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Dynamic Stark ladders in the mesoscopic metallic rings

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Abstract

The quantum theory for mesoscopic electric circuit with charge discreteness is briefly described. The Schrödinger equation of the mesoscopic electric circuit with external source which is the time function have been proposed. The mesoscopic metallic ring is regarded as a "pure" inductance design, and the dependence of the quasienergy spectrum on a parameter called the electric matching ratio is presented.

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After Büttiker, Imry and Landauer predicted the Josephson-like effects in small and strictly one-dimensional rings of normal metal driven by an external magnetic flux [1], several manifestations of the AB effects and AC effects have been predicted and verified in mesoscopic systems [2]. Three experiments have verified the persistent current in mesoscopic metallic rings and many different theories have been proposed [3]. In previous papers [4], the mesoscopic metallic ring is regarded as a "pure" inductance (L) design electrical circuit. The persistent current formula has been proposed by using the quantum theory for mesoscopic electric circuits in accord with the discreteness of electric charge. Likharev et al. had predicted that there are effects including Bloch oscillations [5], Wannier–Stark ladders [6] and Laudau–Zener tunneling [7] in a mesoscopic Josephson junction [8]. Recently, electrons driven by an external field show a variety of very interesting phenomena, the same effects have been observed in high-quality superlattices, optical ring resonators and more recently on ultra cold atoms in accelerating optical potentials [9]. As Imry pointed out the same interesting things occur in a mesoscopic ring threaded by a flux when the flux changes with time [10]. For the special case, the resulting current will show Bloch oscillation with a Josephson-type frequency [11]. When the change of flux is not slow enough, Zener-type transitions may occur among the bands. The dynamic treatment in the mesoscopic metallic rings is very interesting. In this Letter, we briefly demonstrate a quantum mechanical theory for mesoscopic electric circuit with the discreteness of charge, also consider the external source as a time function. The explicit solutions for the quasi-

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energies and the Floquet states are obtained exactly. The results depend on the number-theoretical property of the matching ratio of the ac frequency and the frequency of the Stark ladders associated with the dc field.

The classical equation of motion for an electric circuit of LC design is the same as that for a harmonic oscillator, where the "coordinate" means electric charge [11]. Along with the dramatic achievement in nanotechnology, such as molecular-beam expiatory, atomic-scale fabrication or advanced lithography, mesoscopic physics and nanoelectronics are undergoing a rapid development [12]. The electronic device community has been witnessing a strong and definite trend in the miniaturization of integrated circuits and components toward atomic-scale dimensions [13]. When the transport dimension reaches a characteristic dimension, namely, the charge carrier inelastic coherence length, one must use quantum mechanics to discuss the problems in the mesoscopic systems and also need to consider the charge discreteness. The quantization of the circuit was carried out in the same way as that of a harmonic oscillator [14,15]. In order to take into account the discreteness of electronic charge, we must also impose that the eigenvalues of the self-adjoint operator $\hat{q}$ (electric charge) take discrete values

$$\hat{q}|n\rangle = nq_e|n\rangle, \tag{1}$$

where $n \in Z$ (set of integers) and $q_e = 1.602 \times 10^{-19}$ C, the elementary electric charge [4]. Since the spectrum of charge is discrete, the inner product in charge representation will be a sum instead of the usual integral and the electric current operator will be defined by the discrete derivatives ∇_{q_e}, $\bar{\nabla}_{q_e}$.

$$\nabla_{q_e} = \frac{\hat{Q} - 1}{q_e}, \qquad \bar{\nabla}_{q_e} = \frac{1 - \hat{Q}^+}{q_e}, \tag{2}$$

where $\hat{Q} = e^{iq_e\hat{p}/h}$ is a minimum "shift operator". It is easy to check that $\nabla_{q_e}^+ = -\bar{\nabla}_{q_e}$. Thus for the mesoscopic quantum electric circuit one will have finite differential Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left[-\frac{\hbar^2}{2q_eL}\left(\nabla_{q_e} - \bar{\nabla}_{q_e}\right) + \frac{1}{2C}\hat{q}^2 - \epsilon(t)\hat{q}\right]|\Psi\rangle, \tag{3}$$

where L stands for inductance, C for the capacity and $\epsilon(t)$ for the voltage of an electric source which is always the time's function. The effective Hamiltonian for voltage biased source and current biased source mesoscopic electric circuit can be obtained and they can be exchanged by a time varying canonical transformation [16]. The Schrödinger equation for a L design both in the presence of an adiabatic power source and in the absence of source are solved exactly [4]. A gauge field is introduced and a formula for persistent current that is a periodic function of the magnetic flux is obtained. It provides a formulation of the persistent current in the mesoscopic ring from a different point of view. We can discuss the dynamic characteristics of the mesoscopic metallic ring, regarding the ring as a "pure" L design. It should be pointed out that "pure" means that the energy of L is much larger than that of C so that we can ignore the capacity energy in Eq. (3). One should have in mind, that the capacity always exists in the mesoscopic metallic ring due to structure, impurities or some other reasons. Therefore, we can use the perturbation theory to discuss the mesoscopic metallic rings only using the Hamiltonian of inductance part. We can read the Hamiltonian for a L design in the presence of an external source with time from Eq. (3)

$$H = -\frac{\hbar^2}{2q_eL}\left(\nabla_{q_e} - \bar{\nabla}_{q_e}\right) - \epsilon(t)\hat{q}. \tag{4}$$

The Hamiltonian we consider here is the case for one-dimensional systems with the field free part which describes the "pure" L design circuit [4], $H_0 = -\frac{\hbar^2}{2q_eL}(\nabla_{q_e} - \bar{\nabla}_{q_e})$, under the action of a temporally periodic electric field $\varepsilon(t)$. According to Floquet's theorem, the wave function of Eq. (4) can be expressed as $|\Psi\rangle = \exp(-iEt)u_E(t)$, with $u_E(x, t + T) = u_E(x, t)$, where E is the quasienergy [5,17] and $u_E(t)$ satisfies the Schrödinger equation

$$\left[H - i\hbar\frac{\partial}{\partial t}\right]u_E(t) = Eu_E(t). \tag{5}$$

Assuming that Zener tunneling can be neglected, we only pay attention to the single-band case, writing $u_E(t) = \sum_m u_{E,m}(t)|m\rangle$, where $|m\rangle$ is the single-band Wannier function associated with charge quantum numbers according to Eq. (1). The coefficients $u_{E,m}(t)$

are periodic in time and satisfy the evolution equation

$$i\hbar\frac{d}{dt}u_{E,m}(t) = -\frac{\hbar^2}{2q_e^2L}\big[u_{E,m+1}(t)+u_{E,m-1}(t)\big] - \left[E-\frac{\hbar^2}{q_e^2L}+mq_e\epsilon(t)\right]u_{E,m}(t). \tag{6}$$

By introducing

$$C_m(t) = \exp\left\{\frac{-i}{\hbar}\left[\left(E-\frac{\hbar^2}{q_e^2L}\right)t - mq_e\phi(f)\right]\right\} \times u_{E,m}(t), \tag{7}$$

where $\phi(t)$ is the gauge potential (flux threading the mesoscopic metallic ring) defined $\phi(t) = -\int_0^t dt' \times \epsilon(t')$, we obtain

$$i\hbar\frac{d}{dt}C_m(t) = -\frac{\hbar^2}{2q_e^2L}\left\{C_{m+1}\exp\left[i\frac{q_e}{\hbar}\phi(t)\right] + C_{m-1}\exp\left[-i\frac{q_e}{\hbar}\phi(t)\right]\right\}. \tag{8}$$

Employing the discrete Fourier transform

$$C_k(t) = \sum_m C_m(t)e^{-imk} \quad (-\pi \leqslant k < \pi)$$

leads to the solution of (8) as

$$C_k(t) = C_k(0)\exp\left\{\frac{i}{\hbar}\int_0^t dt'\,\frac{\hbar^2}{q_e^2L} \times \cos\left[k+\frac{q_e}{\hbar}\phi(t')\right]\right\}, \tag{9}$$

where the energy dispersion relation of the parent band is

$$E(k,\phi) = \frac{\hbar^2}{q_e^2L}\left[1-\cos\left(k+\frac{q_e}{\hbar}\phi\right)\right].$$

To derive the quasienergies, we note from (7) that the periodicity of $u_{E,m}(t)$ implies that

$$C_m(T) = \exp\left\{-\frac{i}{\hbar}\left[\left(E-\frac{\hbar^2}{q_e^2L}\right)T - mq_e\phi(T)\right]\right\}C_m(0).$$

The Fourier transform of this relation yields the condition

$$C_k(T) = C_{k-\frac{q_e}{\hbar}\phi(T)}(0)\exp\left\{-\frac{i}{\hbar}\left(E-\frac{\hbar^2}{q_e^2L}\right)T\right\}.$$

Substituting (9) into this condition yields that

$$C_{k-\frac{q_e}{\hbar}\phi(T)}(0)\exp\left\{-i\left(E-\frac{\hbar^2}{q_e^2L}\right)T\right\} = C_k(0)\exp\left\{\frac{i}{\hbar}\int_0^T dt'\,\frac{\hbar^2}{q_e^2L}\cos\left[k+\frac{q_e}{\hbar}\phi(t')\right]\right\}. \tag{10}$$

The quasienergies can then be found from the condition that nonzero solutions for the C's exist. In the pure ac case, we have $\phi(T) = 0$, showing that k is a good quantum number. The quasienergy is simply the band energy

$$\frac{\hbar^2}{q_e^2L}\left[1-\cos\left(k+\frac{q_e}{\hbar}\phi(t)\right)\right]$$

averaged over one time period, whose k-dependence can collapse into a constant under certain conditions [17]. Similar results are obtained if, in addition to the ac field, we include a dc component with field strength ϵ_0 and if the matching ratio $q_e\epsilon_0T/2\pi\hbar = \phi(T)/2\pi$ is an integer, because C_k is periodic in k [18]. If there is only a dc field ϵ_0, but no ac fields, the Wannier–Stark states are

$$u_{E,m} = \sum_n J_{nq_e\epsilon_0+\frac{\hbar^2}{q_e^2L}-E}\big(\hbar^2/q_e^2L\big)|n\rangle, \tag{11}$$

where J is the Bessel function, it constitute a complete set of energy eigenstate [4,19]. Their eigenvalues,

$$E_m = \frac{\hbar^2}{q_e^2L} + mq_e\epsilon_0, \tag{12}$$

form the so-called integral Wannier–Stark ladder. In the case of rational value p/q of the matching ratio, where p and q are relatively prime integers, k is no longer a good quantum number. However, if we write the wave number in the form of $k = s + 2\pi l/q$, where $-\pi/q \leqslant s < \pi/q$ and l is an integer, then s is conserved and we will use it to label the quasienergy states. Since there are q different k states coupled together for a given s, we should obtain q quasienergy bands.

We first substitute $k = s + 2\pi l/q$ into both sides of (10), and make a product on each side over $l = 0, \dots, q-1$, yielding

$$\prod_{l=0}^{q-1} C_{s+\frac{2\pi l}{q}+\frac{2\pi p}{q}}(0)\exp\left\{-\frac{iq}{\hbar}\left(E-\frac{\hbar^2}{q_e^2 L}\right)T\right\}$$
$$= \prod_{l=0}^{q-1} C_{s+\frac{2\pi l}{q}}(0)\exp\left\{\frac{i}{\hbar}\int_0^T dt' \sum_{l=0}^{q-1}\frac{\hbar^2}{q_e^2 L} \times \cos\left(s+\frac{2\pi l}{q}+\frac{q_e}{\hbar}\phi(t)\right)\right\}. \quad (13)$$

Next, since the C's in the product all have the same absolute value according to (10), a nontrivial solution to (10) must correspond to a nonzero product of the C's on each side of (11). Finally, the products of C's on the two sides are equal, because $C_k(0)$ is periodic with period 2π, and because the set $\{0, \dots, q-1\}$ is the same as $\{p, \dots, q-1+p\}$ (mod q). Therefore, the exponential on the quasienergies must have the following form

$$E_{ns} = \frac{\hbar^2}{q_e^2 L} - \frac{1}{qT}\int_0^T dt \sum_{l=0}^{q-1}\frac{\hbar^2}{q_e^2 L}\cos\left(s+\frac{2\pi l}{q}+\frac{q_e}{\hbar}\phi(t)\right) + \frac{2\pi n}{qT} \quad (n \text{ integer}). \quad (14)$$

Since the quasienergy is defined within a Brillouin zone of width $2\pi/T$, we only need to take q consecutive values of n, e.g., $n = 0, \dots, q-1$. Finally, if we make a Fourier expansion of $\cos(x)$ in (14) and carry out the summation over l, the quasienergies then have the simple form

$$E_n = \frac{\hbar^2}{q_e^2 L} + \frac{n q_e \epsilon_0}{p}. \quad (15)$$

Note that the second term on the right-hand side of (15) contains, in addition to the integral Stark ladder $m q_e \epsilon_0$ (when $n/p = m$), the fractional ladder structure $n q_e \epsilon_0/p$ (when $n/p \neq$ integer). It had been called this additional feature the dynamic fractional Stark ladder since the quasienergy gap between adjacent Stark levels is $\Delta E = q_e \epsilon_0/p$.

Because the quasienergy is taken over two Brillouin zones, it is evident that for a given rational p/q there will be $2q-1$ subbands (since the two subbands centered at zero overlap), and the subband centers are evenly spaced with space $q_e\epsilon_0/p$. The fact that a large denominator q of the matching ratio corresponds to a small width of the quasienergy bands can be understood from (15).

Our study should be an important step in understanding the dynamic characteristics of mesoscopic metallic rings. The dynamic fractional Stark ladder arises out of a competition between two fundamental space–time areas, one associated with the dc field and the other with the ac field. Hence, the simultaneous presence of both type of fields is essential. When the ratio between the two fundamental space–time areas, which called the electric matching ratio, is a rational number, p/q, a parent band will split into a series of quasienergy subbands. There are q such subbands if the quasienergy is restricted to a Brillouin zone of length $2\pi/T$. If only the dc field is present, the Hamiltonian (4) becomes time independent, and the "quasienergy" is exactly the energy. In that case, the fractional Stark ladder changes into the integral Stark ladder [4,19].

In conclusion, taking the charge's discreteness into account, we studied the quantization of LC design mesoscopic electric circuit with external source which is the time function. The mesoscopic metallic ring regarded as a "pure" L design electric circuit, and the Stark ladder of the energy band has been obtained explicitly. It is interesting to find these quantum effects in mesoscopic metallic ring experiments.

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