\pm\rangle &= \frac{1}{\sqrt{2}}\big(|01\rangle \pm |10\rangle\big)\\
|\chi_{-1}^\pm\rangle &= \frac{1}{\sqrt{2}}\big(|{-1}0\rangle \pm |0\,{-1}\rangle\big)\\
|\chi_0^\pm\rangle &= \frac{1}{\sqrt{4 \pm 2}}\big(|{-1}1\rangle \pm |00\rangle + |00\rangle \pm |1\,{-1}\rangle\big)\\
|\chi_0^0\rangle &= \frac{1}{\sqrt{3}}\big(|{-1}1\rangle - |00\rangle + |1\,{-1}\rangle\big)\\
|\phi^\pm\rangle &= \frac{1}{\sqrt{2}}\big(|11\rangle \pm |{-1}\,{-1}\rangle\big)
\end{aligned} \tag{19}
$$

It is easy to calculate their concurrence vectors:

$$
\begin{aligned}
\boldsymbol{C}_{|\chi_1^\pm\rangle} &= (\mp 1,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0)\\
\boldsymbol{C}_{|\chi_{-1}^\pm\rangle} &= (0,\, 0,\, 0,\, 0,\, \mp 1,\, 0,\, 0,\, 0,\, 0)\\
\boldsymbol{C}_{|\chi_0^-\rangle} &= (0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 1)
\end{aligned}
$$

$$C_{|\chi_0^+\rangle} = \left(0, -\frac{2}{3}, 0, -\frac{2}{3}, 0, 0, 0, 0, -\frac{1}{3}\right)$$

$$C_{|\chi_0^0\rangle} = \left(0, \frac{2}{3}, 0, \frac{2}{3}, 0, 0, 0, 0, -\frac{2}{3}\right)$$

$$C_{|\phi^\pm\rangle} = (0, 0, 0, 0, 0, 0, 0, 0, \pm 1) \quad (20)$$

The evaluation of the norm of concurrence vector indicates that the singlet $|\chi_0^0\rangle$ is maximally entangled $|\boldsymbol{C}|^2 = 4/3$; the triplet, $|\chi_j^-\rangle$ [hereafter for $SO(3)$ $j = -1, 0, 1$] are entangled but not maximally entangled $|\boldsymbol{C}|^2 = 1$; among the pentads only $|\chi_j^+\rangle$ are entangled states while $|11\rangle$ and $|-1-1\rangle$ are unentangled states whose superposition provides the two entangled states, $|\phi^\pm\rangle$.

3 Concurrence vector for mixed states

In the light of our extension of the measurement of entanglement in terms of concurrence vector for pure states, it is natural to introduce

$$|\tilde{\psi}\rangle_{\alpha\beta} = (E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) \, |\psi^*\rangle \quad (21)$$

here α and β refer to the aforementioned positive roots of A_{N-1} Lie algebra. Similar to the strategy of Wootters [13], we define some matrices given by

$$\tau_{ij}^{\alpha\beta} = \langle v_i \mid (E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) \, |v_j^*\rangle \quad (22)$$

where $\{| v_i\rangle \mid i = 1, 2, \cdots\}$ are the eigenvectors of the density matrix ρ which characterizes a given mixed state. Apparently, the matrices (τ_{ij}) are symmetric. According to Takagi factorization [18], for any symmetric matrix A, there exists a unitary U and a real nonnegative diagonal matrix $d = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ such that $A = Ud\,U^{\mathrm{T}}$ and the diagonal entries of d are the nonnegative square roots of the corresponding eigenvalues of AA^*. Thus, there exists a decomposition $|x_i\rangle = U_{ij}^* \, |v_j\rangle$ satisfying

$$\langle x_i \mid (E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) \, |x_j^*\rangle$$
$$= (U\tau^{\alpha\beta}U^{\mathrm{T}})_{ij} = \lambda_i^{\alpha\beta}\delta_{ij} \quad (23)$$

Here, $\lambda_{ij}^{\alpha\beta}$ is the nonnegative root of the eigenvector of $\tau\tau^*$ and

$$\tau^{\alpha\beta}\tau^{\alpha\beta *} = \sqrt{\rho}(E_\alpha - E_{-\alpha})$$
$$\otimes (E_\beta - E_{-\beta})\rho^*(E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta})\sqrt{\rho} \quad (24)$$

One can make another decomposition of ρ in terms of $\{|y_i\rangle\}$:

$$|y_1\rangle = |x_1\rangle; \quad |u_j\rangle = i|x_j\rangle, \quad j \neq 1 \quad (25)$$

where $\lambda_2 = \max\{\lambda_i, i = 1, \cdots, N\}$. If one repeats the steps given by Wootters in Ref. [16], one can show that for some given positive roots α, β, the concurrence expressed as

$$C^{\alpha\beta} = \lambda_1^{\alpha\beta} - \sum_{i=2}^{n} \lambda_i^{\alpha\beta}$$

Thus, we derive a formula to calculate the concurrence vector for mixed state. Then the norm of such a concurrence vector can be employed to measure the entanglement of mixed state:

$$|\boldsymbol{C}|^2 = \sum_{\alpha\beta} |C_{\alpha\beta}|^2 \quad (26)$$

For the $SU(2)$ case there is only one positive root, which corresponds to a pair of qubits; the concurrence vector (26) is one dimensional, which is just the original definition of Wootters' concurrence.

4 Concurrence surface and entanglement edge

Calculating the norm of the concurrence vectors for an arbitrary normalized vectors in the three-dimensional subspace spanned by either $\{|\psi_1^-\rangle, |\psi_2^-\rangle, |\psi_3^-\rangle\}$, or $\{|\psi_1^+\rangle, |\psi_2^+\rangle, |\psi_3^+\rangle\}$, or $\{|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\}$ respectively, we obtain several conclusions. Any states in the space of the former two cases yield a constant norm of concurrence vectors, which means that the Hilbert subspace manifests a fixed entanglement. The norm of the concurrence vector vanishes in the third case when the expanding coefficients take some particular magnitudes. This indicates that some states in that Hilbert subspace are separable. Braunstein *et al.* [19] analyzed the separability of N-qubit states near the maximally mixed state. Vidal and Tarrach [20] give a separability boundary for the mixture of the maximally mixed state with a pure state. Caves and Milburn [21] discussed the lower and upper bounds on the size of the neighborhood of separable states around the maximally mixed state of qutrits. One will see in the following that all these become easily understandable by making use of the concept of concurrence vectors.

The entanglement features of those three Hilbert subspaces discussed previously arise from the sign properties of the concurrence vectors. Consider a Hilbert subspace constituted by some bases obeying

$$\langle \psi_\mu \mid \tilde{\psi}_\nu \rangle_{\alpha\beta} \propto \delta_{\mu\nu} \quad (27)$$

where $|\tilde{\psi}_\nu\rangle_{\alpha\beta}$ is defined by Eq. (21). One immediate criterion is that the state vectors lying in the Hilbert subspace are all entangled as long as the corresponding nonzero components of the concurrence vectors for those bases have the same signs (i.e., positive or negative). Otherwise, there exists a separable state in the Hilbert subspace spanned by those entangled bases.

It is instructive to observe the entanglement edges in the subspace of the 6 dimensional representation of $SU(3)$. Since the norm of concurrence vectors are constant in the subspace spanned by $|\psi_j^+\rangle$, we let the coefficients of these three bases equal to p. We also choose the coefficients for $|11\rangle$, $|22\rangle$ and $|33\rangle$ to be the same q so as to plot a three-dimensional picture. The curve of the norm of the concurrence vector versus p and q is given in Fig. 2(a). When $p \to \sqrt{2}q$ and $q \to 1/\sqrt{3}$ it approaches the entanglement edge.

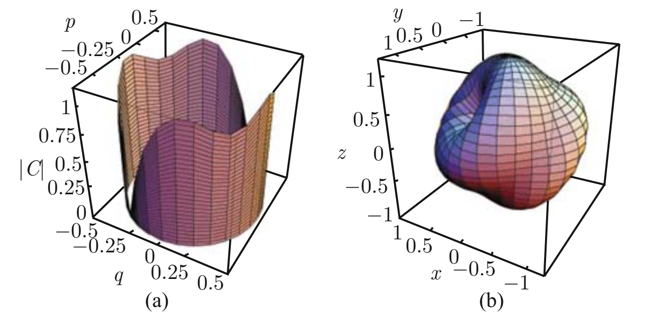


Fig. 2 The left is the relation of $|\boldsymbol{C}|$ versus p and q for state $p(|\psi_1^+\rangle + |\psi_2^+\rangle + |\psi_3^+\rangle) + q(|11\rangle + |22\rangle + |33\rangle)$. The right is the surface $|\boldsymbol{C}(\theta, \phi)|$ for Hilbert subspace $\sin\theta\cos\phi\,|\varphi_1\rangle + \sin\theta\sin\phi\,|\varphi_2\rangle + \cos\theta\,|\varphi_3\rangle)$.

Now we consider the case for $SO(3)$. As nonzero components of concurrence vectors for the triplet are positive and that for the three entangled bases in the pentad are negative, both the whole Hilbert subspace spanned by $\{|\chi_j^-\rangle\}$ and the subspace spanned by $\{|\chi_j^+\rangle\}$ are entangled. Because the nonzero components of concurrence vectors for $|\phi^+\rangle$ is positive but that for $|\phi^-\rangle$ is negative [see Eq. (20)], the five- dimensional Hilbert subspace spanned by $\{|\chi_j^+\rangle\}$ and $|\phi^\pm\rangle$ are not fully entangled.

The parameter space of general states in some three-dimensional Hilbert subspace is described by a two-sphere (θ, ϕ) and a phase factor $\mathrm{e}^{\mathrm{i}\delta}$ which is fixed to a unit without loss of generality. It is illuminative to observe the geometry structures by plotting the norm of the concurrence vector as radial coordinate, i.e., $r(\theta, \phi) = |\boldsymbol{C}_{|\theta,\phi\rangle}|$, we call it *concurrence surface*. The concurrence surfaces for the triplets of either $SO(3)$ or $SU(3)$ is simply a sphere of unit radius, which encloses a spheroid of volume $4\pi/3$. The concurrence surface for the Hilbert subspace spanned by the three maximally entangled states $\{\varphi_j\}$ of $SU(3)$ is plotted in Fig. 2 (b), whose enclosure merely occupies 13.5 % more volume than a unit spheroid does though their bases are maximally entangled. This is obviously due to the presence of entanglement edges. The concurrence surfaces for the Hilbert subspace spanned by the three originally entangled states in the pentad [Fig. 3(a)], and that spanned by the singlet together with $\varphi^\pm$ are plotted in Fig. 3(b). Their enclosures occupy volumes of 3.188 68 and 2.759 16 respectively, the latter is smaller though one of its bases is maximally entangled.

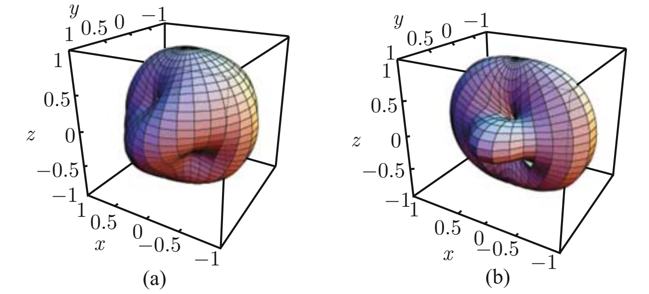


Fig. 3 The surfaces $|\boldsymbol{C}(\theta, \phi)|$ for the Hilbert subspaces $\sin\theta\cos\phi\,|\psi_1^+\rangle + \sin\theta\sin\phi\,|\psi_2^+\rangle + \cos\theta\,|\psi_3^+\rangle)$(**a**), and $\sin\theta\cos\phi\,|\chi_0^0\rangle + \sin\theta\sin\phi\,|\phi^+\rangle + \cos\theta\,|\phi^-\rangle$(**b**).

Quantum entanglement implies correlations between the results of measurements on component subsystems of a larger physical system, which cannot be understood by means of correlations between local classical properties inherent in those subsystems. Wang and Zanardi [22] studied the entanglement of unitary operators on a bipartite quantum system that is related to the entangling power of the associated quantum evolutions. It maybe useful to associate those topics with the evolution of concurrence vectors. Zanardi and Rasetti [23] showed that the notion of generalized Berry phase can be used for enabling quantum computation, which also supports the necessity of the investigation on the parameter manifold related to quantum systems with high rank symmetries.

5 Summary

We proposed a concurrence vector to measure the entanglement of bipartite systems of arbitrary dimension by employing the fundamental representation of A_n ($n = N-1$) Lie algebra. We have shown that the norm of the concurrence vector can be used to evaluate the entanglement of a state, i.e., a separable state has a vanishing norm. Another advantage of concurrence vectors is easy to judge whether a Hilbert subspace spanned by some entangled bases is fully entangled states, or there exists entanglement edge in the subspace. If the corresponding nonzero components of concurrence vectors of the basis states are all positive or all negative, all the state vectors lying in this Hilbert subspace are entangled. Otherwise, there exists a separable state in the Hilbert subspace spanned by those entangled bases. We calculated the concurrence vectors for the states related to $SU(3)$ and $SO(3)$ explicitly. We also discussed concurrence surface and compared the volumes enclosed by the surface for various cases. Their geometry will be useful for understanding the entanglement capacities in various Hilbert subspaces.

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