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Extension of Bethe ansatz to multiple occupancies for one-dimensional SU(4) fermions with δ-function interaction

Zu-Jian Ying[a)]
Department of Physics, Hangzhou Teachers College, Hangzhou 310012, China

You-Quan Li and Shi-Jian Gu
Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China

Bin Chen
Department of Physics, Hangzhou Teachers College, Hangzhou 310012, China

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We consider the problem of consistence between the Bethe ansatz (BA) wave function and the multiparticle (more than two) scattering in one-dimensional δ-function interacting SU(4) fermions, which the approach of BA does not explicitly take into account. We find the scattering conditions of three and four particles located at the same position and show that the conditions can be fulfilled by the two-particle connection conditions of the BA wave function. So the definition of the BA wave function can be exactly extended to those cases with multiple occupancies. The inconsistence between the BA and multiparticle interacting on a same site in the degenerate Hubbard model, which makes the BA fail for the model, is shown to vanish in the limit of small site spacing. A correspondence relation of the BA equation and SU(4) symmetry of the system is also indicated for the fermions. The degeneracy of state with BA eigenenergy is given. Singlet lies in the case when there are equal numbers of particles in each inner component. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1515380]

I. INTRODUCTION

Since the Bethe ansatz (BA) approach was proposed to solve exactly the Heisenberg model,[1] it has been applied to many integrable systems such as one-dimensional δ-function interacting bosons[2,3] and spin-1/2 fermions,[4] Hubbard model,[5] Kondo model,[6] spin ladder,[7] etc., and recently there has been much interest in extending the BA to the study of degenerate electrons systems.[8–14] As is well known, the coordinate BA wave functions for N-particle systems are defined on separated regions denoted by $x_{Q_1} < \cdots < x_{Q_N}$ where Q is a permutation of the particles and x_{Q_i} is the coordinate of the Q_ith particle, the connection boundaries between two adjacent regions are those cases in which two (and no more than two) particles interact at the same position, and the two-particle scattering matrices solve the boundary conditions. The matrices transferring amplitudes in the BA wave function from one region to another factorize into products of two-particle S matrices. The diagonalization of the N-particle transfer matrices with periodic conditions leads to the BA equations that determine the whole solutions. So the original BA, as a matter of fact, does not involve the boundaries with more than two particles scattering at the same position or take into account the corresponding scattering conditions. If the scattering conditions of multiple occupations, which are required by models of the degenerate or multicomponent particle systems, cannot be consistently met by the BA wave function, the BA wave function will not be the solutions of the systems. Such a problem of consistence has been encountered in the degenerate Hubbard model, the BA fails to solve the model at configurations with more than two particles on one site,[10] and the theoretical work in the framework of the BA for degenerate Hubbard model has

[a)]Electronic mail: zjying@zimp.zju.edu.cn

been based on the exclusion of such multiply occupied configurations.[11–14] Therefore, whether the BA wave functions are exactly valid for the degenerate systems is still unconfirmed if the consistency between the BA wave function and multiparticle scattering conditions has not been definitely verified.

In the present paper we shall investigate explicitly the problem of consistence between the BA and the multiparticle scattering conditions for one-dimensional δ-function interacting SU(4) fermions which is a spatially continuous system instead of a lattice model. We work out explicitly the scattering boundary conditions of three and four particles located at the same spatial point. We prove that the multiparticle scattering conditions can be fulfilled by the two-particle connection conditions of the BA wave function. Therefore we can extend exactly the definition of the BA wave function to the configurations with multiple occupancies and, unlike in lattice models, the BA is valid for the spatially continuous model without the exclusion of more than two particles interacting at the same position. The inconsistency in the BA for the degenerate Hubbard model is shown to vanish in the limit of small site spacing. An interesting correspondence between the SU(4) symmetry and the BA equation is also pointed out for the δ-function interacting fermions. The states relating to a BA eigenenergy are degenerate, and the degeneracy can be calculated in terms of the highest weight vector of the corresponding SU(4) multiplet.

II. TWO-PARTICLE SCATTERING AND THE BETHE ANSATZ WAVE FUNCTION

The Schrödinger equation of the N-fermion system we shall consider reads

$$\left(-\frac{1}{2}\sum_i^N \frac{\partial^2}{\partial x_i^2}+c\sum_{i<j}\delta(x_i-x_j)\right)\psi=E\psi. \tag{1}$$

We define the BA wave function in the form of

$$\psi^{(Q)}_{a_1\cdots a_N}(x_1,\cdots,x_N)=\sum_{P\in S_N}A^{(Q)}_{a_1\cdots a_N}(k_{P_1}\cdots k_{P_N})e^{i\Sigma_j k_{P_j}x_j}, \tag{2}$$

where a_j labels the four components of the jth fermions. The inner degree of freedom can be spin–orbital double. Other forms of BA wave function are equivalent, our further proof holds as long as the solution is BA wave function. The BA wave function is defined in the region sector

$$R_Q:x_{Q_1}<\cdots<x_{Q_N}.$$

The wave function (2) surely solves the eigenequation

$$-\frac{1}{2}\sum_i^N \frac{\partial^2}{\partial x_i^2}\psi^{(Q)}=E_{\rm BA}\psi^{(Q)} \tag{3}$$

in R_Q with the eigenenergy

$$E_{\rm BA}=\frac{1}{2}\sum_i^N k_i^2. \tag{4}$$

Let $(Q_1\cdots ij\cdots Q_N)$ denote a sector with $Q_r=i$ and $Q_{r+1}=j$ for some r. The two-particle connection condition across the barrier $x_i=x_j$ is given by

$$\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\psi^{(Q_1\cdots ji\cdots Q_N)}_{a_1\cdots a_N}\Big|_{x_j=x_i-}-\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\psi^{(Q_1\cdots ij\cdots Q_N)}_{a_1\cdots a_N}\Big|_{x_j=x_i+}=2c\,\psi^{(Q_1\cdots ij\cdots Q_N)}_{a_1\cdots a_N}\Big|_{x_i=x_j}, \tag{5}$$

which can be obtained by integrating of the eigenequation over a Gauss box at the hyperplane $x_i=x_j$. And on the right-hand side of Eq. (5) the $\psi^{(Q_1\cdots ij\cdots Q_N)}|_{x_j=x_i}$ can be replaced by $\psi^{(Q_1\cdots ji\cdots Q_N)}|_{x_j=x_i}$ because of the continuity of the wave function. Combining the above condition with the uniqueness of the wave function on the two-particle connection boundary, one can find the two-particle S-matrix,[4]

$$S^{ij}=\frac{(k_{P_i}-k_{P_j})I^{ij}+icP^{ij}}{(k_{P_i}-k_{P_j})+ic}, \quad S^{ij}A^{(Q_1\cdots\ ij\cdots\ Q_N)}_{(k_{P_1}\cdots\ k_{P_i}\cdots\ k_{P_j}\cdots\ k_{P_N})}=A^{(Q_1\cdots\ ji\cdots\ Q_N)}_{(k_{P_1}\cdots\ k_{P_i}\cdots\ k_{P_j}\cdots\ k_{P_N})}, \tag{6}$$

where P^{ij} exchanges inner components a_i, a_j and

$$\begin{aligned} &P^{ij}A^{(Q_1\cdots\ ij\cdots\ Q_N)}_{a_1\cdots\ a_i\cdots\ a_j\cdots\ a_N}(k_{P_1}\cdots\ k_{P_i}\cdots\ k_{P_j}\cdots\ k_{P_N}) \\ &=A^{(Q_1\cdots\ ij\cdots\ Q_N)}_{a_1\cdots\ a_j\cdots\ a_i\cdots\ a_N}(k_{P_1}\cdots\ k_{P_i}\cdots\ k_{P_j}\cdots\ k_{P_N}) \\ &=-A^{(Q_1\cdots\ ji\cdots\ Q_N)}_{a_1\cdots\ a_i\cdots\ a_j\cdots\ a_N}(k_{P_1}\cdots\ k_{P_j}\cdots\ k_{P_i}\cdots\ k_{P_N}), \end{aligned}$$

where the minus sign in the second equation comes from the antisymmetry of fermions.

It should be noted that $\psi^{(Q_1\cdots ij\cdots Q_N)}|_{x_j=x_i-}$ and $\psi^{(Q_1\cdots ji\cdots Q_N)}|_{x_j=x_i+}$ in (5) are, respectively, in two regions $(Q_1\cdots ij\cdots Q_N)$ and $(Q_1\cdots ji\cdots Q_N)$ which are neighbor sectors. The transfer matrices for regions not adjacent are products of the two-particle S-matrices. The S-matrices satisfy the Yang–Baxter relations.[4] The diagonalization of the N-particle transfer matrices combined with periodic conditions

$$\psi(\cdots\ x_j\cdots\ )=\psi(\cdots\ x_j+L\cdots\ )$$

results in the BA equations by means of the methods in Refs. 15 and 16

$$\begin{aligned} e^{ik_jL}&=\prod_{\alpha=1}^{M}\frac{k_j-\lambda_\alpha+ic/2}{k_j-\lambda_\alpha-ic/2}, \\ 1&=-\prod_{j=1}^{N}\frac{\lambda_\alpha-k_j-ic/2}{\lambda_\alpha-k_j+ic/2}\prod_{\alpha'=1}^{M}\frac{\lambda_\alpha-\lambda_{\alpha'}+ic}{\lambda_\alpha-\lambda_{\alpha'}-ic}\prod_{\beta=1}^{M'}\frac{\lambda_\alpha-\mu_\beta-ic/2}{\lambda_\alpha-\mu_\beta+ic/2}, \\ 1&=-\prod_{\alpha=1}^{M}\frac{\mu_\beta-\lambda_\alpha-ic/2}{\mu_\beta-\lambda_\alpha+ic/2}\prod_{\beta'=1}^{M'}\frac{\mu_\beta-\mu_{\beta'}+ic}{\mu_\beta-\mu_{\beta'}-ic}\prod_{\gamma=1}^{M''}\frac{\mu_\beta-\nu_\gamma-ic/2}{\mu_\beta-\nu_\gamma+ic/2}, \\ 1&=-\prod_{\beta=1}^{M'}\frac{\nu_\gamma-\mu_\beta-ic/2}{\nu_\gamma-\mu_\beta+ic/2}\prod_{\gamma'=1}^{M''}\frac{\nu_\gamma-\nu_{\gamma'}+ic}{\nu_\gamma-\nu_{\gamma'}-ic}, \end{aligned} \tag{7}$$

which determine the whole solution in the BA approach. The total particle numbers N_1, N_2, N_3, N_4 with inner components 1,2,3,4 are, respectively,

$$\begin{aligned} N_1&=N-M, \quad N_2=M-M', \\ N_3&=M'-M'', \quad N_4=M''. \end{aligned} \tag{8}$$

Here we do not intend to discuss the corresponding ground states, excitations or thermodynamics. In the following we shall consider the relation of the BA wave function and the system symmetry as well as the state's degeneracy.

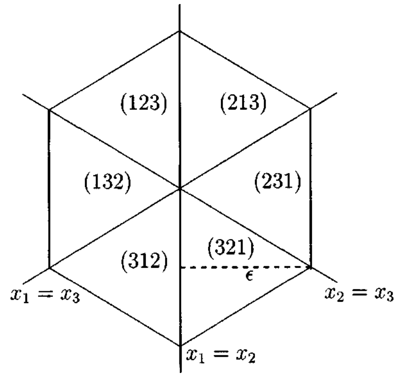


FIG. 1. A cross section of the hexagonal prism in the three-dimensional subspace of x_1, x_2, and x_3. The center axis of the prism is along the intersecting line $x_1 = x_2 = x_3$, and labels of the three interacting particles ijk have been set to be 123.

As one can see from the above procedure, the BA treatment is built up on the basis of solving the separate noninteracting regions R_Q and the two-particle boundaries which are connections of *two* neighbor regions. The BA indeed does not take into account the multiple-occupancy (more than two) boundaries which are joint boundary of *three* or *four* regions, and the corresponding multiparticle scattering conditions become additional ones; what these conditions are is not explicit and whether they are solved by BA remains to be answered. In fact, it is just by examining this kind of multiparticle scattering that Choy and Haldane found the failure of the BA in the degenerate Hubbard model.[10] In the following we shall explicitly investigate the consistence between the BA and those multiparticle scattering out of the consideration of the BA for the four-component fermions (1).

III. THREE-PARTICLE SCATTERING

With the help of the continuity of the wave function at two-particle scattering boundary, which has been used in the Bethe ansatz, it is easy to prove the single valueness of the wave function for three and four particles at the same point. Without guarantee of the wave function single valueness, further discussion on the problem of consistence would be unnecessary since it had been inconsistent here.

To investigate the consistence between the BA and the three-particle scattering we have to obtain the corresponding scattering boundary condition which is not obvious or explicit to write out. Let us consider three particles i, j, and k at the boundary $x_i = x_j = x_k$. We integrate the eigenequation (1) over an arbitrarily shaped volume through which the intersecting line $x_i = x_j = x_k$ is located in the three-dimensional subspace of x_i, x_j, and x_k. The arbitrarily shaped volume can be cut into an ϵ-sized hexagonal prism of which a cross section is shown in Fig. 1. The integration of the eigenequation over the cutoff part cancels automatically, which is a natural result of (3) and (5), respectively, in the sectors and on the two-particle connecting hyperplanes. Therefore, it's sufficient to consider only the prism left.

Set ϵ to be infinitesimal, carry out the expansion of the Gaussian integral in orders of ϵ and keep the lowest order, we find the three-particle condition to be

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)(\psi^{(Q_1\cdots 231\cdots Q_N)} - \psi^{(Q_1\cdots 132\cdots Q_N)})|_{x_1=x_2=x_3} + \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\right)(\psi^{(Q_1\cdots 312\cdots Q_N)} - \psi^{(Q_1\cdots 213\cdots Q_N)})|_{x_1=x_2=x_3} + \left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_1}\right)(\psi^{(Q_1\cdots 123\cdots Q_N)} - \psi^{(Q_1\cdots 321\cdots Q_N)})|_{x_1=x_2=x_3} = 12c\,\psi^{(Q_1\cdots 123\cdots Q_N)}|_{x_1=x_2=x_3}, \tag{9}$$

where we have set the three particles to be 123 for the sake of clarity, and the infinitesimal notations as in $\psi^{(Q_1\cdots 231\cdots Q_N)}|_{x_1=x_3^+,x_2=x_3^-}$ have been dropped since it has been indicated in the corresponding label Q of region sector. Also on the right-hand side of the above relations, $\psi^{(Q_1\cdots 123\cdots Q_N)}|_{x_1=x_2=x_3}$ can be replaced by wave function with other permutation of the particles 123, while the positions of other particles remain unchanged. On the left-hand side of (9), it should be noted that the pairs of wave functions in a bracket are not in neighbor sectors as those of Eq. (5).

We find the condition (9) can be satisfied in terms of the two-particle connection condition (5). As one knows in the BA, the two-particle S-matrices provide to cancel the amplitudes of a same exponential $\exp(i\Sigma_j k_{P_j} x_j)$, the amplitudes $A^{(Q)}_{k_{P_1}\cdots k_{P_N}}$ are independent of the particles coordinates, so it does not affect the cancellation of the amplitudes in (5) when we set the position of a third adjacent particle to be the same as the two particles on the connection boundary and keep all the permutation notation Q's in the amplitudes unchanged. Hence, Eq. (5) still holds at $x_{Q_1}<\cdots<x_i=x_j=x_k<\cdots<x_{Q_N}$,

$$\begin{aligned}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)(\psi^{(Q_1\cdots jik\cdots Q_N)}-\psi^{(Q_1\cdots ijk\cdots Q_N)})|_{x_i=x_j=x_k}&=2c\,\psi^{(Q_1\cdots ijk\cdots Q_N)}|_{x_i=x_j=x_k},\\ \left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)(\psi^{(Q_1\cdots kji\cdots Q_N)}-\psi^{(Q_1\cdots kij\cdots Q_N)})_{x_i=x_j=x_k}&=2c\,\psi^{(Q_1\cdots kij\cdots Q_N)}|_{x_i=x_j=x_k}.\end{aligned}\tag{10}$$

Also note that the wave functions on the left-hand sides are defined on neighbor regions as in (5). We split the differential operation $\partial/\partial x_1-\partial/\partial x_2$ to be $(\partial/\partial x_1-\partial/\partial x_3)+(\partial/\partial x_3-\partial/\partial x_2)$ and similarly for others, thus the left-hand side of (9) becomes

$$\begin{aligned}&\left(\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_3}\right)+\left(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_2}\right)\right)(\psi^{(Q_1\cdots 231\cdots Q_N)}-\psi^{(Q_1\cdots 132\cdots Q_N)})|_{x_1=x_2=x_3}\\ &+\left(\left(\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_1}\right)+\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_3}\right)\right)(\psi^{(Q_1\cdots 312\cdots Q_N)}-\psi^{(Q_1\cdots 213\cdots Q_N)})|_{x_1=x_2=x_3}\\ &+\left(\left(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_2}\right)+\left(\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_1}\right)\right)(\psi^{(Q_1\cdots 123\cdots Q_N)}-\psi^{(Q_1\cdots 321\cdots Q_N)})|_{x_1=x_2=x_3}\end{aligned}$$

which, as a result of (10), equals to $12c\,\psi^{(Q)}|_{x_1=x_2=x_3}$, exactly the right-hand side of (9). Therefore we have proved that the scattering condition of three particles at a same position can be fulfilled by the two-particle connection conditions in the BA solution.

IV. FOUR-PARTICLE SCATTERING

Now let us consider the case of four particles interacting at the same point. For simplicity we omit the notations of other particles. Similar to the three-particle scattering, it is also sufficient to calculate the integral of the eigenequation over an ϵ-sized Gaussian box at the axis $x_1=x_2=x_3=x_4$. A vector in the sector of (1234) can be built on another set of orthonormal basis

$$\mathbf{r}_{1234}\equiv x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3+x_4\mathbf{e}_4=\omega\alpha_{1234}+h\gamma+y\beta_{1234}+y'\beta'_{1234},\tag{11}$$

where the basis vectors are

$$\alpha_{1234}=-\frac{3\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3-3\mathbf{e}_4}{\sqrt{20}},\quad \gamma=\frac{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4}{2},$$
$$\beta_{1234}=\frac{\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+\mathbf{e}_4}{2},\quad \beta'_{1234}=\frac{\mathbf{e}_1-3\mathbf{e}_2+3\mathbf{e}_3-\mathbf{e}_4}{\sqrt{20}}. \tag{12}$$

γ is the direction vector along the center axis $x_1=x_2=x_3=x_4$. The connecting boundary $x_i=x_j$ can be expressed in terms of α_{1234}, γ, β_{1234}, and β'_{1234} according to the above relations. In the (1234) sector the inwards normals of the intersecting hyperplanes $x_1=x_2$, $x_2=x_3$ and $x_3=x_4$ are, respectively, $\mathbf{n}_{21}=\mathbf{e}_2-\mathbf{e}_1$, $\mathbf{n}_{32}=\mathbf{e}_3-\mathbf{e}_2$, and $\mathbf{n}_{43}=\mathbf{e}_4-\mathbf{e}_3$. It is easy to see $\alpha_{1234}\cdot\mathbf{n}_{21}=\alpha_{1234}\cdot\mathbf{n}_{32}=\alpha_{1234}\cdot\mathbf{n}_{43}$. Setting $\omega=\epsilon$, we have a hyperplane which can be the side surface of the Gaussian box in the (1234) region, α_{1234} is the outwards normal direction vector. Other sectors are similar with the subscripts permuted correspondingly. We also set the size parameter ϵ to be infinitesimal, carry out the expansion in the orders of ϵ and keep the lowest order in the integration of the eigenequation. After a careful calculation we find the scattering condition of four particles at one point to be

$$-\frac{1}{2}\sum_{Q\in S_4}\left(3\frac{\partial}{\partial x_{Q_1}}+\frac{\partial}{\partial x_{Q_2}}-\frac{\partial}{\partial x_{Q_3}}-3\frac{\partial}{\partial x_{Q_4}}\right)\psi^{(Q)}(x,x,x,x)=120c\,\psi(x,x,x,x). \tag{13}$$

As we show below, the above four-particle scattering condition is also fulfilled by the two-particle conditions.

Relations in (10) are similarly extended to

$$\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)(\psi^{(Q_1\cdots jikl\cdots Q_N)}-\psi^{(Q_1\cdots ijkl\cdots Q_N)})|_{x_i=x_j=x_k=x_l}=2c\,\psi^{(Q_1\cdots ijkl\cdots Q_N)}|_{x_i=x_j=x_k=x_l},$$
$$\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_k}\right)(\psi^{(Q_1\cdots ikjl\cdots Q_N)}-\psi^{(Q_1\cdots ijkl\cdots Q_N)})|_{x_i=x_j=x_k=x_l}=2c\,\psi^{(Q_1\cdots ijkl\cdots Q_N)}|_{x_i=x_j=x_k=x_l}, \tag{14}$$
$$\left(\frac{\partial}{\partial x_k}-\frac{\partial}{\partial x_l}\right)(\psi^{(Q_1\cdots ijlk\cdots Q_N)}-\psi^{(Q_1\cdots ijkl\cdots Q_N)})|_{x_i=x_j=x_k=x_l}=2c\,\psi^{(Q_1\cdots ijkl\cdots Q_N)}|_{x_i=x_j=x_k=x_l}.$$

And by means of the following trick:

$$\left(3\frac{\partial}{\partial x_{Q_1}}+\frac{\partial}{\partial x_{Q_2}}-\frac{\partial}{\partial x_{Q_3}}-3\frac{\partial}{\partial x_{Q_4}}\right)=\left(3\left(\frac{\partial}{\partial x_{Q_1}}-\frac{\partial}{\partial x_{Q_2}}\right)+4\left(\frac{\partial}{\partial x_{Q_2}}-\frac{\partial}{\partial x_{Q_3}}\right)+3\left(\frac{\partial}{\partial x_{Q_3}}-\frac{\partial}{\partial x_{Q_4}}\right)\right), \tag{15}$$

the differential operations are changed to be of neighbor particles pairs on the left-hand side of (13). To use (14) we note the relation

$$\sum_{Q\in S_4}\left(\frac{\partial}{\partial x_{Q_1}}-\frac{\partial}{\partial x_{Q_2}}\right)\psi^{(Q)}(x,x,x,x)=\frac{1}{2}\sum_{Q\in S_4}\left(\frac{\partial}{\partial x_{Q_1}}-\frac{\partial}{\partial x_{Q_2}}\right)(\psi^{(Q)}(x,x,x,x)-\psi^{(Q')}(x,x,x,x)), \tag{16}$$

where $Q'=(Q_2Q_1Q_3Q_4)$, and the similar relations for the cases of $(\partial/\partial x_{Q_2}-\partial/\partial x_{Q_3})$ and $(\partial/\partial x_{Q_3}-\partial/\partial x_{Q_4})$ with Q' replaced, respectively, by $(Q_1Q_3Q_2Q_4)$ and $(Q_1Q_2Q_4Q_3)$. Applying (14)–(16), we find the four-particle condition (13) is also satisfied. The general case for any four scattering particles among the total N particles can be directly generalized from the results we have obtained.

The configurations with more than four particles at one point are excluded by the Pauli principle (extended to four components), the antisymmetric wave function vanishes in such cases. Thus we have covered all kinds of occupancies and explicitly proven the consistence between the BA and the multiparticle scattering boundary conditions.

V. NARROW-SITES LIMITING IN THE DEGENERATE HUBBARD MODEL

In the above sections we have proven explicitly the consistence between the multiparticle scattering and the BA solution in spatially continuous case. Now let us turn to the lattice case and investigate the limiting process of small site spacing in the Hubbard model. As one knows the Hubbard model, in the limit of small site distance d, approaches to the model of electrons with δ-function interaction which is spatially continuous. The single-band Hubbard model was solved by the SU(2) BA.[5] But the direct generalization of SU(2) BA solution of the single-band Hubbard model to the degenerate SU(N) case for the degenerate Hubbard model fails due to the multiparticle same-site interacting case. In Ref. 10 it is found that there appears to be an unhermition extra interacting term for three bosons occupying the same site in the BA wave function which was proposed with an attempt to solve the boson Hubbard model. This unhermition term extra to the Hubbard on-site interaction $3U$, with the form of

$$\frac{U^2/t}{\cos(k_1+k_2)\cos(k_2+k_3)\cos(k_3+k_1)}, \tag{17}$$

where U is the Hubbard on-site interaction and t is the hopping constant and k_1, k_2, and k_3 are momenta of the interacting bosons, breaks the validity of the BA for boson Hubbard model. The similar thing will happen for degenerate electrons or fermions and the inconsistent extra term has the same form as (17) except a sign difference. According to our explicit proof of consistence between the BA and the multiparticle interacting for δ-function interacting fermions, the inconsistent term should come to vanish in the limit process of site distance going to zero, since the Hubbard model approaches to the model of electrons with δ-function interaction. Let us pick up the site distance d which is usually taken to be unit, and the S-matrix for the Hubbard model is

$$S^{ij}=\frac{(\sin k_s d-\sin k_t d)I^{ij}+iU/2tP^{ij}}{(\sin k_s d-\sin k_t d)+iU/2t} \tag{18}$$

where the notations of P_s, P_t, and P^{ij} are the same as in (6) and $\sin k_s d$ is originated from the discreteness in space coordinate for lattice sites. The above S-matrix becomes

$$\frac{(k_s d-k_t d)I^{ij}+iU/2tP^{ij}}{(k_s d-k_t d)+iU/2t}=\frac{(k_s-k_t)I^{ij}+iU/2tdP^{ij}}{(k_s-k_t)+iU/2td} \tag{19}$$

for small d.

One may mention the large momentum case in which $k_j d$ may not be small. This is true indeed if we have definite total site number. But the problem is naturally solved when we take into account the increasing site number in the limiting process. In fact, the one-dimensional space of the fermions should have a finite length L. So the total site number $N_s=L/d$ should increase with the decreasing d in the limiting process. The momenta are inversely proportional to the site number and site spacing

$$k_j\propto\frac{2\pi}{N_s d}$$

and

$$k_j d \propto \frac{2\pi}{N_s} = \frac{2\pi d}{L} \to 0, \tag{20}$$

where the space length L is definite constant. Thus we still have the limiting result (19).

Besides L, another important parameter

$$c = U/2td$$

should remain to be constant as $d \to 0$ and the S-matrix (19) becomes the same as the S-matrix (6) of δ-function interacting fermions. So we have

$$\frac{U}{t} = 2cd, \tag{21}$$

which means that U/t is proportional to the decreasing site distance d. Therefore the extra term (17) will vanish with the d and inconsistency arising in the lattice case diminishes and disappears in the limiting process to spatially continuous case.

VI. SU(4) SYMMETRY OF THE SYSTEM

Since we have proven that the BA is exactly valid for all occupancies in the four-component fermions with δ-function interactions, we would like to mention an interesting correspondence between the BA equations and system symmetry and then consider the degeneracy of the eigenstates. The system of the four-component fermions, with the second-quantized Hamiltonian

$$H = -\frac{1}{2}\int dx \sum_a \psi_a^*(x) \frac{\partial^2}{\partial x^2} \psi_a(x) + \frac{c}{2}\int dx \sum_{a\neq a'} n_a(x) n_{a'}(x),$$

where $\psi_a^*(x)$ creates a fermion with internal component a at position x and $n_a(x) = \psi_a^*(x)\psi_a(x)$ is the particle number operator, possesses the SU(4) symmetry

$$\mathrm{SU}(4): \{D_m, \quad E_{aa'} | m = 1,2,3; \ a,a' = 1,2,3,4, \ a \neq a'\}, \tag{22}$$

where the generators are

$$E_{aa'} = \int dx\, \psi_a^*(x)\psi_{a'}(x),$$

$$D_m = N_m - N_{m+1}, \quad N_m = \int dx\, \psi_m^*(x)\psi_{m'}(x),$$

and a,m label the inner components. The commutation relations are

$$\begin{aligned}[E_{ss'}, E_{tt'}] &= \delta_{s',t} E_{st'} - \delta_{s,t'} E_{ts'}, \\ [D_m, E_{ss'}] &= (\delta_{m,s} - \delta_{m,s'} - \delta_{m+1,s} + \delta_{m+1,s'}) E_{ss'}\end{aligned} \tag{23}$$

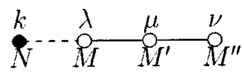
and $\{D_m\}$ forms the commuting Cardan subalgebra of rank 3. As the system is spatially continuous instead of lattice, the symmetry involving pairing creation and annihilation operators as in the simple Hubbard model[17] and spin–orbital case[18,19] does not exist.

The BA equations (7) rewritten as

$$e^{ik_j L} = \prod_{\alpha=1}^{M} \Xi_{1/2}(k_j - \lambda_\alpha),$$

$$
\begin{aligned}
1&=-\prod_{j=1}^{N}\Xi_{-1/2}(\lambda_\alpha-k_j)\prod_{\alpha'=1}^{M}\Xi_1(\lambda_\alpha-\lambda_{\alpha'})\prod_{\beta=1}^{M'}\Xi_{-1/2}(\lambda_\alpha-\mu_\beta),\\
1&=-\prod_{\alpha=1}^{M}\Xi_{-1/2}(\mu_\beta-\lambda_\alpha)\prod_{\beta'=1}^{M'}\Xi_1(\mu_\beta-\mu_{\beta'})\prod_{\gamma=1}^{M''}\Xi_{-1/2}(\mu_\beta-\nu_\gamma),\\
1&=-\prod_{\beta=1}^{M'}\Xi_{-1/2}(\nu_\gamma-\mu_\beta)\prod_{\gamma'=1}^{M''}\Xi_{-1/2}(\nu_\gamma-\nu_{\gamma'}),
\end{aligned}
\tag{24}
$$

where $\Xi_n(x)=(x+inc)/(x-inc)$ is easy to remember by means of the Dynkin diagram of A_3 Lie algebra



where the dark dot is added to represent the charge rapidity k_j which takes an angle of 120° relative to the first simple root r_1. The subscripts of $\Xi_n(x)$ in the above form of BA equations correspond, respectively, to the covariant components of the simple roots which are chosen as nonorthogonal basis, $r_1=(-1/2, 1, -1/2, 0)$, $r_2=(0, -1/2, 1, -1/2)$, $r_3=(0, 0, -1/2, 1)$. This connection exists because the system has the above SU(4) symmetry, the generators of which constitute an A_3 Lie algebra. Such a connection was noticed for correlated electrons with twofold orbital degeneracy in SU(4) lattice case[14] and the electrons with spin–exchange interactions in $\mathrm{SU}(2)\times\mathrm{SU}(2)$ case.[20]

The BA eigenstates are the SU(4) highest weight state. Let us assume the highest weight is $w^h=(w_1^h, w_3^h, w_3^h)$ which labels an irreducible representation of the SU(4) group. We obtain the expression

$$
\begin{aligned}
w_1^h&=N_1-N_2=N-2M+M',\\
w_2^h&=N_2-N_3=M-2M'+M'',\\
w_3^h&=N_3-N_4=M'-2M'',
\end{aligned}
\tag{25}
$$

from the Cartan operators and the relations (8) of N_m and M, M', M''. The lower weight states in the SU(4) representation can be obtained by the lowering operators $E_{-\alpha}$ acting on the highest weight state $|w^h\rangle=|\psi_{\mathrm{BA}}\rangle$,

$$
|w^h-\alpha\rangle=E_{-\alpha}|w^h\rangle,
\tag{26}
$$

where the lowering operator $E_{-\alpha}$ is a product of the SU(4) generators $E_{ss'}$ which commute with the Hamiltonian. The obtained lower-weight states are also the eigenstates of the system

$$
H|w^h-\alpha\rangle=E_{-\alpha}H|w^h\rangle=E_{-\alpha}E_{\mathrm{BA}}|w^h\rangle=E_{\mathrm{BA}}|w^h-\alpha\rangle
\tag{27}
$$

with the same eigenenergy E_{BA} as the BA solution. Therefore the state with energy E_{BA} is degenerate with degeneracy

$$
n_{w^h}=\tfrac{1}{12}(w_1^h+1)(w_2^h+1)(w_3^h+1)(w_1^h+w_2^h+2)(w_2^h+w_3^h+2)(w_1^h+w_2^h+w_3^h+3)
\tag{28}
$$

which corresponds to the total weight number and dimension of the SU(4) representation. The only possible nondegenerate case is

$$
w_1^h=w_2^h=w_3^h=0
\tag{29}
$$

which requires

$$M=\frac{3N}{4},\quad M'=\frac{N}{2},\quad M''=\frac{N}{4},$$

$$N_1=N_2=N_3=N_4=\frac{N}{4}, \tag{30}$$

and the total particle number N should be the integer times 4, i.e., there are an equal number of particles in each state $a=1,2,3,4$. If the condition (30) is not satisfied, all the states will be degenerate including the BA ground state.

VII. BRIEF SUMMERY

In this paper, we have worked out the boundary conditions of three and four particles interacting at the same spatial point for the δ-function interacting SU(4) fermions and shown that they can be solved by the two-particle connection boundary conditions in the BA. Therefore we have explicitly proven the consistence of the BA with the multiparticle scattering boundary conditions and that the BA is valid for all multiple occupancies. Unlike the lattice degenerate Hubbard model, the definition of the BA wave function for the spatially continuous system can be extended to those cases with more than two particles occupying the same positions. We also show that the inconsistence of BA in the degenerate Hubbard model vanishes in the limit of small-site spacing. An interesting correspondence between the BA equations and the system SU(4) symmetry is indicated. The degeneracy is generally given for the state with BA eigenenergy. Singlet exists in the case that there are equal number of particles in each inner component. Our proof for consistence of the BA and the multiparticle scattering is also valid for the δ-function interacting bosons.[2,3] In that case one needs to show the consistency for all the multiple occupancies.

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