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Constructing soluble quantum spin models

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Abstract

By applying the bond operator method, we construct several soluble quantum spin models in two and three dimensions. We first generalize the spin-1/2 bond operators developed by Sachdev and Bhatt [Phys. Rev. B 41 (1990) 9323] to spin $S \geqslant 1$. Then we study two and three-dimensional antiferromagnetic spin-S Heisenberg models and establish conditions on the couplings such that the completely dimerized state is an eigenstate of the model Hamiltonians.

1. Introduction

Study of quantum spin models has a long history back to the exact solution by Bethe for the spin-1/2 antiferromagnetic Heisenberg model (AFH) with nearest-neighbor coupling in one dimension, the famous Bethe ansatz solution [2,3]. Later, the model with anisotropic couplings, the XXZ chain, was solved by the Bethe ansatz and Jordan–Wagner transformation [3]. Although the model Hamiltonian appears simple, one cannot solve it analytically in higher dimensions. On the other hand, with some further neighbor couplings, it is possible to obtain analytical solutions of the model with various geometries. One example is the one-dimensional spin-1/2 Majumdar–Ghosh (MG) model [4], where next-nearest-neighbor coupling was introduced, results in the exact (twofold-degenerate) ground state being a direct product of spin singlet so-called spin dimers. The elementary excitation can be constructed as a pair of unbound spins above the completely dimerized state [5]. Another example is the Shastry–Sutherland model [6], proposed twenty years ago for a special kind of two-dimensional triangular lattice. The model had its experimental realization in compound $SrCu_2(BO_3)_2$ recently [7,8] and stimulated active theoretical

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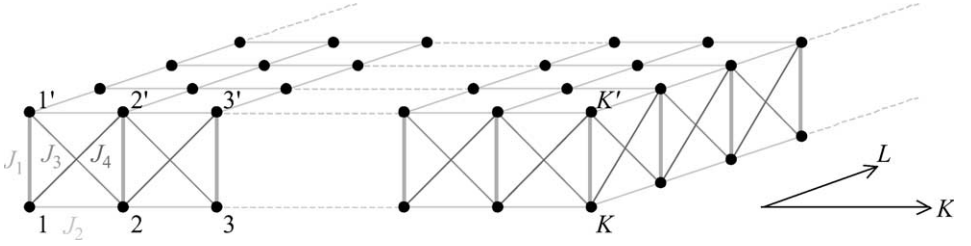

Fig. 1. The net spin layer of $2KL$ spin, where thick solid lines represent singlet dimers.

studies [9–13]. There exist a few soluble quantum spin models for ladders and other systems in literature [14–21].

Recently, we have shown [22] that there exists a class of exact soluble quantum spin model for any spin S in one and two dimensions. We call it the net spin model. As shown in Fig. 1, each layer has $K \times L$ sites where spins interact with their nearest-neighbors via coupling J_2. Between layers, spin $\mathbf{S}$ with $S = |\mathbf{S}| = 1/2, 1, \ldots$, sits on one layer connects to spin $\mathbf{S}'$ on the other layer not only perpendicularly by coupling J_1, but also diagonally by J_3 and J_4. L is the measure of the dimensionality varying from 1 (one dimension) to K (two dimensions). The net spin model is described by the Hamiltonian:

$$\begin{aligned} H = {} & 2J_1 \sum_{k,l=1}^{K,L} \mathbf{S}_{k,l} \cdot \mathbf{S}'_{k,l} \\ & + 2J_2 \sum_{k,l=1}^{K-1,L-1} \mathbf{S}_{k,l} \cdot (\mathbf{S}_{k+1,l} + \mathbf{S}_{k,l+1}) + \mathbf{S}'_{k,l} \cdot (\mathbf{S}'_{k+1,l} + \mathbf{S}'_{k,l+1}) \\ & + 2J_3 \sum_{k,l=1}^{K-1,L-1} \mathbf{S}'_{k,l} \cdot (\mathbf{S}_{k+1,l} + \mathbf{S}_{k,l+1}) \\ & + 2J_4 \sum_{k,l=1}^{K-1,L-1} \mathbf{S}_{k,l} \cdot (\mathbf{S}'_{k+1,l} + \mathbf{S}'_{k,l+1}). \end{aligned} \tag{1}$$

We have shown that under the condition $2J_2 = J_3 + J_4$, the completely dimerized state ψ_D, where spins on the lower layer form dimers with spins on the upper layer, is an eigenstate of Eq. (1). When J_1 is sufficiently large, ψ_D is also the ground state. In addition, we also found that wave functions such as the single magnon state ψ_m, the double magnon state $\psi_{m,m'}$, and four-spin plaquette singlet state $\psi_\square$ are also eigenstates of the model. These particular wave functions are:

$$\psi_D = [1, 1'][2, 2'] \cdots [KL, K'L'], \tag{2}$$

$$\psi_m = [1, 1'] \cdots (i, i') \cdots [KL, K'L'], \tag{3}$$

$$\psi_{m,m'} = [1, 1'] \cdots (i, i')[j, j'](k, k') \cdots [KL, K'L'], \tag{4}$$

$$\psi_\square = [1, 1'] \cdots \{\square\} \cdots [KL, KL'], \tag{5}$$

where $[i, i']$ denotes a normalized spin singlet with $|\mathbf{S}_i + \mathbf{S}'_i| = 0$, e.g., for the spin-$1/2$ case,

$$[i, i'] = \frac{1}{\sqrt{2}}\big(|\uparrow_i\downarrow_{i'}\rangle - |\downarrow_i\uparrow_{i'}\rangle\big), \tag{6}$$

(i, i') denotes a normalized non-singlet such as triplet or pentad with $|\mathbf{S}_i + \mathbf{S}'_i| = m \neq 0$, e.g., for the spin-$1/2$ case and $m = 1$

$$\begin{aligned}(i, i') = &-\frac{1}{\sqrt{2}}\big(|\uparrow_i\uparrow_{i'}\rangle - |\downarrow_i\downarrow_{i'}\rangle\big), \quad \text{or}\\ &\frac{i}{\sqrt{2}}\big(|\uparrow_i\uparrow_{i'}\rangle + |\downarrow_i\downarrow_{i'}\rangle\big), \quad \text{or}\\ &\frac{1}{\sqrt{2}}\big(|\uparrow_i\downarrow_{i'}\rangle + |\downarrow_i\uparrow_{i'}\rangle\big),\end{aligned} \tag{7}$$

and $\{\square\}$ denotes the four-spin plaquette singlet with $|\mathbf{S}_i + \mathbf{S}'_i + \mathbf{S}_{i+1} + \mathbf{S}'_{i+1}| = 0$, e.g., for the spin-$1/2$ case,

$$\{\square\} = \frac{1}{\sqrt{12}}\left(\left|\begin{matrix}\downarrow\downarrow\\ \uparrow\uparrow\end{matrix}\right\rangle + \left|\begin{matrix}\downarrow\uparrow\\ \uparrow\downarrow\end{matrix}\right\rangle + \left|\begin{matrix}\uparrow\downarrow\\ \downarrow\uparrow\end{matrix}\right\rangle + \left|\begin{matrix}\uparrow\uparrow\\ \downarrow\downarrow\end{matrix}\right\rangle - 2\left|\begin{matrix}\downarrow\uparrow\\ \downarrow\uparrow\end{matrix}\right\rangle - 2\left|\begin{matrix}\uparrow\downarrow\\ \uparrow\downarrow\end{matrix}\right\rangle\right), \tag{8}$$

with $\uparrow$ and $\downarrow$ representing the z-component of single spin being equal to $1/2$ and $-1/2$, respectively.

Thus a subset of the Hilbert of the net spin model is exactly known. We investigated the ground state and the excitation gap in detail for a wide range of parameters and varying dimensionality L from $L = 1$ (one dimension we call it net spin ladder) to $L = K$ (two dimensions we call it net spin layer). Our main findings were: (i) let coupling J_1 define the energy scale of the model, when other couplings increase, the ground state changes from the completely dimerized state ψ_D to the ground state of the spin-σ ($\sigma = 2S$) model defined on the single layer; (ii) the model exhibits rich excitation phases with distinct behaviors in one and two dimensions; (iii) dimensional crossover for spin $S = 1/2$ was identified. Finally, we note that at a particular point in the parameter space: $S = 1/2$, $L = 1$, $J_2 = \frac{1}{2}J_1$, $J_3 = J_1$, and $J_4 = 0$ the net spin model goes back to the MG model [4]. Therefore, the net spin model is a generalization of the MG model for any spin S in one and two dimensions.

In this paper, we search for other exactly soluble spin models in two and three dimensions. To this purpose, we apply an operator approach, the bond operator method. In spirit, this approach is similar to techniques used for constructing integrable fermionic models [23]. First, in Section 2 we review the bond operator for the spin-$1/2$ case and introduce the bond operator for the spin-1 case, with briefly discussion of the bond operator for the general spin-S case. Then in Section 3 we use the bond operator to rewrite the Hamiltonian Eq. (1) in such a way that one can obtain the condition(s) for the completely dimerized state being an eigenstate of the Hamiltonian immediately. Afterwards, we present several two-dimensional (single layer) and three-dimensional soluble quantum spin models in Sections 4 and 5, followed by a short summary in Section 6.

2. Bond operator

2.1. Spin-1/2 bond operator

In this section, the so called "bond operator" representation is discussed. For a dimer consisting of two spins **S** and **S**′, instead of using the spin operator representation for S_α and S'_α, where $\alpha = x, y, z$, an alternative representation, known as the bond operator could also be used. Such representation was first introduced by Sachdev and Bhatt in 1990 [1], in which the original two spin-1/2 operators, **S** and **S**′, are represented by four bosonic operators $s^\dagger, t_x^\dagger, t_y^\dagger$ and $t_z^\dagger$, which create the singlet state $|s\rangle$ and triplet states $|t_x\rangle$, $|t_y\rangle$, and $|t_z\rangle$ out of the vacuum state $|\text{vac}\rangle$, respectively,

$$|s\rangle \equiv s^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big), \tag{9}$$

$$|t_x\rangle \equiv t_x^\dagger|\text{vac}\rangle = \frac{-1}{\sqrt{2}}\big(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle\big), \tag{10}$$

$$|t_y\rangle \equiv t_y^\dagger|\text{vac}\rangle = \frac{i}{\sqrt{2}}\big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\big), \tag{11}$$

$$|t_z\rangle \equiv t_z^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow + |\downarrow\uparrow\rangle\big), \tag{12}$$

and conversely

$$\begin{aligned} S_\alpha &= \frac{1}{2}\big(s^\dagger t_\alpha + t_\alpha^\dagger s - i\epsilon_{\alpha\beta\gamma} t_\beta^\dagger t_\gamma\big), \\ S'_\alpha &= \frac{1}{2}\big(-s^\dagger t_\alpha - t_\alpha^\dagger s - i\epsilon_{\alpha\beta\gamma} t_\beta^\dagger t_\gamma\big), \end{aligned} \tag{13}$$

where α, β, and γ take the values of x, y, and z. The Levi-Civita symbol ϵ is the totally antisymmetric tensor, and all repeated indices are summed over. For physical state there is either a singlet or triplet, so the following constraint should be imposed,

$$s^\dagger s + t_\alpha^\dagger t_\alpha = 1. \tag{14}$$

2.2. Spin-1 bond operator

Similar to the spin-1/2 case, we can construct the spin-1 bond operators. We leave their constructions to Appendix A. The original two spin-1 operators, **S** and **S**′, are represented by nine bosonic operators $s^\dagger$, $t_1^\dagger$, $t_0^\dagger$, $t_{-1}^\dagger$, $p_2^\dagger$, $p_1^\dagger$, $p_0^\dagger$, $p_{-1}^\dagger$ and $p_{-2}^\dagger$ which create the singlet state $|s\rangle$, triplet states $|t_\alpha\rangle$, and pentad states $|p_\beta\rangle$ out of the vacuum state $|\text{vac}\rangle$, respectively.

The singlet $|s\rangle$ is defined as

$$|s\rangle \equiv s^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{3}}\big(|\uparrow\downarrow\rangle - |00\rangle + |\downarrow\uparrow\rangle\big). \tag{15}$$

The triplets $|t_\alpha\rangle$ with $\alpha = \pm 1$ and 0 being the z-component of the total spin are defined as

$$|t_1\rangle \equiv t_1^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{2}}\big(|0\uparrow\rangle - |\uparrow 0\rangle\big), \tag{16}$$

$$|t_0\rangle \equiv t_0^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{2}}\big(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\big), \tag{17}$$

$$|t_{-1}\rangle \equiv t_{-1}^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{2}}\big(|\downarrow 0\rangle - |0\downarrow\rangle\big). \tag{18}$$

The fivefold multiplets pentad $|p_\beta\rangle$ with $\beta = \pm 2, \pm 1$, and 0 are defined as

$$|p_2\rangle \equiv p_2^\dagger|\text{vac}\rangle = |\uparrow\uparrow\rangle, \tag{19}$$

$$|p_1\rangle \equiv p_1^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{2}}\big(|0\uparrow\rangle + |\uparrow 0\rangle\big), \tag{20}$$

$$|p_0\rangle \equiv p_0^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{6}}\big(|\uparrow\downarrow\rangle + 2|00\rangle + |\downarrow\uparrow\rangle\big), \tag{21}$$

$$|p_{-1}\rangle \equiv p_{-1}^\dagger|\text{vac}\rangle = \frac{1}{\sqrt{2}}\big(|0\downarrow\rangle + |\downarrow 0\rangle\big), \tag{22}$$

$$|p_{-2}\rangle \equiv p_{-2}^\dagger|\text{vac}\rangle = |\downarrow\downarrow\rangle. \tag{23}$$

Conversely,

$$\begin{aligned}
S^z &= p_2^\dagger p_2 + \frac{1}{2}p_1^\dagger p_1 - \frac{1}{2}p_{-1}^\dagger p_{-1} - p_{-2}^\dagger p_{-2} + \frac{1}{2}t_1^\dagger t_1 - \frac{1}{2}t_{-1}^\dagger t_{-1} \\
&\quad - \frac{1}{2}\big(p_1^\dagger t_1 + t_1^\dagger p_1\big) - \frac{1}{\sqrt{3}}\big(p_0^\dagger t_0 + t_0^\dagger p_0\big) - \frac{1}{2}\big(p_{-1}^\dagger t_{-1} + t_{-1}^\dagger p_{-1}\big) \\
&\quad - \sqrt{\frac{2}{3}}\big(t_0^\dagger s + s^\dagger t_0\big), \\
S'^z &= p_2^\dagger p_2 + \frac{1}{2}p_1^\dagger p_1 - \frac{1}{2}p_{-1}^\dagger p_{-1} - p_{-2}^\dagger p_{-2} + \frac{1}{2}t_1^\dagger t_1 - \frac{1}{2}t_{-1}^\dagger t_{-1} \\
&\quad + \frac{1}{2}\big(p_1^\dagger t_1 + t_1^\dagger p_1\big) + \frac{1}{\sqrt{3}}\big(p_0^\dagger t_0 + t_0^\dagger p_0\big) + \frac{1}{2}\big(p_{-1}^\dagger t_{-1} + t_{-1}^\dagger p_{-1}\big) \\
&\quad + \sqrt{\frac{2}{3}}\big(t_0^\dagger s + s^\dagger t_0\big), \\
S^+ &= p_2^\dagger p_1 + p_{-1}^\dagger p_{-2} + \sqrt{\frac{3}{2}}\big(p_1^\dagger p_0 + p_0^\dagger p_{-1}\big) + \frac{1}{\sqrt{2}}\big(t_1^\dagger t_0 + t_0^\dagger t_{-1}\big) \\
&\quad + p_2^\dagger t_1 - t_{-1}^\dagger p_{-2} + \frac{1}{\sqrt{2}}\big(p_1^\dagger t_0 - t_0^\dagger p_{-1}\big) + \frac{1}{\sqrt{6}}\big(p_0^\dagger t_{-1} - t_1^\dagger p_0\big) \\
&\quad + \frac{2}{\sqrt{3}}\big(t_1^\dagger s - s^\dagger t_{-1}\big),
\end{aligned}$$

$$S'^{+} = p_2^\dagger p_1 + p_{-1}^\dagger p_{-2} + \sqrt{\frac{3}{2}}\left(p_1^\dagger p_0 + p_0^\dagger p_{-1}\right) + \frac{1}{\sqrt{2}}\left(t_1^\dagger t_0 + t_0^\dagger t_{-1}\right) - p_2^\dagger t_1 + t_{-1}^\dagger p_{-2} - \frac{1}{\sqrt{2}}\left(p_1^\dagger t_0 - t_0^\dagger p_{-1}\right) - \frac{1}{\sqrt{6}}\left(p_0^\dagger t_{-1} - t_1^\dagger p_0\right) - \frac{2}{\sqrt{3}}\left(t_1^\dagger s - s^\dagger t_{-1}\right). \tag{24}$$

Since physical states are either singlet, triplet, or pentad, the following constraint is imposed

$$s^\dagger s + t_\alpha^\dagger t_\alpha + p_\beta^\dagger p_\beta = 1, \tag{25}$$

where $\alpha = \pm 1, 0$ and $\beta = \pm 2, \pm 1, 0$.

For general spin-S, its bond operator representation could also be obtained, in principle. However, when spin S increases, the number of bosonic operators required to represent the spin operators $\mathbf{S}$ and $\mathbf{S}'$ is $(2S+1)^2$, which increases exponentially. More discussions are given in Appendix B.

3. Bond operator representation of the net spin model

By using the bond operator representation, it is easy to see that the condition $2J_2 = J_3 + J_4$ makes the completely dimerized state ψ_D be the eigenstate of the net spin model. As an illustration, we study the spin-1/2 case in two dimensions. Substituting bosonic operators for the spin operators, as defined in Eq. (13), we rewrite the net spin Hamiltonian Eq. (1) as

$$H = 2J_1 h_1 + 2(2J_2 - J_3 - J_4)h_2 + 2(J_3 - J_4)h_3 + 2(J_2 + J_3 + J_4)h_4,$$

where

$$h_1 = \sum_{k,l=1}^{K,L}\left(-\frac{3}{4}s_{k,l}^\dagger s_{k,l} + \frac{1}{4}t_{\alpha k,l}^\dagger t_{\alpha k,l}\right), \tag{26}$$

$$h_2 = \sum_{k,l=1}^{K-1,L-1}\frac{1}{4}\left(s_{k,l}^\dagger t_{\alpha k,l} s_{k+1,l}^\dagger t_{\alpha k+1,l} + s_{k,l}^\dagger t_{\alpha k,l} t_{\alpha k+1,l}^\dagger s_{k+1,l} + t_{\alpha k,l}^\dagger s_{k,l} s_{k+1,l}^\dagger t_{\alpha k+1,l} + t_{\alpha k,l}^\dagger s_{k,l} t_{\alpha k+1,l}^\dagger s_{k+1,l} + s_{k,l}^\dagger t_{\alpha k,l} s_{k,l+1}^\dagger t_{\alpha k,l+1} + s_{k,l}^\dagger t_{\alpha k,l} t_{\alpha k,l+1}^\dagger s_{k,l+1} + t_{\alpha k,l}^\dagger s_{k,l} s_{k,l+1}^\dagger t_{\alpha k,l+1} + t_{\alpha k,l}^\dagger s_{k,l} t_{\alpha k,l+1}^\dagger s_{k,l+1}\right), \tag{27}$$

$$h_3 = \sum_{k,l=1}^{K-1,L-1}\frac{i}{4}\epsilon_{\alpha\beta\gamma}\left(s_{k,l}^\dagger t_{\alpha k,l} t_{\beta k+1,l}^\dagger t_{\gamma k+1,l} + t_{\alpha k,l}^\dagger s_{k,l} t_{\beta k+1,l}^\dagger t_{\gamma k+1,l} - t_{\beta k,l}^\dagger t_{\gamma k,l} s_{k+1,l}^\dagger t_{\alpha k+1,l} - t_{\beta k,l}^\dagger t_{\gamma k,l} t_{\alpha k+1,l}^\dagger s_{k+1,l} + s_{k,l}^\dagger t_{\alpha k,l} t_{\beta k,l+1}^\dagger t_{\gamma k,l+1} + t_{\alpha k,l}^\dagger s_{k,l} t_{\beta k,l+1}^\dagger t_{\gamma k,l+1} - t_{\beta k,l}^\dagger t_{\gamma k,l} s_{k,l+1}^\dagger t_{\alpha k,l+1} - t_{\beta k,l}^\dagger t_{\gamma k,l} t_{\alpha k,l+1}^\dagger s_{k,l+1}\right), \tag{28}$$

$$h_4 = \sum_{k,l=1,\alpha\neq\beta}^{K-1,L-1} \frac{1}{4} t^\dagger_{\alpha k,l} t_{\beta k,l} \big(t_{\alpha k+1,l} t^\dagger_{\beta k+1,l} - t^\dagger_{\alpha k+1,l} t_{\beta k+1,l} + t_{\alpha k,l+1} t^\dagger_{\beta k,l+1} - t^\dagger_{\alpha k,l+1} t_{\beta k,l+1} \big). \tag{29}$$

In Eq. (26), h_1 is diagonal in boson number operators. All the remaining four-operator terms (h_2, h_3 and h_4) give zero when they act on the completely dimerized state ψ_D except the terms $t^\dagger_{\alpha k,l} s_{k,l} t^\dagger_{\alpha k+1,l} s_{k+1,l}$ and $t^\dagger_{\alpha k,l} s_{k,l} t^\dagger_{\alpha k,l+1} s_{k,l+1}$ in h_2. Since $t^\dagger_{\alpha i} s_i t^\dagger_{\alpha j} s_j$ will destroy a singlet and create a triplet on both sites i and j, a new state is created, i.e.,

$$t^\dagger_{\alpha i} s_i t^\dagger_{\alpha j} s_j \psi_D = [1, 1'] \cdots (i, i')(j, j') \cdots [KL, K'L']. \tag{30}$$

So to make the completely dimerized state ψ_D be an eigenstate, the coefficient of h_2 must be zero, which leads to the condition $2J_2 = J_3 + J_4$.

If $J_2 = J_3 = J_4$, the Hamiltonian becomes

$$H = 2J_1 \sum_{k,l=1}^{K,L} \left(-\frac{3}{4} s^\dagger_{k,l} s_{k,l} + \frac{1}{4} t^\dagger_{\alpha k,l} t_{\alpha k,l} \right) + 2J_2 \sum_{k,l=1,\alpha\neq\beta}^{K-1,L-1} t^\dagger_{\alpha k,l} t_{\beta k,l} \big(t_{\alpha k+1,l} t^\dagger_{\beta k+1,l} + t_{\alpha k,l+1} t^\dagger_{\beta k,l+1} - t^\dagger_{\alpha k+1,l} t_{\beta k+1,l} - t^\dagger_{\alpha k,l+1} t_{\beta k,l+1} \big). \tag{31}$$

By inspecting the above simplified Hamiltonian, we immediately see that the eigenvalue of the completely dimerized state E_D is $-2J_1 \times 3/4 \times N$ where $N = KL$ is the total number of singlets on the double layer. Moreover, in the large J_1 limit, the completely dimerized state is the ground state and the first excited state should be a single magnon created by breaking a dimer at any site, with eigenvalue $E_D + 2J_1$. Thus, without solving the Hamiltonian explicitly, useful information are already revealed by the bond operator representation of the model.

For general spin-S case, similar approach will lead to the same conclusions. In order not to bore the readers, the derivations is not shown here. Instead, we present the derivations for the spin-1 case for a three-dimensional model in Section 5 and leave the derivations for general spin-S case to Appendix B.

4. Two-dimensional exactly soluble model

In this section, we construct other exactly soluble models in two dimensions by applying the bond operator method. Obviously, there could be a lot. A simple case would be a natural continuation of the net spin model where one connects a set of ladders by inter-ladder couplings J_5, J_6, J_7, and J_8 as shown in Fig. 2.

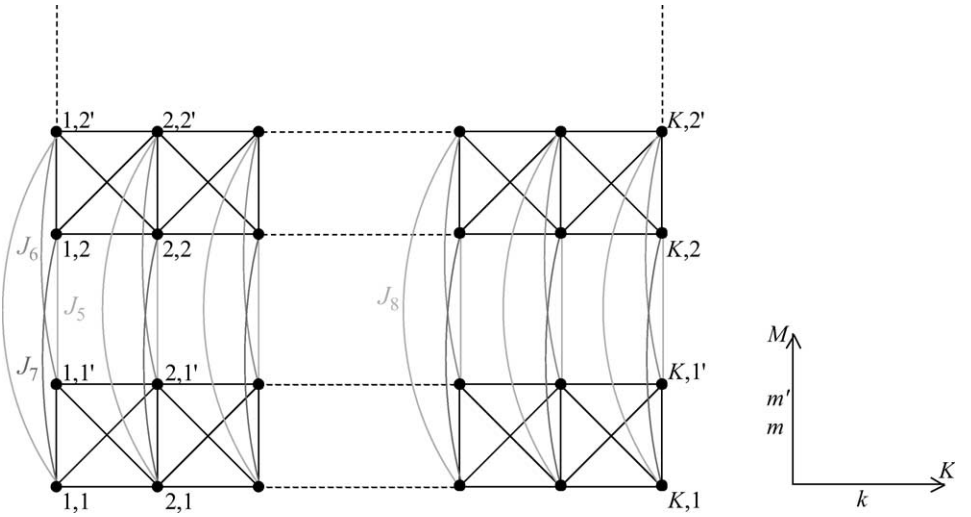

Fig. 2. The nested-ladder model.

In Fig. 2, m and m' label the lower and upper hand of ladder, respectively, and there are total of M ladders. The Hamiltonian of the model in Fig. 2 is

$$\begin{aligned} H = {} & 2J_1 \sum_{k,m=1}^{K,M} \mathbf{S}_{k,m} \cdot \mathbf{S}'_{k,m} + 2J_2 \sum_{k,m=1}^{K-1,M} (\mathbf{S}_{k,m} \cdot \mathbf{S}_{k+1,m} + \mathbf{S}'_{k,m} \cdot \mathbf{S}'_{k+1,m}) \\ & + 2J_3 \sum_{k,m=1}^{K-1,M} \mathbf{S}_{k+1,m} \cdot \mathbf{S}'_{k,m} + 2J_4 \sum_{k,m=1}^{K-1,M} \mathbf{S}_{k,m} \cdot \mathbf{S}'_{k+1,m} \\ & + 2J_5 \sum_{k,m=1}^{K,M-1} \mathbf{S}'_{k,m} \cdot \mathbf{S}_{k,m+1} + 2J_6 \sum_{k,m=1}^{K,M-1} \mathbf{S}'_{k,m} \cdot \mathbf{S}'_{k,m+1} \\ & + 2J_7 \sum_{k,m=1}^{K,M-1} \mathbf{S}_{k,m} \cdot \mathbf{S}_{k,m+1} + 2J_8 \sum_{k,m=1}^{K,M-1} \mathbf{S}_{k,m} \cdot \mathbf{S}'_{k,m+1}. \end{aligned} \tag{32}$$

To be definitive, we refer the bond operator for spin $\mathbf{S}_{k,m}$ and $\mathbf{S}'_{k,m}$. Substituting Eq. (13) into the Hamiltonian Eq. (32), we get

$$\begin{aligned} H = {} & 2J_1 h_1 + 2(2J_2 - J_3 - J_4) h_{2a} + 2(J_6 + J_7 - J_5 - J_8) h_{2b} + 2(J_3 - J_4) h_{3a} \\ & + 2(J_8 - J_5) h_{3b} + 2(2J_2 + J_3 + J_4) h_{4a} + 2(J_5 + J_6 + J_7 + J_8) h_{4b}, \end{aligned} \tag{33}$$

where

$$h_1 = \sum_{k,m=1}^{K,M} \left(-\frac{3}{4} s^{\dagger}_{k,m} s_{k,m} + \frac{1}{4} t^{\dagger}_{\alpha k,m} t_{\alpha k,m} \right), \tag{34}$$

$$h_{2a} = \sum_{k,m=1}^{K-1,M} \frac{1}{4}\big(s^\dagger_{k,m} t_{\alpha k,m} s^\dagger_{k+1,m} t_{\alpha k+1,m} + s^\dagger_{k,m} t_{\alpha k,m} t^\dagger_{\alpha k+1,m} s_{k+1,m} + t^\dagger_{\alpha k,m} s_{k,m} s^\dagger_{k+1,m} t_{\alpha k+1,m} + t^\dagger_{\alpha k,m} s_{k,m} t^\dagger_{\alpha k+1,m} s_{k+1,m}\big), \tag{35}$$

$$h_{2b} = \sum_{k,m=1}^{K,M-1} \frac{1}{4}\big(s^\dagger_{k,m} t_{\alpha k,m} s^\dagger_{k,m+1} t_{\alpha k,m+1} + s^\dagger_{k,m} t_{\alpha k,m} t^\dagger_{\alpha k,m+1} s_{k,m+1} + t^\dagger_{\alpha k,m} s_{k,m} s^\dagger_{k,m+1} t_{\alpha k,m+1} + t^\dagger_{\alpha k,m} s_{k,m} t^\dagger_{\alpha k,m+1} s_{k,m+1}\big), \tag{36}$$

$$h_{3a} = \sum_{k,m=1}^{K-1,M} \frac{i}{4}\epsilon_{\alpha\beta\gamma}\big(s^\dagger_{k,m} t_{\alpha k,m} t^\dagger_{\beta k+1,m} t_{\gamma k+1,m} + t^\dagger_{\alpha k,m} s_{k,m} t^\dagger_{\beta k+1,m} t_{\gamma k+1,m} - t^\dagger_{\beta k,m} t_{\gamma k,m} s^\dagger_{k+1,m} t_{\alpha k+1,m} - t^\dagger_{\beta k,m} t_{\gamma k,m} t^\dagger_{\alpha k+1,m} s_{k+1,m}\big), \tag{37}$$

$$h_{3b} = \sum_{k,m=1}^{K,M-1} \frac{i}{4}\epsilon_{\alpha\beta\gamma}\big(s^\dagger_{k,m} t_{\alpha k,m} t^\dagger_{\beta k,m+1} t_{\gamma k,m+1} + t^\dagger_{\alpha k,m} s_{k,m} t^\dagger_{\beta k,m+1} t_{\gamma k,m+1} - t^\dagger_{\beta k,m} t_{\gamma k,m} s^\dagger_{k,m+1} t_{\alpha k,m+1} - t^\dagger_{\beta k,m} t_{\gamma k,m} t^\dagger_{\alpha k,m+1} s_{k,m+1}\big), \tag{38}$$

$$h_{4a} = \sum_{k,m=1,\alpha\neq\beta}^{K-1,M} \frac{1}{4} t^\dagger_{\alpha k,m} t_{\beta k,m}\big(t_{\alpha k+1,m} t^\dagger_{\beta k+1,m} - t^\dagger_{\alpha k+1,m} t_{\beta k+1,m}\big), \tag{39}$$

$$h_{4b} = \sum_{k,m=1,\alpha\neq\beta}^{K,M-1} \frac{1}{4} t^\dagger_{\alpha k,m} t_{\beta k,m}\big(t_{\alpha k,m+1} t^\dagger_{\beta k,m+1} - t^\dagger_{\alpha k,m+1} t_{\beta k,m+1}\big). \tag{40}$$

Then, based on the arguments we have in previous sections, the conditions for the completely dimerized state to be an eigenstate are

$$2J_2 = J_3 + J_4, \qquad J_6 + J_7 = J_5 + J_8. \tag{41}$$

Again, as in the previous section, same conditions hold for cases with spin S other than $1/2$. Conditions Eq. (41) could have various forms. For example, we can let $J_6 = J_7$ due to symmetry and let $J_8 = 0$ to get $J_5 = 2J_6$, or let $J_5 = J_6 = J_7 = J_8$ and study their energy specturms. In fact, then the model could be maped into coupled spin chains with spatial anisotropy. Quantum phase transitions are expected. We also remark that one can obtain soluble bilayer net spin model with inter-bilayer couplings under similar conditions.

5. Three-dimensional exactly soluble model

In this section, the bond operator method is used to determine the condition(s) required for the completely dimerized state to be an eigenstate of a three-dimensional model H_{3D}, defined by the Hamiltonian,

$$H_{3D} = 2J_1 \sum_{k,l,m=1}^{K,L,M} \mathbf{S}_{k,l,m} \cdot \mathbf{S}'_{k,l,m} + 2\tilde{J}_1 \sum_{k,l,m=1}^{K,L,M-1} \mathbf{S}'_{k,l,m} \cdot \mathbf{S}_{k,l,m+1}$$

$$
\begin{aligned}
&+ 2J_2 \sum_{k,l,m=1}^{K-1,L-1,M} \mathbf{S}_{k,l,m} \cdot (\mathbf{S}_{k+1,l,m} + \mathbf{S}_{k,l+1,m}) \\
&\qquad + \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}'_{k+1,l,m} + \mathbf{S}'_{k,l+1,m}) \\
&+ 2J_3 \sum_{k,l,m=1}^{K-1,L-1,M} \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}_{k+1,l,m} + \mathbf{S}_{k,l+1,m}) \\
&+ 2\tilde{J}_3 \sum_{k,l,m=1}^{K-1,L-1,M-1} \mathbf{S}_{k,l,m+1} \cdot (\mathbf{S}'_{k+1,l,m} + \mathbf{S}'_{k,l+1,m}) \\
&+ 2J_3^{\parallel} \sum_{k,l,m=1}^{K-1,L-1,M} \mathbf{S}_{k+1,l,m} \cdot \mathbf{S}_{k,l+1,m} + \mathbf{S}'_{k+1,l,m} \cdot \mathbf{S}'_{k,l+1,m} \\
&+ 2J_4 \sum_{k,l,m=1}^{K-1,L-1,M} \mathbf{S}_{k,l,m} \cdot (\mathbf{S}'_{k+1,l,m} + \mathbf{S}'_{k,l+1,m}) \\
&+ 2\tilde{J}_4 \sum_{k,l,m=1}^{K-1,L-1,M-1} \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}_{k+1,l,m+1} + \mathbf{S}_{k,l+1,m+1}) \\
&+ 2J_4^{\parallel} \sum_{k,l,m=1}^{K-1,L-1,M} \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+1,l+1,m} + \mathbf{S}'_{k,l,m} \cdot \mathbf{S}'_{k+1,l+1,m} \\
&+ 2J_5 \sum_{k,l,m=1}^{K-1,L-1,M} \mathbf{S}_{k,l,m} \cdot \mathbf{S}'_{k+1,l+1,m} + \mathbf{S}_{k+1,l,m} \cdot \mathbf{S}'_{k,l+1,m} \\
&\qquad + \mathbf{S}_{k,l+1,m} \cdot \mathbf{S}'_{k+1,l,m} + \mathbf{S}_{k+1,l+1,m} \cdot \mathbf{S}'_{k,l,m} \\
&+ 2\tilde{J}_5 \sum_{k,l,m=1}^{K-1,L-1,M-1} \mathbf{S}'_{k,l,m} \cdot \mathbf{S}_{k+1,l+1,m+1} + \mathbf{S}'_{k+1,l,m} \cdot \mathbf{S}_{k,l+1,m+1} \\
&\qquad + \mathbf{S}'_{k,l+1,m} \cdot \mathbf{S}_{k+1,l,m+1} + \mathbf{S}'_{k+1,l+1,m} \cdot \mathbf{S}_{k,l,m+1} \\
&+ 2J_6 \sum_{k,l,m=1}^{K,L,M-1} \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k,l,m+1} + \mathbf{S}'_{k,l,m} \cdot \mathbf{S}'_{k,l,m+1} \\
&+ 2J_6^{\parallel} \sum_{k,l,m=1}^{K-2,L-2,M} \mathbf{S}_{k,l,m} \cdot (\mathbf{S}_{k+2,l,m} + \mathbf{S}_{k,l+2,m}) \\
&\qquad + \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}'_{k+2,l,m} + \mathbf{S}'_{k,l+2,m}) \\
&+ 2J_7 \sum_{k,l,m=1}^{K-2,L-2,M} \mathbf{S}_{k,l,m} \cdot (\mathbf{S}'_{k+2,l,m} + \mathbf{S}'_{k,l+2,m}) \\
&\qquad + \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}_{k+2,l,m} + \mathbf{S}_{k,l+2,m})
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\tilde{J}_7 \sum_{k,l,m=1}^{K-2,L-2,M-1} \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}_{k+2,l,m+1} + \mathbf{S}_{k,l+2,m+1}) \\
&\qquad + \mathbf{S}_{k,l,m+1} \cdot (\mathbf{S}'_{k+2,l,m} + \mathbf{S}'_{k,l+2,m}) \\
&+ 2J_7^{\parallel} \sum_{k,l,m=1}^{K-2,L-2,M} \mathbf{S}_{k,l,m} \cdot (\mathbf{S}_{k+2,l+1,m} + \mathbf{S}_{k+1,l+2,m}) \\
&\qquad + \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}'_{k+2,l+1,m} + \mathbf{S}'_{k+1,l+2,m}) \\
&\qquad + \mathbf{S}_{k,l+2,m} \cdot (\mathbf{S}_{k+1,l,m} + \mathbf{S}_{k+2,l+1,m}) \\
&\qquad + \mathbf{S}'_{k,l+2,m} \cdot (\mathbf{S}'_{k+1,l,m} + \mathbf{S}'_{k+2,l+1,m}) \\
&+ 2J_8 \sum_{k,l,m=1}^{K-1,L-1,M-1} \mathbf{S}_{k,l,m} \cdot (\mathbf{S}_{k+1,l,m+1} + \mathbf{S}_{k,l+1,m+1}) \\
&\qquad + \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}'_{k+1,l,m+1} + \mathbf{S}'_{k,l+1,m+1}) \\
&\qquad + \mathbf{S}_{k,l,m+1} \cdot (\mathbf{S}_{k+1,l,m} + \mathbf{S}_{k,l+1,m}) \\
&\qquad + \mathbf{S}'_{k,l,m+1} \cdot (\mathbf{S}'_{k+1,l,m} + \mathbf{S}'_{k,l+1,m}) \\
&+ 2J_9 \sum_{k,l,m=1}^{K-1,L-1,M-1} \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+1,l+1,m+1} + \mathbf{S}_{k+1,l,m} \cdot \mathbf{S}_{k,l+1,m+1} \\
&\qquad + \mathbf{S}_{k+1,l+1,m} \cdot \mathbf{S}_{k,l,m+1} + \mathbf{S}_{k,l+1,m} \cdot \mathbf{S}_{k+1,l,m+1} \\
&\qquad + \mathbf{S}'_{k,l,m} \cdot \mathbf{S}'_{k+1,l+1,m+1} + \mathbf{S}'_{k+1,l,m} \cdot \mathbf{S}'_{k,l+1,m+1} \\
&\qquad + \mathbf{S}'_{k+1,l+1,m} \cdot \mathbf{S}'_{k,l,m+1} + \mathbf{S}'_{k,l+1,m} \cdot \mathbf{S}'_{k+1,l,m+1} \\
&+ 2J_{10} \sum_{k,l,m=1}^{K-2,L-1,M-1} \mathbf{S}_{k,l,m} \cdot \mathbf{S}'_{k+2,l+1,m} + \mathbf{S}_{k+2,l,m} \cdot \mathbf{S}'_{k,l+1,m} \\
&\qquad + \mathbf{S}_{k+2,l+1,m} \cdot \mathbf{S}'_{k,l,m} + \mathbf{S}_{k,l+1,m} \cdot \mathbf{S}'_{k+2,l,m} \\
&+ 2\tilde{J}_{10} \sum_{k,l,m=1}^{K-2,L-1,M-1} \mathbf{S}'_{k,l,m} \cdot \mathbf{S}_{k+2,l+1,m+1} + \mathbf{S}'_{k+2,l,m} \cdot \mathbf{S}_{k,l+1,m+1} \\
&\qquad + \mathbf{S}'_{k+2,l+1,m} \cdot \mathbf{S}_{k,l,m+1} + \mathbf{S}'_{k,l+1,m} \cdot \mathbf{S}_{k+2,l,m+1} \\
&+ 2J_{11} \sum_{k,l,m=1}^{K-2,L-2,M} \mathbf{S}_{k,l,m} \cdot (\mathbf{S}_{k+2,l,m+1} + \mathbf{S}_{k,l+2,m+1}) \\
&\qquad + \mathbf{S}'_{k,l,m} \cdot (\mathbf{S}'_{k+2,l,m+1} + \mathbf{S}'_{k,l+2,m+1}) \\
&\qquad + \mathbf{S}_{k,l,m+1} \cdot (\mathbf{S}_{k+2,l,m} + \mathbf{S}_{k,l+2,m}) \\
&\qquad + \mathbf{S}'_{k,l,m+1} \cdot (\mathbf{S}'_{k+2,l,m} + \mathbf{S}'_{k,l+2,m}) \\
&+ 2J_{11}^{\parallel} \sum_{k,l,m=1}^{K-2,L-2,M} \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+2,l+2,m} + \mathbf{S}'_{k,l,m} \cdot \mathbf{S}'_{k+2,l+2,m} \\
&\qquad + \mathbf{S}_{k,l+2,m} \cdot \mathbf{S}_{k+2,l,m} + \mathbf{S}'_{k,l+2,m} \cdot \mathbf{S}'_{k+2,l,m}. \qquad (42)
\end{aligned}
$$

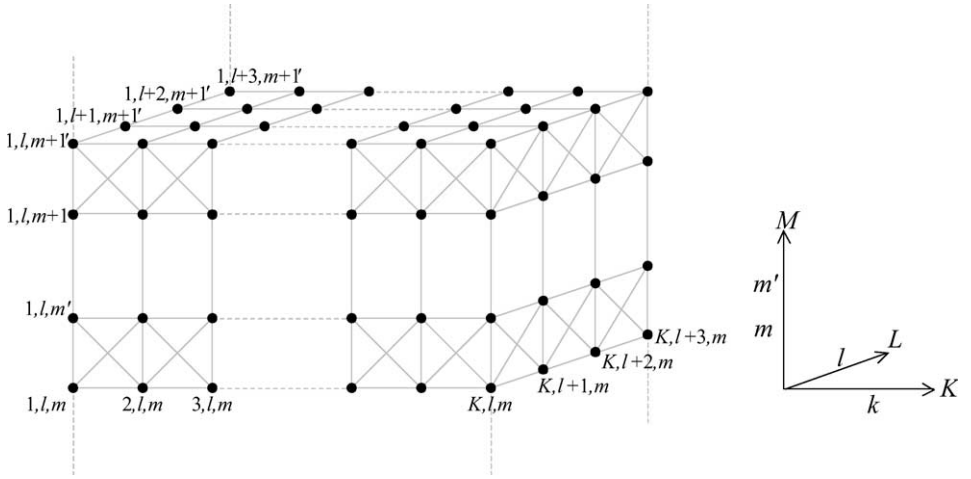

Fig. 3. The labelling scheme of the three-dimensional model.

We draw the three-dimensional model considered in Fig. 3. Let the lattice separation be one and we include all couplings with separation no larger than $2\sqrt{2}$. There are total of twenty-two couplings. Since the couplings J_{10} and $\tilde{J}_{10}$ cannot be grouped in the bond representation, we have to discard them. The Hamiltonian looks rather complicated but it is basically built up by the bilayer net spin model. Couplings between the bilayers are specified with tilde (˜), while those lie on the xy-plane are labelled with parallel sign ($\|$). Specifically:

Separation	Coupling(s) included
1	$J_1, \tilde{J}_1, J_2$
$\sqrt{2}$	$J_3, \tilde{J}_3, J_3^{\|}, J_4, \tilde{J}_4, J_4^{\|}$
$\sqrt{3}$	$J_5, \tilde{J}_5$
2	$J_6, J_6^{\|}$
$\sqrt{5}$	$J_7, \tilde{J}_7, J_7^{\|}, J_8$
$\sqrt{6}$	$J_9, (J_{10}, \tilde{J}_{10})$
$2\sqrt{2}$	$J_{11}, J_{11}^{\|}$

The bond operator representations of the Hamiltonian Eq. (42) for the spin-1/2 case is too tedious to present here, readers who are interested could ask us for manuscript. Basically we combine terms according to combinations of creation and annihilation boson operators. Straightforward manipulations lead to the following conditions which make the completely dimerized state be an eigenstate of the Hamiltonian Eq. (42):

$$2J_2 = J_3 + J_4, \qquad 2J_5 = J_3^{\|} + J_4^{\|}, \qquad J_6^{\|} = J_7, \qquad 2J_6 = \tilde{J}_1,$$
$$2J_8 = \tilde{J}_3 = \tilde{J}_4, \qquad 2J_9 = \tilde{J}_5, \qquad 2J_{11} = \tilde{J}_7, \qquad J_7^{\|}, J_{11}^{\|} = 0. \tag{43}$$

As a demonstration, we present derivations for the spin-1 case here. Obviously, expression of the Hamiltonian in terms of bond operators is even more complicated. However, since the main concern is about how H_{3D} Eq. (42) acts on the completely dimerized state ψ_D, only terms consisting of singlet operator s need to be considered. Hence, one can simplify the expression of the bond operator representation by keeping the terms with singlet operator s only,

$$S^z = -S'^z \to -\sqrt{\frac{2}{3}}\big(t_0^\dagger s + s^\dagger t_0\big), \tag{44}$$

$$S^+ = -S'^+ \to \sqrt{\frac{2}{3}}\big(t_1^\dagger s - s^\dagger t_{-1}\big), \tag{45}$$

$$\mathbf{S}_i = -\mathbf{S}'_i. \tag{46}$$

and

$$\mathbf{S}_i \cdot \mathbf{S}_j \psi_D = \mathbf{S}'_i \cdot \mathbf{S}'_j \psi_D = -\mathbf{S}_i \cdot \mathbf{S}'_j \psi_D = -\mathbf{S}'_i \cdot \mathbf{S}_j \psi_D. \tag{47}$$

Thus, the bond operator representation for the four spin interactions, $(\mathbf{S}_i \cdot \mathbf{S}_j, \mathbf{S}'_i \cdot \mathbf{S}'_j, \mathbf{S}_i \cdot \mathbf{S}'_j, \mathbf{S}'_i \cdot \mathbf{S}_j)$ are now the same upto a sign. So we only need to consider one of them, say, $\mathbf{S}_i \cdot \mathbf{S}_j$,

$$\mathbf{S}_i \cdot \mathbf{S}_j = S_i^z S_j^z + \frac{1}{2}\big(S_i^+ S_j^- + S_i^- S_j^+\big), \tag{48}$$

when for $i \neq j$

$$S_i^z S_j^z = \frac{2}{3}\big(t_{0i}^\dagger t_{0j}^\dagger s_i s_j + t_{0i}^\dagger s_j^\dagger s_i t_{0j} + s_i^\dagger t_{0j}^\dagger t_{0i} s_j + s_i^\dagger s_j^\dagger t_{0i} t_{0j}\big), \tag{49}$$

$$S_i^+ S_j^- = \frac{4}{3}\big(-t_{1i}^\dagger t_{-1j}^\dagger s_i s_j + t_{1i}^\dagger s_j^\dagger s_i t_{1j} + s_i^\dagger t_{-1j}^\dagger t_{-1i} s_j - s_i^\dagger s_j^\dagger t_{-1i} t_{1j}\big). \tag{50}$$

When $i = j$, some four-operator terms will reduce to two-operator terms, e.g., $t_{1i}^\dagger t_{0i} t_{1j}^\dagger t_{0j} \to t_{1i}^\dagger t_{1i}(1 + t_{0i}^\dagger t_{0i}) \to t_{1i}^\dagger t_{1i}$. Thus, we have to keep terms with t operators but neglect terms with p operators, i.e.,

$$S_i^z S_i^z \to \frac{1}{2} t_{1i}^\dagger t_{1i} + t_{0i}^\dagger t_{0i} + \frac{1}{2} t_{-1i}^\dagger t_{-1i} + \frac{2}{3} s_i^\dagger s_i, \tag{51}$$

$$S_i^+ S_i^- \to 2 t_{1i}^\dagger t_{1i} + t_{0i}^\dagger t_{0i} + t_{-1i}^\dagger t_{-1i} + \frac{4}{3} s_i^\dagger s_i, \tag{52}$$

$$S_i^- S_i^+ \to t_{1i}^\dagger t_{1i} + t_{0i}^\dagger t_{0i} + 2 t_{-1i}^\dagger t_{-1i} + \frac{4}{3} s_i^\dagger s_i, \tag{53}$$

To simplify the expression, let us denote $\mathbf{O}_{i;j}$ as

$$\begin{aligned}\mathbf{O}_{i;j} \equiv \frac{2}{3}\big(&t_{0i}^\dagger t_{0j}^\dagger s_i s_j + t_{0i}^\dagger s_j^\dagger s_i t_{0j} + s_i^\dagger t_{0j}^\dagger t_{0i} s_j + s_i^\dagger s_j^\dagger t_{0i} t_{0j} \\ &- t_{1i}^\dagger t_{-1j}^\dagger s_i s_j + t_{1i}^\dagger s_j^\dagger s_i t_{1j} + s_i^\dagger t_{-1j}^\dagger t_{-1i} s_j - s_i^\dagger s_j^\dagger t_{-1i} t_{1j} \\ &- t_{-1i}^\dagger t_{1j}^\dagger s_i s_j + t_{-1i}^\dagger s_j^\dagger s_i t_{-1j} + s_i^\dagger t_{1j}^\dagger t_{1i} s_j - s_i^\dagger s_j^\dagger t_{1i} t_{-1j}\big).\end{aligned} \tag{54}$$

Finally, the Hamiltonian of the three-dimensional model Eq. (42) becomes (only the terms with s-operator are included),

$$
\begin{aligned}
H_{3\mathrm{D}} \to 2J_1 \sum_{k,l,m=1}^{K,L,M} \left(-2s^{\dagger}_{k,l,m}s_{k,l,m} - t^{\dagger}_{1k,l,m}t_{1k,l,m} - t^{\dagger}_{0k,l,m}t_{0k,l,m} - t^{\dagger}_{-1k,l,m}t_{-1k,l,m}\right)
&+ 2(2J_2 - J_3 - J_4) \sum_{k,l,m=1}^{K-1,L-1,M} (\mathbf{O}_{k,l,m;k+1,l,m} + \mathbf{O}_{k,l,m;k,l+1,m}) \\
&+ 2(2J_6 - \tilde{J}_1) \sum_{k,l,m=1}^{K,L,M-1} \mathbf{O}_{k,l,m;k,l,m+1} \\
&+ 4\left(J_6^{\parallel} - J_7\right) \sum_{k,l,m=1}^{K-2,L-2,M} (\mathbf{O}_{k,l,m;k+2,l,m} + \mathbf{O}_{k,l,m;k,l+2,m}) \\
&+ 2(2J_8 - \tilde{J}_3) \sum_{k,l,m=1}^{K-1,L,M-1} (\mathbf{O}_{k,l,m+1;k+1,l,m} + \mathbf{O}_{k,l,m+1;k,l+1,m}) \\
&+ 2(2J_8 - \tilde{J}_4) \sum_{k,l,m=1}^{K-1,L,M-1} (\mathbf{O}_{k,l,m;k+1,l,m+1} + \mathbf{O}_{k,l,m;k,l+1,m+1}) \\
&+ 2(2J_{11} - \tilde{J}_7) \sum_{k,l,m=1}^{K-2,L-2,M-1} (\mathbf{O}_{k,l,m;k+2,l,m+1} + \mathbf{O}_{k,l,m;k,l+2,m+1} \\
&\qquad + \mathbf{O}_{k,l,m+1;k+2,l,m} + \mathbf{O}_{k,l,m+1;k,l+2,m}) \\
&+ 4J_7^{\parallel} \sum_{k,l,m=1}^{K-2,L-2,M} (\mathbf{O}_{k,l,m;k+2,l+1,m} + \mathbf{O}_{k+2,l,m;k,l+1,m} \\
&\qquad + \mathbf{O}_{k,l,m;k+1,l+2,m} + \mathbf{O}_{k,l+2,m;k+1,l,m}) \\
&+ 2\left(J_3^{\parallel} + J_4^{\parallel} - 2J_5\right) \sum_{k,l,m=1}^{K,L,M} (\mathbf{O}_{k,l,m;k+1,l+1,m} + \mathbf{O}_{k+1,l,m;k,l+1,m}) \\
&+ 4J_{11}^{\parallel} \sum_{k,l,m=1}^{K-2,L-2,M} (\mathbf{O}_{k,l,m;k+2,l+2,m} + \mathbf{O}_{k,l+2,m;k+2,l,m}) \\
&+ 2(2J_9 - \tilde{J}_5) \sum_{k,l,m=1}^{K-1,L-1,M-1} (\mathbf{O}_{k,l,m;k+1,l+1,m+1} + \mathbf{O}_{k+1,l,m;k,l+1,m+1} \\
&\qquad + \mathbf{O}_{k,l+1,m;k+1,l,m+1} + \mathbf{O}_{k+1,l+1,m;k,l,m+1}).
\end{aligned}
\tag{55}
$$

With the same arguments used before, only the terms $t^{\dagger}_{0i}t^{\dagger}_{0j}s_is_j, t^{\dagger}_{1i}t^{\dagger}_{-1j}s_is_j$ and $t^{\dagger}_{-1i}t^{\dagger}_{1j}s_is_j$ lead to non-dimer bases, which make the completely dimerized state not an

eigenstate. Hence, the coefficients of those terms are set to zero, which is Eq. (43) as for the spin-1/2 case. If all conditions are satisfied, the eigenenergy of the completely dimerized state is $-2J_1 \times 2 \times N$ where $N = KLM$ is the total number of dimers. In the limit of large J_1, the excitation gap is $2J_1$ and it is from singlet to triplet.

Among these twenty-two couplings, J_1 is independent of the others, four of them ($J_7^{\parallel}, J_{10}, \tilde{J}_{10}, J_{11}^{\parallel}$) should be zero if there is no further neighbor couplings considered, and the rests seventeen couplings are connected by the seven conditions as stated in Eq. (43). Various combinations of them will lead to a variety of quantum phase transitions. Note that those seven conditions on couplings are not all related for some of them are independent of the others. In the followings, we discuss a few soluble cases. The objective is to obtain soluble models with minimum number of couplings. For all the cases, the condition $2J_2 = J_3 + J_4$ is always held.

5.1. *Double layer model I:* $2J_5 = J_3^{\parallel} + J_4^{\parallel}$

This condition leads to another double layer model for there is no connections between layers, as shown in Fig. 4. In this model, the effect of $J_3^{\parallel}$ and $J_4^{\parallel}$ on the dimers are exactly cancelled out by J_5 which is very similar to the net spin model we proposed before where the effects of J_2 are cancelled out by J_3 and J_4.

5.2. *Double layer model II:* $J_6^{\parallel} = J_7$

Another double layer model can be constructed by the condition $J_6^{\parallel} = J_7$ (Fig. 5). The quantum fluctuation due to $J_6^{\parallel}$ is cancelled out exactly by J_7, results in the completely dimerized state being an eigenstate.

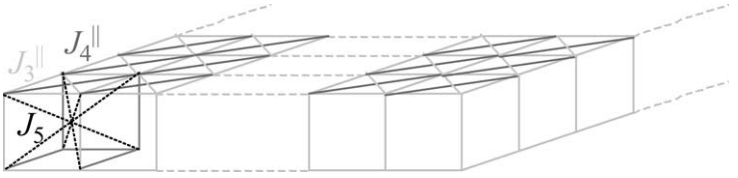

Fig. 4. The new double layer model constructed by conditions $2J_2 = J_3 + J_4$ and $2J_5 = J_3^{\parallel} + J_4^{\parallel}$.

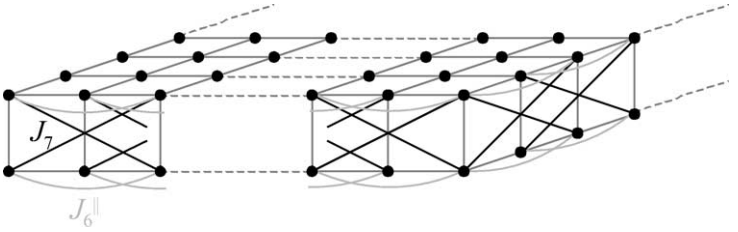

Fig. 5. The new double layer model constructed by conditions $2J_2 = J_3 + J_4$ and $J_6^{\parallel} = J_7$.

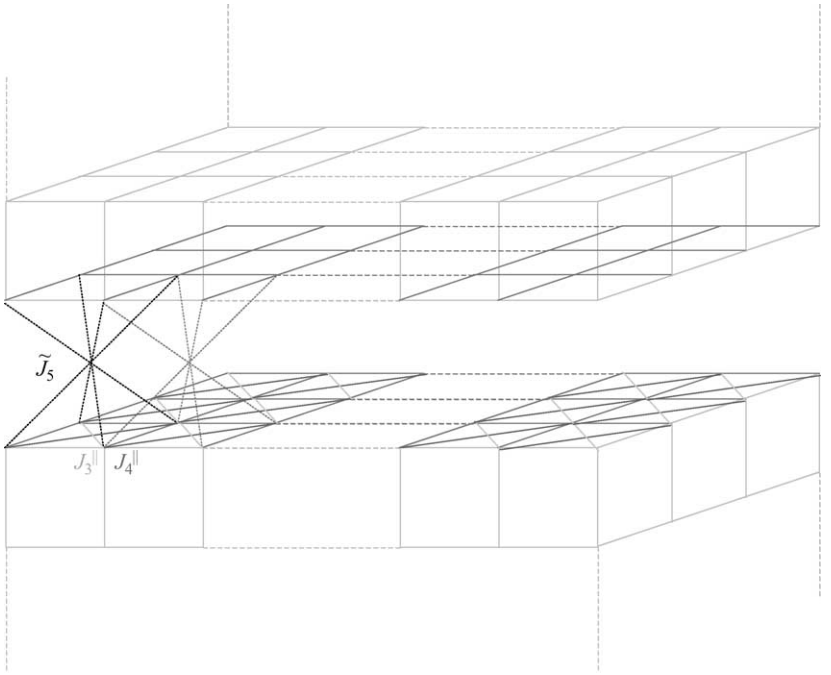

Fig. 6. The three-dimensional model constructed by conditions $2J_2 = J_3 + J_4$ and $2\tilde{J}_5 = J_3^{\parallel} + J_4^{\parallel}$.

5.3. *Three-dimensional model I:* $2\tilde{J}_5 = J_3^{\parallel} + J_4^{\parallel}$

Actually, coupling $\tilde{J}_5$ could take the role of J_5 to cancel the effects of $J_3^{\parallel}$ and $J_4^{\parallel}$ on the dimers. So previous condition is modified to $J_5 = 0, J_9 = 0, 2\tilde{J}_5 = J_3^{\parallel} + J_4^{\parallel}$. This is a simple three-dimensional model as shown in Fig. 6.

5.4. *Three-dimensional model II:* $J_6^{\parallel} = \tilde{J}_7$

Similarly to Section 5.2, the role of J_7 in Section 5.3 can be replaced by $\tilde{J}_7$ with J_{11} set to zero, results in a soluble 3D model as shown in Fig. 7. Note that we use open boundary conditions so there exist no $J_6^{\parallel}$ on the bottom and top layers.

5.5. *Three-dimensional model III:* $\tilde{J}_1 = 2J_6$

The model is shown in Fig. 8, which can be considered as an extension of the nested-ladder model discussed in Section 4. A generalization of this case is that if there exists coupling J_a, which connects $\mathbf{S}_{k,l,m}$ and $\mathbf{S}'_{k,l,m+1}$, then the condition for the completely dimerized state to be an eigenstate of the Hamiltonian would be $J_a + \tilde{J}_1 = 2J_6$.

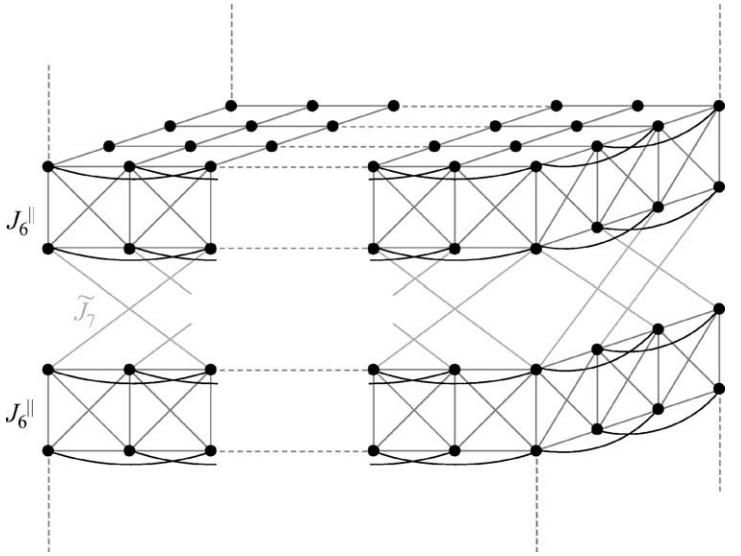

Fig. 7. The three-dimensional model constructed by conditions $2J_2 = J_3 + J_4$ and $J_6^{\parallel} = \tilde{J}_7$.

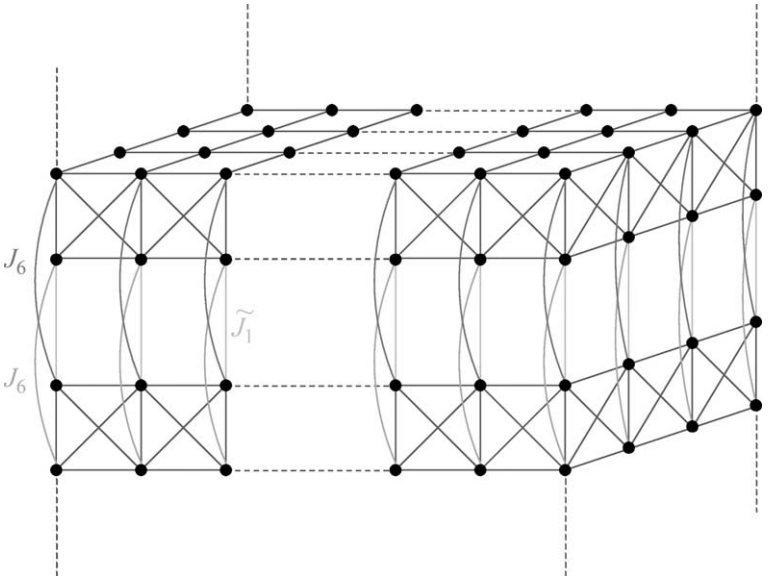

Fig. 8. The three-dimensional model constructed by conditions $2J_2 = J_3 + J_4$ and $\tilde{J}_1 = 2J_6$.

5.6. *More three-dimensional models*

In fact, each independent set of conditions as specified in Eq. (43) will lead to a three-dimensional soluble model in which the completely dimerized state is an eigenstate of

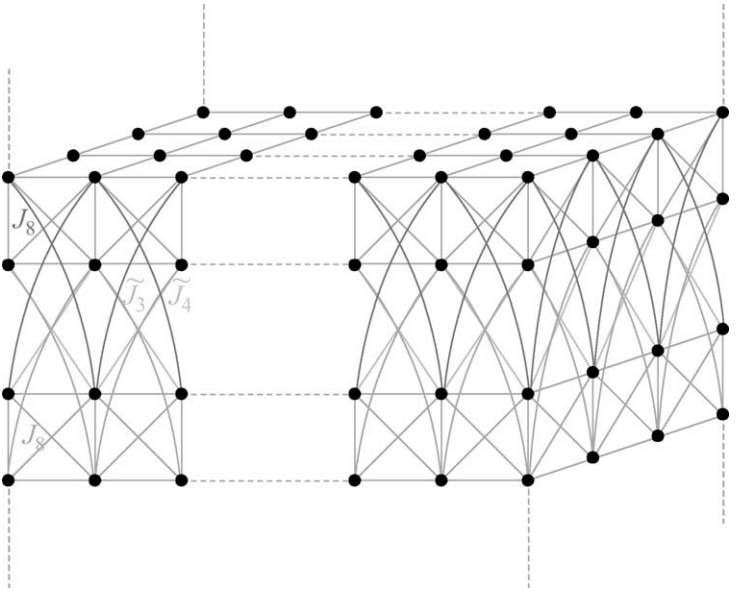

Fig. 9. The three-dimensional model constructed by conditions $2J_2 = J_3 + J_4$ and $\tilde{J}_3 = \tilde{J}_4 = 2J_8$.

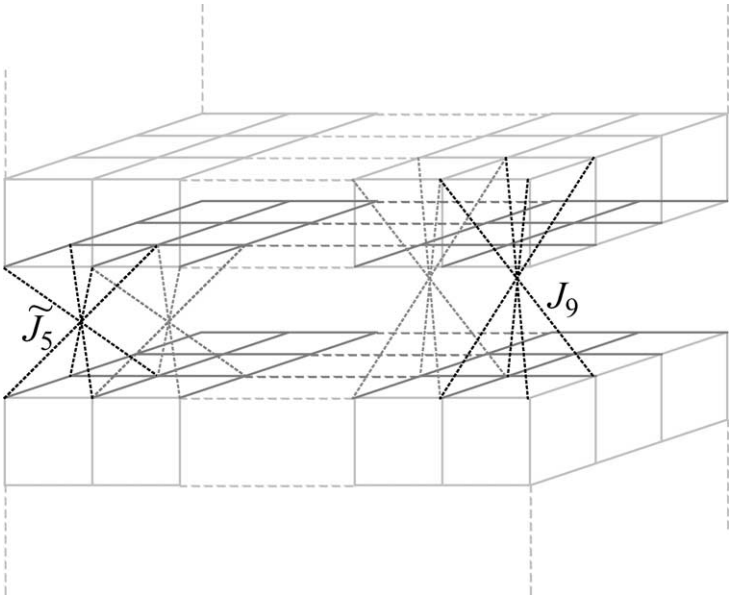

Fig. 10. The three-dimensional model constructed by conditions $2J_2 = J_3 + J_4$ and $\tilde{J}_5 = 2J_9$.

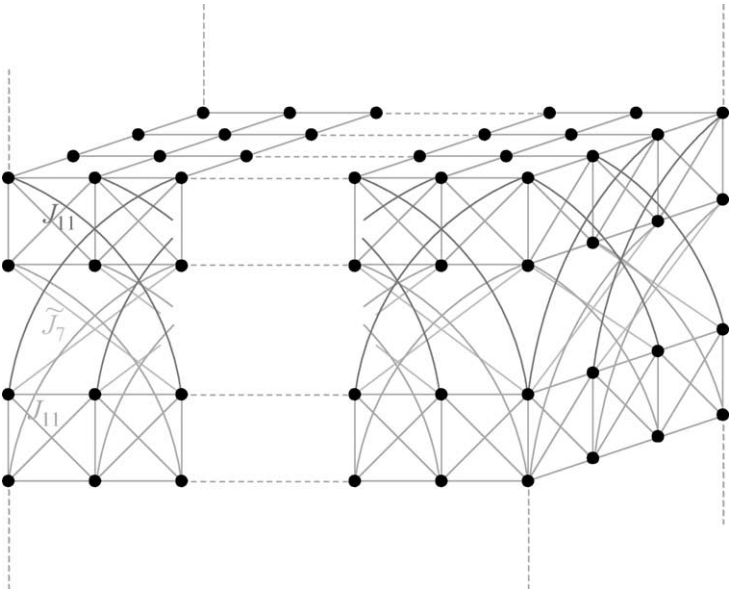

Fig. 11. The three-dimensional model constructed by conditions $2J_2 = J_3 + J_4$ and $\tilde{J}_7 = 2J_{11}$.

the Hamiltonian. One can also combine those independent set of conditions together to construct a new three-dimensional model. Of course, the more couplings one introduces, the more relations one needs and more complicated models will be obtained. To end this section, let us just mention three more of them:

3D model IV: $2J_8 = \tilde{J}_3 = \tilde{J}_4$, shown in Fig. 9;
3D model V: $2J_9 = \tilde{J}_5$, shown in Fig. 10;
3D model VI: $2J_{11} = \tilde{J}_7$, shown in Fig. 11.

Let us specify lattice symmetry for models considered. Because these models are built on the double layer, each unit cell is a tetragonal contains two spins. For models represented by Figs. 4 and 6, the space symmetry group is C_4 if $J_3^{\parallel} = J_4^{\parallel}$. If we let $J_3 = J_4$, then they have all of the space symmetries for the tetragonal lattice: C_4, S_4, C_{4h}, C_{4v}, D_4, D_{4h}, and D_{2d}. For models represented by Figs. 5 and 7–11, the space symmetry group is C_4 for any parameter combinations. If we have $J_3 = J_4$, then they also possess all space symmetries of the tetragonal lattice: C_4, S_4, C_{4h}, C_{4v}, D_4, D_{4h}, and D_{2d}.

6. Summary

In summary, we obtained bond operator representation for the spin-1 dimers and applied the bond operator method for any spin S to construct various two and three-dimensional quantum spin models where the completely dimerized state, whose wave function is a direct product of all spin dimers, is an eigenstate. We presented three two-dimensional

models: the nested-ladder model and two double layer models, and six three-dimensional models. These soluble quantum spin models are shown in Figs. 4–11. As shown in Ref. [22], all these models would have the completely dimerized state as the ground state in the limit of large J_1. The eigenvalue of the completely dimerized state is a product of (i) the energy scale $-2J_1$, (ii) the magnitude of spin $S(S+1)$, and (iii) the total number of singlet dimers in the model KL (for the double layer cases) or KLM (for the three-dimensional cases). Moreover, as demonstrated in Ref. [22], with these soluble models, one can analytically study quantum phase transitions and obtain rich excitation spectrums.

Acknowledgements

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Appendix A. Spin-1 bond operator

To obtain the spin-1 bond operator, the matrix representation of Eq. (15) to Eq. (23) is

$$\begin{bmatrix} |p_2\rangle \\ |p_1\rangle \\ |p_0\rangle \\ |p_{-1}\rangle \\ |p_{-2}\rangle \\ |t_1\rangle \\ |t_0\rangle \\ |t_{-1}\rangle \\ |s\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \end{bmatrix} \begin{bmatrix} |\uparrow\uparrow\rangle \\ |\uparrow 0\rangle \\ |\uparrow\downarrow\rangle \\ |0\uparrow\rangle \\ |00\rangle \\ |0\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow 0\rangle \\ |\downarrow\downarrow\rangle \end{bmatrix}, \tag{A.1}$$

or shortly

$$\mathbf{e} = \mathbf{M}^{-1}\tilde{\mathbf{e}}. \tag{A.2}$$

Then, $\tilde{\mathbf{e}} = \mathbf{M}\mathbf{e}$ leads to

$$|\uparrow\uparrow\rangle = |p_2\rangle, \tag{A.3}$$

$$|\uparrow 0\rangle = \frac{1}{\sqrt{2}}\big(|p_1\rangle - |t_1\rangle\big), \tag{A.4}$$

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{6}}|p_0\rangle - \frac{1}{\sqrt{2}}|t_0\rangle + \frac{1}{\sqrt{3}}|s\rangle, \tag{A.5}$$

$$|0\uparrow\rangle = \frac{1}{\sqrt{2}}\big(|p_1\rangle + |t_1\rangle\big), \tag{A.6}$$

$$|00\rangle = \sqrt{\frac{2}{3}}|p_0\rangle - \frac{1}{\sqrt{3}}|s\rangle, \tag{A.7}$$

$$|0\downarrow\rangle = \frac{1}{\sqrt{2}}\big(|p_{-1}\rangle - |t_{-1}\rangle\big), \tag{A.8}$$

$$|\downarrow\uparrow\rangle = \frac{1}{\sqrt{6}}|p_0\rangle + \frac{1}{\sqrt{2}}|t_0\rangle + \frac{1}{\sqrt{3}}|s\rangle, \tag{A.9}$$

$$|\downarrow 0\rangle = \frac{1}{\sqrt{2}}\big(|p_{-1}\rangle + |t_{-1}\rangle\big), \tag{A.10}$$

$$|\downarrow\downarrow\rangle = |p_{-2}\rangle. \tag{A.11}$$

Introducing the matrix representations of Spin-1 operators

$$\mathbf{S}^z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{S}^+ = \sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{S}^- = \sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{A.12}$$

the bond operator representations of the original spin operator S^z and S^+ are given by

$$\begin{aligned} S^z &= \tilde{\mathbf{e}}^T \cdot \big(\mathbf{S}^z \otimes \mathbf{I}\big) \cdot \tilde{\mathbf{e}} = \mathbf{e}^T \cdot \mathbf{M}^T \cdot \big(\mathbf{S}^z \otimes \mathbf{I}\big) \cdot \mathbf{M} \cdot \mathbf{e} \\ &= p_2^\dagger p_2 + \frac{1}{2}p_1^\dagger p_1 - \frac{1}{2}p_1^\dagger t_1 - \frac{1}{\sqrt{3}}p_0^\dagger t_0 - \frac{1}{2}p_{-1}^\dagger p_{-1} - \frac{1}{2}p_{-1}^\dagger t_{-1} - p_{-2}^\dagger p_{-2} \\ &\quad - \frac{1}{2}t_1^\dagger p_1 + \frac{1}{2}t_1^\dagger t_1 - \frac{1}{\sqrt{3}}t_0^\dagger p_0 - \frac{1}{2}t_{-1}^\dagger p_{-1} - \frac{1}{2}t_{-1}^\dagger t_{-1} - \sqrt{\frac{2}{3}}\big(t_0^\dagger s + s^\dagger t_0\big), \end{aligned} \tag{A.13}$$

where $\mathbf{I}$ is an 3×3 identity matrix, and

$$\begin{aligned} S^+ &= \tilde{\mathbf{e}}^T \cdot \big(\mathbf{S}^+ \otimes \mathbf{I}\big) \cdot \tilde{\mathbf{e}} \\ &= p_2^\dagger p_1 + p_2^\dagger t_1 + \sqrt{\frac{3}{2}}p_1^\dagger p_0 + \frac{1}{\sqrt{2}}p_1^\dagger t_0 + \sqrt{\frac{3}{2}}p_0^\dagger p_{-1} + \frac{1}{\sqrt{6}}p_0^\dagger t_{-1} + p_{-1}^\dagger p_{-2} \\ &\quad - \frac{1}{\sqrt{6}}t_1^\dagger p_0 + \frac{1}{\sqrt{2}}t_1^\dagger t_0 - \frac{1}{\sqrt{2}}t_0^\dagger p_{-1} + \frac{1}{\sqrt{2}}t_0^\dagger t_{-1} - t_{-1}^\dagger p_{-2} \\ &\quad + \frac{2}{\sqrt{3}}\big(t_1^\dagger s - s^\dagger t_{-1}\big). \end{aligned} \tag{A.14}$$

Similarly,

$$\begin{aligned} S'^z &= \tilde{\mathbf{e}}^T \cdot \big(\mathbf{I} \otimes \mathbf{S}^z\big) \cdot \tilde{\mathbf{e}} = \mathbf{e}^T \cdot \mathbf{M}^T \cdot \big(\mathbf{I} \otimes \mathbf{S}^z\big) \cdot \mathbf{M} \cdot \mathbf{e} \\ &= p_2^\dagger p_2 + \frac{1}{2}p_1^\dagger p_1 + \frac{1}{2}p_1^\dagger t_1 + \frac{1}{\sqrt{3}}p_0^\dagger t_0 - \frac{1}{2}p_{-1}^\dagger p_{-1} + \frac{1}{2}p_{-1}^\dagger t_{-1} - p_{-2}^\dagger p_{-2} \\ &\quad + \frac{1}{2}t_1^\dagger p_1 + \frac{1}{2}t_1^\dagger t_1 + \frac{1}{\sqrt{3}}t_0^\dagger p_0 + \frac{1}{2}t_{-1}^\dagger p_{-1} - \frac{1}{2}t_{-1}^\dagger t_{-1} + \sqrt{\frac{2}{3}}\big(t_0^\dagger s + s^\dagger t_0\big), \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} S'^{+} &= \tilde{\mathbf{e}}^{T} \cdot \left(\mathbf{I} \otimes \mathbf{S}^{+}\right) \cdot \tilde{\mathbf{e}} \\ &= p_2^{\dagger} p_1 - p_2^{\dagger} t_1 + \sqrt{\frac{3}{2}} p_1^{\dagger} p_0 - \frac{1}{\sqrt{2}} p_1^{\dagger} t_0 + \sqrt{\frac{3}{2}} p_0^{\dagger} p_{-1} - \frac{1}{\sqrt{6}} p_0^{\dagger} t_{-1} + p_{-1}^{\dagger} p_{-2} \\ &\quad + \frac{1}{\sqrt{6}} t_1^{\dagger} p_0 + \frac{1}{\sqrt{2}} t_1^{\dagger} t_0 + \frac{1}{\sqrt{2}} t_0^{\dagger} p_{-1} + \frac{1}{\sqrt{2}} t_0^{\dagger} t_{-1} + t_{-1}^{\dagger} p_{-2} \\ &\quad - \frac{2}{\sqrt{3}} \left(t_1^{\dagger} s - s^{\dagger} t_{-1}\right). \end{aligned} \tag{A.16}$$

Appendix B. General spin S

Theoretically, exactly soluble model can be constructed for any spin-S with this bond operator method, provided that the bond operator representation is given for that spin-S. However, when spin-S increases, the number of bosonic operators required to represent the spin operators $\mathbf{S}$ and $\mathbf{S}'$ is $(2S+1)^2$, increases exponentially.

We can no longer list all the number of terms of the Hamiltonian in the bond operator representation. However, it is also unnecessary to do so. Since, the singlet is only directly connected to the triplet upon the action of raising or lowing operators $S^{\pm}$, the number of terms we need to consider to keep the complete dimerized state as an eigenstate is limited to three, like what we show explicitly in the spin-1/2 ($t_{\alpha i}^{\dagger} s_i t_{\alpha j}^{\dagger} s_j$ with $\alpha = x, y, z$) and spin-1 case ($t_{0i}^{\dagger} t_{0j}^{\dagger} s_i s_j$, $t_{1i}^{\dagger} t_{-1\,j}^{\dagger} s_i s_j$ and $t_{-1i}^{\dagger} t_{1j}^{\dagger} s_i s_j$).

When only the singlet $[i, i']$ is concerned, we have

$$\mathbf{S}_i^2 + \mathbf{S}_i'^2 + 2\mathbf{S}_i \cdot \mathbf{S}_i' = \left(\mathbf{S}_i + \mathbf{S}_i'\right)^2 = S_{ii'}(S_{ii'} + 1) = 0. \tag{B.1}$$

Hence

$$\mathbf{S}_i \cdot \mathbf{S}_i' = -S(S+1), \tag{B.2}$$

and compare it with

$$\mathbf{S}_i \cdot \mathbf{S}_i = S(S+1). \tag{B.3}$$

We get

$$\mathbf{S}_i = -\mathbf{S}_i'. \tag{B.4}$$

Put the above relation into the Hamiltonian Eq. (32), we have

$$
\begin{aligned}
H\psi_D = {} & -2J_1 \sum_{k,m=1}^{K,M} \mathbf{S}_{k,m} \cdot \mathbf{S}_{k,m}\psi_D \\
& + 2(2J_2 - J_3 - J_4) \sum_{k,m=1}^{K-1,M} \mathbf{S}_{k,m} \cdot \mathbf{S}_{k+1,m}\psi_D \\
& + 2(J_6 + J_7 - J_5 - J_8) \sum_{k,m=1}^{K,M-1} \mathbf{S}_{k,m} \cdot \mathbf{S}_{k,m+1}\psi_D . \qquad \text{(B.5)}
\end{aligned}
$$

It is clear that the complete dimerized state ψ_D is an eigenstate with eigenenergy $E_D = -2J_1 KMS(S+1)$ when the conditions in Eq. (41) hold.

For the three-dimensional model,

$$
\begin{aligned}
H_{3\mathrm{D}} = {} & 2J_1 \sum_{k,l,m=1}^{K,L,M} \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k,l,m} \\
& + 2(2J_2 - J_3 - J_4) \sum_{k,l,m=1}^{K-1,L-1,M} (\mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+1,l,m} + \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k,l+l,m}) \\
& + 2(2J_6 - \tilde{J}_1) \sum_{k,l,m=1}^{K,L,M-1} \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k,l,m+1} \\
& + 4\big(J_6^{\parallel} - J_7\big) \sum_{k,l,m=1}^{K-2,L-2,M} (\mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+2,l,m} + \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k,l+2,m}) \\
& + 2(2J_8 - \tilde{J}_3) \sum_{k,l,m=1}^{K-1,L-1,M-1} (\mathbf{S}_{k,l,m+1} \cdot \mathbf{S}_{k+1,l,m} + \mathbf{S}_{k,l,m+1} \cdot \mathbf{S}_{k,l+1,m}) \\
& + 2(2J_8 - \tilde{J}_4) \sum_{k,l,m=1}^{K-1,L-1,M-1} (\mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+1,l,m+1} + \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k,l+1,m+1}) \\
& + 2(2J_{11} - \tilde{J}_7) \sum_{k,l,m=1}^{K-2,L-2,M-1} (\mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+2,l,m+1} + \mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k,l+2,m+1} \\
& \qquad + \mathbf{S}_{k,l,m+1} \cdot \mathbf{S}_{k+2,l,m} + \mathbf{S}_{k,l,m+1} \cdot \mathbf{S}_{k,l+2,m}) \\
& + 2(2J_9 - \tilde{J}_5) \sum_{k,l,m=1}^{K-1,L-1,M-1} (\mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+1,l+1,m+1} + \mathbf{S}_{k+1,l,m} \cdot \mathbf{S}_{k,l+1,m+1} \\
& \qquad + \mathbf{S}_{k,l+1,m} \cdot \mathbf{S}_{k+1,l,m+1} + \mathbf{S}_{k+1,l+1,m} \cdot \mathbf{S}_{k,l,m+1}) \\
& + 4J_7^{\parallel} \sum_{k,l,m=1}^{K-2,L-1,M} (\mathbf{S}_{k,l,m} \cdot \mathbf{S}_{k+2,l+1,m} + \mathbf{S}_{k+2,l,m} \cdot \mathbf{S}_{k,l+1,m})
\end{aligned}
$$

$$+2\left(J_3^{\parallel}+J_4^{\parallel}-2J_5\right)\sum_{k,l,m=1}^{K-1,L-1,M}\left(\mathbf{S}_{k+1,l,m}\cdot\mathbf{S}_{k,l+1,m}+\mathbf{S}_{k,l,m}\cdot\mathbf{S}_{k+1,l+1,m}\right)$$
$$-4J_7^{\parallel}\sum_{k,l,m=1}^{K-1,L-2,M}\left(\mathbf{S}_{k,l,m}\cdot\mathbf{S}_{k+1,l+2,m}+\mathbf{S}_{k,l+2,m}\cdot\mathbf{S}_{k+1,l,m}\right)$$
$$+4J_{11}^{\parallel}\sum_{k,l,m=1}^{K-2,L-2,M}\left(\mathbf{S}_{k,l,m}\cdot\mathbf{S}_{k+2,l+2,m}+\mathbf{S}_{k,l+2,m}\cdot\mathbf{S}_{k+2,l,m}\right). \tag{B.6}$$

From the Hamiltonian Eq. (B.6), the complete dimerized state gives eigenenergy $E_D = -2J_1KLMS(S+1)$ when the conditions in Eq. (43) hold. That is, we can obtain the conditions for a general spin-S model which has dimerized ground state, by simply using the relation $\mathbf{S}_i = -\mathbf{S}_i'$.

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