

Modern Physics Letters B, Vol. 16, No. 26 (2002) 1–5
 © World Scientific Publishing Company

MESOSCOPIC CIRCUIT WITH LINEAR DISSIPATION*

TING LU[†] and YOU-QUAN LI^{‡,§}

*Zhejiang Institute of Modern Physics,
 Zhejiang University, Hangzhou 310027, China*

[†]*Department of Physics, University of California, San Diego, CA 92093, USA*

[†]*tlu@nssgi-1.ucsd.edu*

[§]*yqli@zimp.zju.edu.cn*

Received 00 Month 2002

We firstly demonstrate the main ideas of quantum theory for mesoscopic electric circuits, which we proposed several years ago. In the theory, the importance of the charge discreteness in a mesoscopic electric circuit is addressed. As a further development, we discuss the mesoscopic electric circuit in the presence of linear dissipation as well as magnetic flux. We propose a quantum Kirchoff equation for the LCR circuit and discuss the oscillations for different criterion factors. We also solve the energy spectra and eigenvalues of the physical current under a soluble symmetry hypothesis.

1. Introduction

The rapid achievements in nanotechnology have brought about tremendous developments in experimental physics on a mesoscopic scale. Miniaturization of integrated circuits is undoubtedly a persistent trend in the future electronic device community. A theory for mesoscopic circuits was proposed recently by one of present authors and co-worker, in which a charge discreteness is first introduced in the quantization of electric circuits.¹

In the theory, a new kind of commutation relation for electric charge and current occurred inevitably, which gives rise to a deformation of the uncertainty relation which recovers the conventional one once the charge discreteness vanishes. In addition to the concept of charge representation, the so-called canonical current representation and pseudo-current representation differ from each other as long as the charge discreteness exists though they have the same continuous limit. It not only provides a concrete realization of mathematical models which discuss the space quantization in high energy physics^{2,3} and quantum gravity,⁴ but also presents a sequence of applications in condensed matter physics from a different point of view.

*This work is supported by the trans-century project of the Ministry of Education and the NSFC for distinguished youth.

[‡]Talk presented at the International Symposium on Frontiers of Science — In Celebration of the 80th Birthday of C. N. Yang (17–19 June 2002), Tsinghua University, China.

2 *T. Lu & Y.-Q. Li*

For example, the persistent current is solved by regarding the mesoscopic metal ring as the circuit of a pure L-design.¹ Application of the theory to a pure C-design gives rise to the Coulomb blockade solution.⁵ This theory was also employed to study quantum transmission lines⁶ as well as quantum current magnification.⁷ A possible generalization to coupled circuits is also proposed.⁸ In terms of the density matrix formulation, a possible method was suggested⁹ to take account of dissipation in mesoscopic circuits but has not been applied to concrete examples. In the next section, we will study the quantum LRC circuit on the basis of Bateman's strategy, where linear dissipation is taken into account. The oscillations for various criterion factors are discussed. In Sec. 4, we solve the energy spectra and the eigenvalues under a soluble symmetry hypothesis.

2. Quantum Kirchoff Equation

In the previous work, we proposed the Hamiltonian¹ for a quantum mesoscopic LC in the presence of flux, where we did not consider the dissipation in the circuit. It is known that the conventional quantization fails once a imaginary dissipation term is directly introduced in the Hamiltonian. However, electrical resistance always exists in electrical circuits, whence it is important to extend the idea for mesoscopic circuit¹ to LCR circuits. According to Bateman's strategy,¹⁰ classical linear dissipation could be included in the Lagrangian formulation by introducing a time-decaying exponential factor multiplying the Lagrangian function, which make it possible to write out the classical Hamiltonian in a standard way. Thus, the Hamiltonian that we are going to study reads

$$\hat{H} = -\frac{\hbar^2}{2q_e L} e^{-\frac{\mathbb{R}}{L}t} (D_{q_e} - \bar{D}_{q_e}) + \frac{1}{2C} e^{\frac{\mathbb{R}}{L}t} \hat{q}^2 - \varepsilon(t) e^{\frac{\mathbb{R}}{L}t} \cdot \hat{q}, \quad (1)$$

where L , C , R and q_e stand for inductance, capacity, resistance and elementary electronic charge, respectively. The covariant discrete derivatives in Eq. (1) are defined by

$$D_{q_e} := \frac{e^{-i(q_e/\hbar)\phi} \hat{Q} - 1}{q_e},$$

$$\bar{D}_{q_e} := \frac{1 - e^{-i(q_e/\hbar)\phi} \hat{Q}^\dagger}{q_e},$$

where the minimum shift operator in charge space is given by $\hat{Q} = e^{-iq_e \hat{p}/\hbar}$, the canonical current \hat{p} operator is the Dirac conjugation of the charge operator \hat{q} , i.e., $[\hat{q}, \hat{p}] = i\hbar$. Substituting Eq. (1) into the Heisenberg equation $d\hat{F}/dt = \partial\hat{F}/\partial t + i[\hat{H}, \hat{F}]/\hbar$, we obtain the quantum Kirchoff equation:

$$L \frac{d^2 \hat{q}}{dt^2} + R \frac{d\hat{q}}{dt} + \frac{1}{C} \hat{M} \hat{q} = \cos\left(\frac{q_e(\hat{p} - \phi)}{\hbar}\right) \varepsilon(t), \quad (2)$$

where

$$\hat{M} = \frac{1}{2} \left\{ \hat{q} \cos \left(\frac{q_e(\hat{p} - \phi)}{\hbar} \right) \hat{q}^{-1} + \cos \left(\frac{q_e(\hat{p} - \phi)}{\hbar} \right) \right\}. \quad (3)$$

Because $\hat{M} \rightarrow 1$ when $q_e \rightarrow 0$, the above quantum Kirchoff equation recovers the well-known classical one:

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = \varepsilon(t).$$

3. Quantum LCR Oscillation

Taking average to the quantum Kirchoff equation (2), we have

$$L \frac{d^2 \langle \hat{q} \rangle}{dt^2} + R \frac{d \langle \hat{q} \rangle}{dt} + \frac{1}{C} \langle \hat{M} \hat{q} \rangle = \left\langle \cos \left(\frac{q_e(\hat{p} - \phi)}{\hbar} \right) \varepsilon(t) \right\rangle, \quad (4)$$

where

$$\left\langle \frac{d \hat{F}}{dt} \right\rangle = \frac{d}{dt} \langle \hat{F} \rangle$$

was adopted. We will discuss the LCR oscillations in the absence and in the presence of an external source, respectively.

3.1. In the absence of a source

In the absence of an external source, Eq. (4) becomes

$$\frac{d^2 \langle \hat{q} \rangle}{dt^2} + \frac{R}{L} \frac{d \langle \hat{q} \rangle}{dt} + \frac{1}{LC} \frac{\langle \hat{M} \hat{q} \rangle}{\langle \hat{q} \rangle} \langle \hat{q} \rangle = 0. \quad (5)$$

Let $\beta = (R/2L)$, $\omega^2 = (1/LC)(\langle \hat{M} \hat{q} \rangle / \langle \hat{q} \rangle)$, ($w \geq 0$) and $G \equiv \beta^2 - \omega^2$, then β and G can be regarded as the criterion factors for the quantum LCR oscillation:

1. If $\beta = 0$, Eq. (5) solves

$$\langle \hat{q} \rangle = A_1 \cos(\omega t + \varphi_1).$$

This is the infinitely large inductance limit or vanishing resistance limit.

2. If $\beta \neq 0$, there will be three cases:

- (i) For $G < 0$, we have

$$\langle \hat{q} \rangle = A_2 e^{-\beta t} \cos(\omega_r t + \varphi_2),$$

where $\omega_r = \sqrt{\omega^2 - \beta^2} = \sqrt{-G}$. It behaves as a damped oscillator with amplitude decreasing ratio $\lambda = 2\pi\beta/\omega_r$

- (ii) For $G = 0$,

$$\langle \hat{q} \rangle = (A_3 + A_4 t) e^{-\beta t},$$

which represents critical damping.

4 *T. Lu & Y.-Q. Li*

(iii) For $G > 0$, we obtain

$$\langle \hat{q} \rangle = A_5 e^{-(\beta+\beta_r)t} + A_6 e^{(\beta+\beta_r)t}$$

with $\beta_r = \sqrt{\beta^2 - \omega^2} = \sqrt{G}$. The aforementioned A 's and φ 's are undetermined integral constants.

In comparison to the classical criterion factor $g = (R^2/4L^2) - (1/LC)$, the quantum criterion factor $G = (R^2/4L^2) - (1/LC)(\langle \hat{M}\hat{q} \rangle / \langle \hat{q} \rangle)$ recovers the classical one when $q_e \rightarrow 0$.

3.2. In the presence of a source

In the presence of a source and if $\langle \hat{f} \rangle \equiv \langle \cos(q_e(\hat{p} - \phi)/\hbar)\varepsilon(t) \rangle \neq 0$, Eq. (4) reads

$$\frac{d^2 \langle \hat{q} \rangle}{dt^2} + 2\frac{R}{2L} \frac{d \langle \hat{q} \rangle}{dt} + \frac{1}{LC} \frac{\langle \hat{M}\hat{q} \rangle}{\langle \hat{q} \rangle} \langle \hat{q} \rangle = \langle \hat{f} \rangle. \quad (6)$$

We consider an AC source $\langle \hat{f} \rangle = F \cos(\mu t)$, which allows us to employ the complex number method by supposing $\langle \hat{q} \rangle = \tilde{A}_q e^{i\mu t}$, yielding

$$\tilde{A}_q = \frac{F}{L(\omega^2 - \mu^2 + i2\beta\mu)}. \quad (7)$$

Then, we obtain

$$\begin{aligned} |\tilde{A}_q| &= \frac{F}{L\sqrt{(\omega^2 - \mu^2)^2 + (2\beta\mu)^2}} \\ \arg(\tilde{A}) &= \arctan \frac{-2\beta\mu}{\omega^2 - \mu^2} \end{aligned} \quad (8)$$

4. Energy Spectra and Current Eigenvalues

We now consider an adiabatic approximation so that the change of the source $\varepsilon(t)$ is very slow, while $R/L \ll 1$, thus $e^{\frac{R}{L}t}$ changes slowly. Then, we can deal with the equation in a similar way as did for LC circuits. In charge representation, any state $|\psi\rangle$ can be expanded as $|\psi\rangle = \sum_n \psi_n |n\rangle$. It is easy to calculate that the Schrödinger equation:

$$\sum_n \langle m | \hat{H} | n \rangle \langle n | \psi \rangle = E_m \langle m | \psi \rangle \quad (9)$$

gives rise to

$$\frac{\hbar^2}{2q_e L} e^{-\frac{R}{L}t} (e^{-i(\alpha_m + q_e \phi/\hbar)} \psi_{m-1} + e^{-i(\alpha_{m+1} + q_e \phi/\hbar)} \psi_{m+1}) \quad (10)$$

$$= \left(\frac{m^2 q_e^2}{2C} e^{\frac{R}{L}t} - m q_e \varepsilon e^{\frac{R}{L}t} + \frac{\hbar^2}{q_e L} e^{-\frac{R}{L}t} - E \right) \psi_m. \quad (11)$$

This equation can be solved under a hypothesis that

$$\hat{Q}|\psi\rangle = e^{i\theta}|\psi\rangle, \quad (12)$$

where θ is an undetermined phase. This means that the system has a symmetry under translation in charge space. In this case the energy spectrum can be obtained explicitly:

$$E_m = \frac{m^2 q_e^2}{2C} e^{\frac{\Phi}{L}t} - m q_e e^{\frac{\Phi}{L}t} \varepsilon + \frac{2\hbar^2}{q_e^2 L} e^{-\frac{\Phi}{L}t} \sin^2 \left[\frac{1}{2}(\theta - q_e \phi / \hbar) \right]. \quad (13)$$

The physics current operator is derived from the Heisenberg equation⁸:

$$\begin{aligned} \hat{I} &= \frac{\partial \hat{q}}{\partial t} + \frac{i}{\hbar} [\hat{H}, \hat{q}] \\ &= -\frac{i\hbar}{2q_e L} e^{-\frac{\Phi}{L}t} (e^{-iq_e \phi / \hbar} \hat{Q} - e^{iq_e \phi / \hbar} \hat{Q}^+), \end{aligned} \quad (14)$$

which gives an explicit eigenvalue under hypothesis (12):

$$I = \frac{\hbar}{q_e L} e^{-\frac{\Phi}{L}t} \sin \left(\theta - \frac{q_e \phi}{\hbar} \right), \quad (15)$$

which oscillates according to the changes of flux and its amplitude decreases over time due to the existence of resistance.

Remarks

It is reasonable to take into account the discreteness of electrical charge in mesoscopic circuits due to the fact that the number of charge carriers involved is no longer very large. It has been shown that the theory addressing the charge quantization at the beginning naturally explains some phenomena occurring in the mesoscopic system.

References

1. Y. Q. Li and B. Chen, *Phys. Rev.* **B53**, 4027–4032 (1996).
2. H. S. Snyder, *Phys. Rev.* **71**, 38–41 (1947).
3. S. Mantecinos, I. Saavedra and O. Kunstmann, *Phys. Lett.* **A109**, 139–142 (1985).
4. L. J. Garay, *Int. J. Mod. Phys.* **A10**, 145–166 (1995) and references therein.
5. Y. Q. Li, in *Proc. 5th Wigner Symposium*, eds. P. Kasperkovitz and D. Grau (World Scientific, Singapore, 1998), pp. 307–310.
6. J. C. Flores, *Phys. Rev.* **B64**, 235309 (2001).
7. J. C. Flores and C. A. Utreras *Phys. Rev.* **B66**, 153410 (2002).
8. Y. Q. Li, in *Spin-Statistical Connection and Communtation Relations: Experimental Tests and Theoretical Implications*, eds. R. C. Hilborn and Guglielmo M. Tino (AIP, Melville, NY, 2000) pp. 85
9. Y. Q. Li and B. Chen, in *Advances in Intelligent Systems and Computer Science*, ed. N. Mastorakis (WSES Press, 1999), pp. 267–272.
10. H. Bateman, *Phys. Rev.* **38**, 815 (1931).