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$SU(2) \times U(1)$ unified theory for charge, orbit and spin currents

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Abstract
Spin and charge currents in systems with Rashba or Dresselhaus spin–orbit couplings are formulated in a unified version of four-dimensional $SU(2)\times U(1)$ gauge theory, with $U(1)$ being the Maxwell field and $SU(2)$ being the Yang–Mills field. While the bare spin current is non-conserved, it is compensated by a contribution from the $SU(2)$ gauge field, which gives rise to a spin torque in the spin transport, consistent with the semi-classical theory of Culcer *et al*. Orbit current is shown to be non-conserved in the presence of electromagnetic fields. Similar to the Maxwell field inducing forces on charge and charge current, we derive forces acting on spin and spin current induced by the Yang–Mills fields such as the Rashba and Dresselhaus fields and the sheer strain field. The spin density and spin current may be considered as a source generating Yang–Mills field in certain condensed matter systems.

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1. Introduction

Spintronics or spin-based electronics offers opportunities for a new generation of electronic devices for information process and storage [1, 2]. One of the recent developments in this field is the study of the spin Hall effect, which has potential applications to generate and manipulate spin polarization and spin currents by applying an electric field [3–9, 11–17]. The intrinsic transverse spin current induced by an electric field was predicted by Murakami *et al* [4] in the Luttinger Hamiltonian and by Sinova *et al* [5] in two-dimensional electron systems with a Rashba spin–orbit coupling. A resonant spin Hall conductance was also predicted by Shen *et al* [6] in the latter system when a strong perpendicular magnetic field is applied. It has been generally agreed by now that the spin Hall conductivity vanishes in Rashba systems with impurities in the absence of a magnetic field [7, 8]. On the experimental side, the coherent spin manipulation and the electrically induced spin accumulation have been observed [9–11].

One of the current issues in the study of spin Hall effect is the non-conservation of the bare spin current in systems with spin–orbit interaction, which is unfamiliar to the community and requires better understanding and interpretation. Recently, Culcer *et al* [12] have developed a semi-classical theory of spin transport and introduced a torque dipole moment to the spin current. In this paper, we present a unified $SU(2) \times U(1)$ theory for spin and charge currents. In our theory, both the Rashba and Dresselhaus interactions are described by an $SU(2)$ Yang–Mills field, and the total spin current contains an additional contribution from the field-strength tensor of the Yang–Mills field and is conserved. Our theory provides a microscopic understanding of the spin torque introduced in the semi-classical spin transport theory. The orbital-angular-momentum current or the orbit current is shown to be non-conserved in the presence of an electromagnetic $U(1)$ field. We also derive the forces induced by the Yang–Mills field such as the Rashba and Dresselhaus fields and the sheer strain field acting on spin and spin current. Finally, we argue that the spin density and spin current may be regarded as a source to generate Yang–Mills fields.

2. Partially conserved spin current

It is well known that the Dirac equation for an electron moving in an external potential V (as well as the Maxwell vector potential $\mathbf{A}$) in the non-relativistic limit up to the order of $1/c^2$ leads to the Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A} + \frac{2m}{\hbar^2}\hat{\mathbf{s}} \times \vec{\lambda}\right)^2 + \frac{1}{2}\nabla \cdot \vec{\lambda} - \frac{2}{\hbar}\mu_B \hat{\mathbf{s}} \cdot \mathbf{B} + \frac{m\lambda^2}{4\hbar^2} + V, \qquad (1)$$

where $\vec{\lambda} = \frac{\hbar^2}{4m^2c^2}\nabla V$, $\mathbf{B} = \nabla \times \mathbf{A}$ and e denotes the charge of the carriers under consideration.

The first term in equation (1) indicates the Rashba spin–orbit coupling (e.g., $\vec{\lambda} = \alpha\hat{z}$), which represents the dynamical momentum involving the interaction of an electron with both the $U(1)$ Maxwell field and $SU(2)$ Yang–Mills field, the second is the Darwin term [20], the third term is the Zeeman energy and the last one is a higher order term [21].

For an electron system confined in the x–y plane, the strength vector of the Rashba coupling is given by $\vec{\lambda} = b\langle\mathbf{E}\rangle$ [22], with $\mathbf{E}$ being the electric field along the z-direction and b the linear coefficient. The component of the $SU(2)$ Yang–Mills field potentials in this system can be expressed in terms of the $U(1)$ Maxwell fields $\mathbf{E}$ and $\mathbf{B}$, hence the $SU(2)$ Yang–Mills field strength is related to the Maxwell-field strength and their derivatives. This appears to be a realistic example of Yang–Mills field in condensed matter system.

To start with general formalism, let us consider the Schrödinger equation for a particle moving in an external $U(1)$ Maxwell field and an $SU(2)$ Yang–Mills gauge field,

$$\begin{aligned} &\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t) = H\Psi(\mathbf{r}, t) \\ &H = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A} - \eta\boldsymbol{\mathcal{A}}^a\tau^a\right)^2 + eA_0 + \eta\mathcal{A}_0^a\tau^a, \end{aligned} \qquad (2)$$

where Ψ is a two-component wavefunction. Clearly, in comparison to equation (1), A_0 and $\mathcal{A}_0^a$ refer to $\frac{m\lambda^2}{4e\hbar^2} + \frac{V}{e}$ and $-\frac{2}{\hbar}\mu_B\hat{\mathbf{s}} \cdot \mathbf{B}$, respectively, but our further discussion is valid for the general case. In this paper, the Greek superscripts/subscripts stand for 0, 1, 2, 3, the Latin ones for 1, 2, 3 and repeated indices are summed over. Let us denote the vector potential of the Maxwell electromagnetic field by $A_\mu = (A_0, A_i)$ and the vector potential of the Yang–Mills field [18] by $\mathbb{A}_\mu = \mathcal{A}_\mu^a\tau^a$, with τ^a being the generators of $SU(2)$ Lie group. Similar to the field-strength tensor of the electromagnetic field (Maxwell field), which is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the field-strength tensor of the Yang–Mills field is defined by $\mathbb{F}_{\mu\nu} = \mathcal{F}^a_{\mu\nu}\tau^a$, whose components are given by $\mathcal{F}^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu - \eta\epsilon^{abc}\mathcal{A}^b_\nu \mathcal{A}^c_\mu$.

Let $\hat{s}^a = (\hat{s}^1, \hat{s}^2, \hat{s}^3)$ be the spin operators, then the local spin density is given by $\sigma^a(\mathbf{r}, t) = \Psi^\dagger(\mathbf{r}, t)\hat{s}^a\Psi(\mathbf{r}, t)$. Using the method similar to the derivation of the continuity equation for the charge current, one obtains a *continuity-like* equation

$$\left(\frac{\partial}{\partial t} - \eta\vec{\mathcal{A}}_0 \times\right)\vec{\sigma}(\mathbf{r}, t) + \left(\frac{\partial}{\partial x_i} + \eta\vec{\mathcal{A}}_i \times\right)\vec{J}_i(\mathbf{r}, t) = 0 \tag{3}$$

where the spin-current density is defined as

$$\vec{\mathbf{J}}(\mathbf{r}, t) = \tfrac{1}{2}\,\mathrm{Re}\,\Psi^\dagger(\vec{s}\mathbf{v} + \mathbf{v}\vec{s})\Psi \tag{4}$$

with the velocity operator $\hat{\mathbf{v}}$ given by

$$\hat{\mathbf{v}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[\mathbf{r}, H] = \frac{1}{m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A} - \eta\mathcal{A}^a\tau^a\right) \tag{5}$$

and $\eta\tau^a = \hat{s}^a$ which are just the spin operators if $\eta = \hbar$. In order to avoid ambiguities, the indices i, j, k, μ, ν refer to the components of a vector in spatial space while a, b, c, α refer to those in the intrinsic space (Lie algebra space or spin space). Moreover, a vector in spatial space is denoted by a bold face while that in intrinsic space is specified by an overhead arrow, e.g., $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, $\mathbf{A} = (A_1, A_2, A_3)$, $\vec{\mathcal{A}}_i = \left(\mathcal{A}^1_i, \mathcal{A}^2_i, \mathcal{A}^3_i\right)$, $\vec{\mathbf{J}} = (\mathbf{J}^1, \mathbf{J}^2, \mathbf{J}^3)$ and $\mathbf{J}^3 = \left(J^3_1, J^3_2, J^3_3\right)$, etc.

Let us discuss the physics implications of equation (3). In the static case $\vec{\mathbf{J}} = 0$, equation (3) reduces to $\partial_t\vec{\sigma} = \eta\vec{\mathcal{A}}_0 \times \vec{\sigma}$, which implies that the 0th component of the Yang–Mills field $\vec{\mathcal{A}}_0$ induces a torsion on the local spin density $\vec{\sigma}$. From equation (3), we can also see that the spin current is non-conserved even in the limit $\vec{\mathcal{A}}_0 = 0$ provided that the spatial components $\vec{\mathcal{A}}_i \neq 0$. This is because that the Yang–Mills field $\vec{\mathcal{A}}_i$ produces a torsion $\eta\vec{\mathcal{A}}_i \times \vec{J}_i$ on the spin current, which results in an additional spin precession.

Note that the continuity-like equation for the partially conserved spin current, equation (3), can be written in a matrix form

$$\left(\frac{\partial}{\partial t} - \Lambda_0\right)\begin{pmatrix}\sigma^1\\ \sigma^2\\ \sigma^3\end{pmatrix} + \left(\frac{\partial}{\partial x_i} + \Lambda_i\right)\begin{pmatrix}J^1_i\\ J^2_i\\ J^3_i\end{pmatrix} = 0 \tag{6}$$

with $\Lambda_\mu = \eta\mathcal{A}^a_\mu\ell^a$. Here, ℓ^a stand for a representation matrices of the generators of $SO(3)$ Lie group, namely,

$$\ell_1 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}, \qquad \ell_2 = \begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \qquad \ell_3 = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

This provides a geometric picture that the non-conservation of spin current implies the existence of a non-trivial connection characterizing parallel displacements in spacetime manifold, i.e., the local intrinsic frame rotates from point to point.

3. Unified operators for charge and spin currents

As we shall see below, it will be instructive to consider spin and charge in a four-dimensional intrinsic space with three dimensions in spin and one in charge. The continuity equation related to the conventional charge density $\rho = e\Psi^\dagger\Psi$ reads

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{7}$$

where $\mathbf{j} = e\,\mathrm{Re}\,\Psi^\dagger \hat{\mathbf{v}}\Psi$ is the charge current density. We introduce a four-dimensional velocity and a four-dimensional current valued on the $U(1) \times SU(2)$ gauge group. Let $\tau^\alpha = (\tau^0, \tau^a)$ with τ^0 being the identity matrix, the four-dimensional velocity operator is defined as $\hat{v}_\mu = v_\mu^\alpha \tau^\alpha$, which is determined by the Heisenberg equation of motion

$$\hat{v}_\mu = \frac{1}{\mathrm{i}\hbar}[x_\mu, H]. \tag{8}$$

In the above equation, we have taken the commutation relation between $x_0 = ct$ and the Hamiltonian to be $\mathrm{i}c\hbar$. For the Hamiltonian (2), we can write

$$\hat{v}_\mu \equiv (\hat{v}_0, \hat{v}_i) = \left(c, v_i^0 + v_i^a \tau^a\right), \tag{9}$$

with $v_0^0 = 1$, $v_0^a = 0$; $v_i^0 = \left(\hat{p}_i - \frac{e}{c}A_i\right)/m$ and $v_i^a = -\eta \mathcal{A}_i^a/m$. Using these velocity operators, a unified four-dimensional 'current-tensor operator' including both charge and spin degrees of freedom is then defined as

$$\hat{J}_\mu^\alpha \equiv \tfrac{1}{2}\{\hat{v}_\mu, \hat{s}^\alpha\}. \tag{10}$$

The anti-commutator guarantees the defined tensor to be Hermitian. To make the stipulation consistent with what we have discussed above, $\alpha = a$ with $a = 1, 2, 3$ refer to spin and $\alpha = 0$ refers to charge. Accordingly, $\hat{s}^0 = e\tau^0$. From equation (10) we obtain, for $\mu = 0$,

$$\hat{J}_0^\alpha = \begin{cases} \hat{J}_0^0 = ec\tau^0, \\ \hat{J}_0^a = c\hat{s}^a \end{cases} \tag{11}$$

which recovers the usual definitions of charge and spin densities,

$$\begin{aligned} \rho(\mathbf{r}) &\equiv \frac{1}{c}J_0^0(\mathbf{r}) = \frac{1}{c}\Psi^\dagger \hat{J}_0^0 \Psi = e\Psi^\dagger\Psi, \\ \sigma^a(\mathbf{r}) &\equiv \frac{1}{c}J_0^a(\mathbf{r}) = \frac{1}{c}\Psi^\dagger \hat{J}_0^a \Psi = \Psi^\dagger \hat{s}^a \Psi. \end{aligned} \tag{12}$$

For $\mu = i$, we have

$$\hat{J}_i^\alpha = \tfrac{1}{2}\{\hat{v}_i, \hat{s}^\alpha\} = \begin{cases} \hat{J}_i^0 = e\hat{v}_i \\ \hat{J}_i^a = \frac{1}{2}(\hat{v}_i\hat{s}^a + \hat{s}^a\hat{v}_i) \end{cases}$$

which defines the charge and spin current densities,

$$\begin{aligned} j_i(\mathbf{r}) &\equiv J_i^0(\mathbf{r}) = \mathrm{Re}\,\Psi^\dagger \hat{J}_i^0 \Psi = e\,\mathrm{Re}\,\Psi^\dagger \hat{v}_i \Psi, \\ J_i^a(\mathbf{r}) &\equiv J_i^a(\mathbf{r}) = \mathrm{Re}\,\Psi^\dagger \hat{J}_i^a \Psi = \mathrm{Re}\,\Psi^\dagger \tfrac{1}{2}(\hat{v}_i\hat{s}^a + \hat{s}^a\hat{v}_i)\Psi. \end{aligned} \tag{13}$$

The continuity-like equations (3) can be written in a four-dimensional form

$$\frac{\partial}{\partial x_\mu}\mathbb{J}_\mu + \mathrm{i}\eta[\mathbb{A}^\mu, \mathbb{J}_\mu] = 0, \tag{14}$$

where $\mathbb{J}_\mu = J_\mu^a \tau^a$ denotes the spin-current vector which is a matrix-valued vector, and $\mathbb{A}^\mu$ is obtained from $\mathbb{A}_\mu$ by the Minkowski metric tensor $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. This shows that equation (3) is an $SU(2)$ covariant extension of traditional continuity equation. By comparing the above equation with the continuity equation for the charge current, we see that the non-vanishing terms in equation (3) or equation (14) in systems with Rashba spin–orbit coupling are just the spin torque in the semi-classical theory of Culcer *et al* [12]. The spin torque represents the spin procession caused by an external magnetic field and the Rashba interaction. Since the rotational symmetry in spin space is broken, the spin density is not conserved. In our theory, the non-conservation of spin current in the presence of Yang–Mills field is due to its non-Abelian feature.

Since the spin current defined in equation (4) is non-conserved, it will be interesting to examine the origin of the non-conservation by means of constructing a conserved 'total current'.

4. Conservation of a total spin current

An electron has two important intrinsic properties: its charge and spin. As we have illustrated, the Rashba or Dresselhaus coupling describes the interaction between an electron and some particular $SU(2)$ gauge fields. The $SU(2)$ gauge potential was also adopted in the discussion of the quantum interference of a magnetic moment in magnetic fields [19]. We are therefore motivated to construct a theory with the gauge field coupled to spin and spin current in Lagrangian formalism.

The non-relativistic Lagrangian density of the system in the symmetric form is given by

$$\begin{aligned}\mathcal{L}_{NR} = {} & \frac{\mathrm{i}}{2}(\dot{\Psi}^\dagger\Psi - \Psi^\dagger\dot{\Psi}) + \Psi^\dagger\left(eA_0 + \eta\mathcal{A}_0^a\right)\Psi \\ & + \frac{1}{2m}\left[\left(\mathbf{p} - \frac{e}{c}\mathbf{A} - \eta\boldsymbol{\mathcal{A}}^a\tau^a\right)\Psi\right]^\dagger \cdot \left[\mathbf{p} - \frac{e}{c}\mathbf{A} - \eta\boldsymbol{\mathcal{A}}^a\tau^a\right]\Psi \\ & - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu},\end{aligned} \tag{15}$$

where $\dot{\Psi} = \frac{\partial}{\partial t}\Psi$. According to the Euler–Lagrangian equation, this Lagrangian density gives the same equation of motion as equation (2)

$$\mathrm{i}\frac{\partial}{\partial t}\Psi = \left[\frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A} - \eta\mathcal{A}^a\tau^a\right)^2 + eA_0 + \eta\mathcal{A}_0^a\tau^a\right]\Psi. \tag{16}$$

For the system with a Rashba spin–orbit coupling in some semiconductors, the $SU(2)$ gauge potential is antisymmetric $\mathcal{A}_i^a = -\mathcal{A}_a^i$, namely,

$$\mathcal{A}_i^a = \frac{\hbar}{8\eta mc^2}\epsilon^{iab}\partial_b V. \tag{17}$$

However, our formalism here in terms of the Yang–Mills field covers more general cases, such as systems with a Dresselhaus coupling.

Noether's theorem suggests that the invariance of the system under a continuous transformations will lead to a corresponding conservation quantity, hence a conserved total current can be defined by

$$\mathcal{J}_\mu^\alpha = \frac{\delta\mathcal{L}}{\delta\mathcal{A}^{\mu\alpha}} \equiv J_\mu^\alpha + J'^\alpha_\mu, \tag{18}$$

which obeys

$$\frac{\partial}{\partial x_\mu}\mathcal{J}_\mu^\alpha = 0, \tag{19}$$

where J'^α_μ refers to the contribution from the Yang–Mills field and Maxwell field. We adopted a notation $\mathcal{A}_\mu^\alpha = \left(\mathcal{A}_\mu^0, \mathcal{A}_\mu^a\right) = \left(A_\mu, \mathcal{A}_\mu^a\right)$ in equation (18) and the explicit components of J_μ^α can be easily read out as

$$J_0^0 = e\Psi^\dagger\Psi, \qquad J_i^0 = e\,\mathrm{Re}\,\Psi^\dagger\hat{v}_i\Psi, \tag{20}$$

for the charge case, and

$$J_0^a = \eta\Psi^\dagger\tau^a\Psi, \qquad J_i^a = \eta\,\mathrm{Re}\,\Psi^\dagger\tfrac{1}{2}(\tau^a\hat{v}_i + \hat{v}_i\tau^a)\Psi, \tag{21}$$

for the spin case. These results are consistent with the unified four-dimensional currents we have introduced in the previous section. Equations (20) and (21) are just non-relativistic limit of $\bar{\psi}\gamma_\mu\psi$ and $\bar{\psi}\gamma_5\gamma_\mu\gamma^a\psi$, respectively, in which $\psi = \binom{\Psi}{\chi}$ refers to Dirac fermions.

Now let us turn to the extra part J'^α_μ. For $\alpha = 0$, J'^0_μ is null and the charge current $\mathbf{j} = e\,\mathrm{Re}\,\Psi^\dagger\hat{\mathbf{v}}\Psi$ (i.e., J_i^0) is just the total current and hence is conserved, which is consistent with equation (7). Note that the field-strength tensor of the $SU(2)$ Yang–Mills field $\mathcal{F}^a_{\mu\nu}$ depends on both the derivatives of the potential $\mathcal{A}^a_\mu$ and the potential itself. This property is very different from the Maxwell field, which only contains the derivatives of the potential A_μ. Thus, the contribution from the Yang–Mills field $J'^a_\mu = \eta\epsilon^{abc}\mathcal{A}^b_\nu\mathcal{F}^c_{\mu\nu}$ is not null, namely,

$$\vec{J'}_0 = -\eta\vec{\mathcal{A}}_j \times \vec{\mathcal{E}}_j, \qquad \vec{J'}_i = \eta\vec{\mathcal{A}}_0 \times \vec{\mathcal{E}}_i + \eta\epsilon_{ijk}\vec{\mathcal{A}}_j \times \vec{\mathcal{B}}_k, \tag{22}$$

where $\mathcal{E}$ and $\mathcal{B}$ are the Yang–Mills 'electric' and 'magnetic' fields. The above density of (intrinsic) angular momentum J'^a_0 and its flow J'^a_i cancel the non-conservation term brought about by the bare spin current after substituting them into the continuity equation (19) for the total current.

The Lagrangian in equation (15) is shown suitable for an electron subjected to the Rashba spin–orbit interaction (17), while that formalism can be constructed for other systems with spin–orbit interaction. It is our conjecture that the non-conservation of the spin current of the electron system may always be associated with corresponding Yang–Mills field. In this picture, the Yang–Mills field induced by spin and spin current carries a fraction of the intrinsic angular momentum, in analogy to the Maxwell field induced by charge and charge current which carries orbital angular momentum. Finally, we argue that spin and spin current may be regarded as a source generating Yang–Mills fields, which in turn interact with the electron. With the help of Lagrangian formalism similar to equation (15), one should be able to study various physical properties of the system, including the effect of the Yang–Mills field to the observable. Since the effect of the Yang–Mills field is weaker than that of the Maxwell field, the bare spin current has been suggested to be employed in detection [23].

5. Forces acting on spin and spin current

Using the covariant formalism, we can easily determine the force induced by the Yang–Mills field on the spin density and spin current (named as spin force for simplicity). Analogous to the Lorentz force evaluated by $j_\mu F_{\mu i}$ in electromagnetism, the general form of the force provided by a Yang–Mills field is given by

$$f_i = J^a_\mu \mathcal{F}^a_{\mu i} = \sigma^a \mathcal{E}^a_i - \epsilon_{ijk} J^a_j \mathcal{B}^a_k \tag{23}$$

where σ^a and J^a_i stand for the spin density and spin current, respectively.

The Yang–Mills fields corresponding to the Rashba and Dresselhaus spin–orbit couplings, the Zeeman term and the sheer strain field for a system in x–y plane are given by

$$\begin{aligned} \vec{\mathcal{A}}_0 &= -\frac{2\mu_B}{\eta}(B_x, B_y, B_z), & \vec{\mathcal{A}}_1 &= \frac{2m}{\eta\hbar}(\beta, \alpha, \gamma y), \\ \vec{\mathcal{A}}_2 &= \frac{2m}{\eta\hbar}(-\alpha, -\beta, -\gamma x), & \vec{\mathcal{A}}_3 &= (0, 0, 0), \end{aligned} \tag{24}$$

where $\mathbf{B} \equiv (B_x, B_y, B_z)$ stands for the magnetic (Maxwell) field, α and β refer to the Rashba and Dresselhaus coupling strengths and γ is related to sheer strain field, respectively. The 'electric' Yang–Mills field is given by $\mathbb{E}_i = \mathcal{E}^a_i \tau^a$ with

$$\vec{\mathcal{E}}_i = \frac{2\mu_B}{\eta}\partial_i \vec{B} + 2\mu_B \vec{\mathcal{A}}_i \times \vec{B}. \tag{25}$$

The 'magnetic' Yang–Mills field reads

$$\mathbb{B}_3 = -\frac{2m}{\eta\hbar}(\partial_1\alpha + \partial_2\beta)\tau^1 - \frac{2m}{\eta\hbar}(\partial_1\beta + \partial_2\alpha)\tau^2 + \left(\frac{4m^2}{\eta\hbar^2}(\beta^2 - \alpha^2) - \frac{4m}{\eta\hbar}\gamma\right)\tau^3, \tag{26}$$

while $\mathbb{B}_1 = \mathbb{B}_2 = 0$. In the above equations, α and β are non-uniform in general. The force can be derived explicitly as

$$
\begin{aligned}
f_1 &= \frac{2\mu_B}{\eta}\vec{\sigma}\cdot\partial_1\vec{B} + \frac{4m\mu_B}{\eta\hbar}[\beta(\vec{\sigma}\times\vec{B})_1 - \alpha(\vec{\sigma}\times\vec{B})_2] - J_2^1\mathcal{B}_3^1 - J_2^2\mathcal{B}_3^2 - J_2^3\mathcal{B}_3^3,\\
f_2 &= \frac{2\mu_B}{\eta}\vec{\sigma}\cdot\partial_2\vec{B} - \frac{4m\mu_B}{\eta\hbar}[\alpha(\vec{\sigma}\times\vec{B})_1 - \beta(\vec{\sigma}\times\vec{B})_2] + J_1^1\mathcal{B}_3^1 + J_1^2\mathcal{B}_3^2 + J_1^3\mathcal{B}_3^3,\\
f_3 &= \frac{2\mu_B}{\eta}\vec{\sigma}\cdot\partial_3\vec{B}.
\end{aligned}
\tag{27}
$$

The first term in these equations corresponds to the force due to inhomogeneity of the magnetic field, which is the same as in the Stern–Gerlach apparatus. Clearly, there are more forces acting on the spin, related to both the magnetic field and the Rashba and Dresselhaus couplings. Furthermore, the spin current will be subjected to transverse forces. Note that the inhomogeneous magnetic field generates a force of Stern–Gerlach type on spin while the non-uniform Rashba and Dresselhaus coupling introduces a force on the spin current. In what follows, we consider some special cases.

5.1. Uniform Rashba and Dresselhaus fields

If the Rashba and Dresselhaus coupling strengths are uniform (α and β are constants) and the magnetic field is absent, we simply have

$$
\begin{aligned}
f_1 &= -\frac{2m}{\hbar}(\beta^2-\alpha^2)J_2^3,\\
f_2 &= \frac{2m}{\hbar}(\beta^2-\alpha^2)J_1^3,\\
f_3 &= 0.
\end{aligned}
\tag{28}
$$

Clearly, the forces arising from the Rashba and Dresselhaus couplings are along the opposite direction. The magnitudes of the forces are related to the perpendicular component ($a = 3$) of the spin current only. Equation (28) was also obtained in a semi-classical approach [26].

5.2. Non-uniform Rashba and Dresselhaus fields

If α [25] and β are non-uniform, there will be a transverse force whose magnitude is related to the in-plane components ($a = 1, 2$) of the spin current. For example, in the case $|\alpha| = |\beta|$ and $B = 0$, we have

$$
\begin{aligned}
f_1 &= \frac{2m}{\eta\hbar}\left[J_2^1(\partial_1\alpha+\partial_2\beta) + J_2^2(\partial_1\beta+\partial_2\alpha)\right],\\
f_2 &= -\frac{2m}{\eta\hbar}\left[J_1^1(\partial_1\alpha+\partial_2\beta) + J_1^2(\partial_1\beta+\partial_2\alpha)\right],\\
f_3 &= 0.
\end{aligned}
\tag{29}
$$

5.3. Pure sheer strain field

For electrons subjected to a sheer strain field [24] only, which corresponds to $\alpha = \beta = B = 0$ but $\gamma \neq 0$, the forces are given by

$$
f_1 = \frac{4m}{\eta\hbar}\gamma J_2^3, \qquad f_2 = -\frac{4m}{\eta\hbar}\gamma J_1^3, \qquad f_3 = 0,
\tag{30}
$$

where $\gamma \propto C_3/(\hbar e)$ in the notation of [24]. Clearly, there is a transverse force acting on the spin current.

6. Orbit density and orbit current

It is worthwhile to investigate the continuity-like equation for orbit density and orbit current. We can define a local density of orbital angular momentum $\omega^a(\mathbf{r}, t) = \frac{1}{2}\Psi^\dagger \hat{L}^a \Psi + \frac{1}{2}(\hat{L}^a\Psi)^\dagger\Psi$. Here, L is the dynamical angular momentum, $\hat{L}^a = \epsilon^{abc} x_b(\hat{p}_c - (e/c)A_c - \eta\mathcal{A}^a\tau^a)$ for the system of an electron moving in a Maxwell field and a Yang–Mills field. The orbit density so defined is gauge covariant. Using the Shchrödinger equation (2), we can derive a continuity-like equation

$$\frac{\partial}{\partial t}\omega^a + \frac{\partial}{\partial x_i} I_i^a + \epsilon^{abc} x_b \mathcal{F}^\alpha_{cv} J^\alpha_v = 0 \tag{31}$$

where the flow of the orbital angular momentum, namely the orbit current, is given by

$$I_i^a(\mathbf{r}, t) = \mathrm{Re}\,\Psi^\dagger \hat{I}_i^a \Psi - \frac{1}{4e}\hat{I}_i^a\bigg|_{A=0}\rho \tag{32}$$

and

$$\begin{aligned} \hat{I}_i^a &= (\hat{v}_i\hat{L}^a + \hat{L}^a\hat{v}_i)/2 \\ \hat{v}_i &= (\hat{p}_i - (e/c)A_i - \eta\mathcal{A}^a\tau^a)/m. \end{aligned} \tag{33}$$

In equation (31), $\mathcal{F}^0_{ci} = F_{ci}$ refers to the field-strength tensor of the Maxwell field and $\mathcal{F}^a_{ci}$ to that of the Yang–Mills field; $J_i^0 = j_i$ refers to the charge current and J_i^a the spin current. Clearly, the orbit current (32) so defined is also gauge covariant. Note that the definition of the orbit current is not a simple extension of the spin current (4) by replacing the spin operator by the orbital-angular-momentum operator. Actually, there is one more term in the orbit current, which involves change density ρ.

It is worthwhile to point out that the third term in equation (31) is related to both charge current and spin current. In the absence of Maxwell and Yang–Mills fields, the third term in equation (31) vanishes and the orbit current is thus conserved. Using notation $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, $\vec{I}_i = \left(I_i^1, I_i^2, I_i^3\right)$ and $\vec{r} = (x_1, x_2, x_3)$, we can write equation (31) in the following form:

$$\frac{\partial}{\partial t}\vec{\omega} + \frac{\partial}{\partial x_i}\vec{I}_i = \vec{r}\times\vec{F} + \vec{r}\times\vec{f}. \tag{34}$$

The physics implication of the right-hand side represents torque produced by Lorentz force $\vec{F}$ due to Maxwell field and the aforementioned spin force $\vec{f}$ due to Yang–Mills field.

7. Summary

We have introduced a four-dimensional charge and spin current tensor for systems coupled with Yang–Mills fields. The Rashba spin–orbit interaction and Dressenlhaus interaction can be regarded as particular Yang–Mills fields. The current tensor is related to the $SU(2)\times U(1)$ gauge potential. We have also provided a precise definition of orbital-angular-momentum current. Using the Lagrangian formalism, we have constructed a conserved total current, which consists of a conventional bare spin current and a non-vanishing term contributed from the Yang–Mills field. The latter provides a microscopic interpretation of the presence of a spin precession resulted in the non-conservation of the bare spin current. We have derived a general formula describing the forces acting on the spin and spin current. We have proposed that the spin density and spin current can be regarded as a source generating Yang–Mills fields in a similar way as the Maxwell field generated by charge density and current.

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