Physics of 2D Cold Bose Gas

based mainly on

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• Role of dimensionality in phase transition and the type of order present (density of states, dynamics)

• Robustness of order in high dimension vs thermal and quantum fluctuation in low dimensions

• 2D is marginal (e.g. quasi-long range order, algebraic instead of exponential decay of the first order correlation function $g_1(\vec{r}) = \langle \psi^+(\vec{r})\psi(0) \rangle$)

• Mermin-Wagner theorem: LRO is impossible in any non-zero temperature in 1D and 2D system with short range interaction with a continuous Hamiltonian symmetry. LRO is associated with the spontaneously breaking of a continuous symmetry. The LRO is destructed by long wave length thermal fluctuations in low dimensions and the symmetry is restored.
• I Infinite Uniform Cold Bose Gas

• Density of states

\[ D(\varepsilon) = \frac{V}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \varepsilon^{1/2} \]

2D \[ D(\varepsilon) = mL^2 \left/ \left( 2\pi \hbar^2 \right) \right. \text{ energy independent} \]

* Chemical potential of classical ideal gas \( n \ll n_Q \)

\[ \mu = k_B T \ln \left( \frac{n}{n_Q} \right), \quad n_Q = \left( \frac{mk_B T}{2\pi \hbar^2} \right)^{3/2} \]

the quantum concentration

It approaches zero when \( T \) is decreased \( \mu \approx -kT / N \) and below BE temperature it is fixed at 0.

Fugacity \( Z = e^{\beta \mu}, \beta = 1 / k_B T \)

The Bose-Einstein distribution \( \bar{n} = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} = \frac{1}{e^{\beta\varepsilon} / Z - 1} \)

\( \sum_l \bar{n}(\varepsilon_l) = N \) determines \( \mu \)
1. The ideal gas

- The phase space density (dimensionless): the capacity of containing particles

$$3D \quad D = n_3 \lambda^3, \quad \lambda = \left( \frac{2\pi \hbar}{mk_B T} \right)^{1/2} = n_Q^{-1/3}$$

Thermal de Broglie wave length

In 3D \( D=2.612 \) for \( Z=1 \). For a definite \( T \), the density has an upper limit (the critical density) BEC takes place. Condition for BEC: saturation of single particle excited states at a critical temperature

$$2D \quad D = n_2 \lambda^2$$

- Relation between \( Z \) and \( D \) In absence of condensation

$$N = \left( \frac{mL^2}{2\pi \hbar^2} \right) \int_0^\infty \frac{d\varepsilon}{e^{\beta (\varepsilon - \mu)} - 1}, \quad D = n_2 \lambda^2 = \int_0^\infty \frac{dx}{e^x / Z - 1} = -\ln(1-Z)$$

$$Z = 1 - e^{-n_2 \lambda^2}$$
• We will study the 2D system as
• The homogeneous ideal gas and interacting gas, in which at sufficiently low temperature the BKT transition leads to quasi-LRO and superfluidity.
• The finite system (box, harmonic trap) of ideal gas and interacting gas. In both cases subtle interplay of BKT transition and Bose-Einstein condensation happens.
In 2D the relation can be satisfied at any finite temperature, hence the density in unlimited. BEC does not occur at finite temperature.

- Distribution of Momentum States and the first order correlation function

\[ n_k^- = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}, \varepsilon_k = \frac{\hbar^2 k^2}{2m} \]

\[ g_1(\vec{r}) = \langle \psi^* (\vec{r}) \psi(0) \rangle = \frac{1}{(2\pi)^2} \int_0^\infty e^{i\vec{k} \cdot \vec{r}} n_k d^2 k \]

Fourier transform of the momentum distribution

Non-degenerate and degenerate regimes have different momentum distributions and correlation functions.

\[ \text{Re} : \sum_{k,k'} \langle a_k^\dagger e^{i\vec{k} \cdot \vec{r}} a_{k'} \rangle = \sum_k n_k e^{i\vec{k} \cdot \vec{r}} \]
• The non-degenerate regime

\[ Z = 1 - e^{-n_2 \lambda^2} \]

\[ D = n \lambda^2 \ll 1 \Rightarrow Z \approx n \lambda^2 \ll 1, \left| \mu \right| / k_B T \gg 1 \]

In this regime all states are weakly populated

\[ n_k \approx Z e^{-\beta \varepsilon_k} \approx n \lambda^2 e^{-k^2 \lambda^2 / 4\pi}, \quad \beta \varepsilon_k = k^2 \lambda^2 / 4\pi \]

and the correlation function is Gaussian

\[ g_1(r) \approx n e^{-\pi r^2 / \lambda^2} \]

with a correlation length \( \xi = \lambda / \sqrt{\pi} \)

. The degenerate regime

\[ D > 1, \quad Z \approx 1, \quad \left| \mu \right| / k_B T \ll 1 \]

In this regime the momentum distribution is bimodal, depending on the energy of states
• For the high energy states

\[ \beta \varepsilon_k = \lambda^2 k^2 / 4\pi \gg 1, \quad k^2 \gg 4\pi / \lambda^2 \]

\[ Z \approx 1, \quad n_k \approx e^{-\lambda^2 k^2 / 4\pi} \ll 1 \text{ Gaussian} \]

The states are weakly populated.

. For the low energy states

\[ \beta \varepsilon_k = \lambda^2 k^2 / 4\pi \ll 1, \quad k^2 \ll 4\pi / \lambda^2 \]

\[ e^{\beta(\varepsilon_k - \mu)} = e^{\beta(\varepsilon_k + |\mu|)} \approx 1 + \beta(\varepsilon_k + |\mu|) \]

\[ n_k = \frac{k_B T}{\varepsilon_k + |\mu|} = \frac{4\pi}{\lambda^2} \frac{1}{k^2 + k_c^2}, \quad k_c = \sqrt{2m\mu / \hbar} \]

Lorentz form. States are strongly populated.
• The correlation function is bimodal. At short distance (high energy states important) $r$ up to $\lambda$

$g_1(r)$ is still Gaussian. The Lorentzian form corresponds to exponential decay for $r >> \lambda$

$g_1(r) \approx e^{-r/l}$, with a correlation length $l = k_c^{-1} \approx \frac{\lambda}{\sqrt{4\pi}} e^{n\lambda^2/2}$

Actually the Lorentz mode is much more strongly populated than the Gaussian mode.

2. Interaction in 2D Bose gas at low temperatures

\[ V(\vec{r}_i - \vec{r}_j) = \frac{4\pi\hbar^2}{m} a_s \delta^3(\vec{r}_i - \vec{r}_j), \quad E_{\text{int}} = \frac{g^{3D}}{2} \int n^2(\vec{r})d^3r, \quad g^{3D} = \frac{4\pi\hbar^2}{m} a_s \]
Real 2D scattering amplitude is logarithmically energy dependent

- Quasi 2D: \( z \)-degree of freedom is frozen, but the scattering is actually 3D: \( a_z = \sqrt{\hbar / m \omega_z} \gg a_s \)

We attempt to have the interaction strength for 2D case independent of energy, as in the 3D case:

\[
E_{\text{int}} = \frac{g}{2} \int n^2(\vec{r})d^2r, \quad n(\vec{r}) \text{ 2D local density}, \quad n_3 = n_2 / a_z
\]

based on dimensional analysis, \( g = \frac{\hbar^2}{m} \tilde{g}, \quad \tilde{g} \) dimensionless

Equating to the 3D interaction energy, we obtain

\( \tilde{g} = \frac{2\pi a_s}{a_z} \). Actually \( \tilde{g} = \sqrt{8\pi a_s / a_z} \) (Petrov et al)

2D healing length \( \xi = \hbar / \sqrt{mgn} = 1 / \sqrt{\tilde{g} n} \)
• Strong interaction limit defined:
• $E_{\text{int}}$ of $N$ particles = $E_{\text{kin}}$ of $N$ non-interacting particles distributed equally over lowest $N$ single particle levels: the interaction is strong enough to keep the particles residing on different states

2D DOS: $D(\varepsilon) = mL^2 / 2\pi\hbar^2$ Let $E_N$ be the energy of the $N^{th}$ single particle state, then $D = N / E_N \Rightarrow E_N = 2\pi\hbar^2 n / m$
$E_{\text{kin}} = NE_N / 2 = \pi\hbar^2 Nn / m$ equated to $E_{\text{int}} = \hbar^2 \tilde{g} Nn / m$

strong interaction limit: $\tilde{g} = 2\pi$

This limit is independent of density, as contrasted to 3D where the dimensionless measure is $n_3a_s^3$
3. Suppression of density fluctuations and the low-energy Hamiltonian

We assume that the weakly interacting 2d Bose gas at \( T = 0 \) is described by \( \psi = \sqrt{n} e^{i\theta} \), where \( n \) and \( \theta \) are classical fields. At finite \( T \) assuming that the system is enclosed in a box of length \( L \) such that the wave function description is still valid, both number and phase fluctuation exist. We argue that for repulsive interaction at sufficiently low temperature the number fluctuation is suppressed, because it is tied up with the interaction energy.

\[
\frac{g}{2} \int n^2(\vec{r}) d^2r = \frac{gL^2}{2} \langle n^2(\vec{r}) \rangle
\]

The number fluctuation is

\[
(\Delta n)^2 = \langle n^2(\vec{r}) \rangle - n^2 = (g_2(0) - 1)n^2
\]

where \( g_2(\vec{r}) = \langle n(\vec{r})n(0) \rangle / n^2 \) is the density-density correlation function (normalized).

At fixed density \( n \), minimizing the interaction is equivalent to minimizing the number fluctuation.
In ideal Bose gas $g(0)=2$ (bunching) and in absence of correlation $g(0)=1$. We can estimate the minimum cost of density fluctuation by equating it to the interaction energy increase by adding a single particle. Setting $g(0)=1$, assuming the suppression of the fluctuation. we have 

$$E_{\text{int}} = gn^2 L^2 / 2 = gN^2 / 2L^2.$$ 

Comparing with the thermal energy of the particle, we get 

$$\frac{gn}{k_B T} = \frac{\tilde{g}}{2\pi} D.$$ 

At sufficiently low temperatures given by $k_B T \ll gn$ or $D \gg 2\pi / \tilde{g}$ the thermal energy is far less than necessary to cause a density fluctuation and
the density fluctuation is strongly suppressed. In numerical calculations it turns out that the number fluctuation is already suppressed for $D > 1$. In this limit the interaction energy becomes a constant $g n^2 L^2 / 2$.

The kinetic energy arises from the phase $\theta$

$$H = \frac{\hbar^2}{2m} \int |\nabla \psi|^2 \, d^2r \Rightarrow H_{\theta} = \frac{\hbar^2}{2m} n_s \int (\nabla \theta)^2 \, d^2r,$$

This gives a good description of long range physics for $r \gg \xi, \lambda$, albeit phenomenological in nature

superfluid velocity $\bar{v} = \frac{\hbar}{m} \nabla \theta$

kinetic energy $= \int \frac{1}{2} m \left( \frac{\hbar}{m} \nabla \theta \right)^2 n_s \, d^2r$
An exact formulation is given by the Bogoliubov analysis,
\[ \hat{\psi} = \psi_0 \text{ (the condensate)} + \delta\psi \text{ (the Bogoliubov modes)} \]
aiming at the justification of the density fluctuation suppression and the superfluidity of the state.

The classical field Hamiltonian

\[ H = \frac{\hbar^2}{2m} \int (\nabla \psi^*(r))(\nabla \psi(r)) \, d^2r + \frac{g}{2} \int (\psi^*(r))^2(\psi(r))^2 \, d^2r. \]

Treating \( \psi \) and \( \psi^* \) as canonical variables, one can derive the G.P. equation. Here we use the order parameter, which seems absent in the 2D case. We can consider the system as finite in size, where the decay of correlation function is still not significant, and deal with \( \psi \) and \( \psi^* \) as order parameters. The applicability of the Bogoliubov approach is justified by Castin Phys. Rev. A 67, 053615, 2007.
\[ \psi(\vec{r}, t) = |\psi(\vec{r}, t)| e^{i\theta(\vec{r}, t)}, \]
\[ |\psi(\vec{r}, t)| = \sqrt{n(1 + \eta(\vec{r}, t))}, \eta \ll 1 \text{ real} \]
\[ |\psi(\vec{r}, t)|^2 = n(1 + 2\eta(\vec{r}, t)). \int \eta \, d^2r = 0. \]
\[ H = \frac{\hbar^2}{2m} n \int (\nabla \theta(r))^2 \, d^2r + \int \left[ \frac{\hbar^2}{2m} n (\nabla \eta(r))^2 + 2gn^2(\eta(r))^2 \right] \, d^2r, \]

\( H \) leads to equations of motion for the phase and density fluctuation. The canonical variables are now \( \eta \) and \( \theta \) (both real)

\[ \theta(r, t) = \sum_{k} c_k(t) e^{ik \cdot r}, \quad \eta(r, t) = \sum_{k} d_k(t) e^{ik \cdot r}, \]

From the Hamiltonian for the field one obtains \( c_k^* = c_{-k} \), \( d_k^* = d_{-k} \), \( d_0 = 0 \)

\[ \dot{c}_k = - \left( \frac{\hbar^2 k^2}{2m} + \frac{2gn}{\hbar} \right) d_k, \quad \dot{d}_k = \frac{\hbar k^2}{2m} c_k \text{ (for } k \neq 0): \]

\( c \) is the “momentum” \quad \( d \) is the “position”
Bogoliubov modes are simple harmonic oscillator states with eigenfrequencies

$$\omega_k = \sqrt{\frac{\hbar k^2}{2m} \left( \frac{\hbar k^2}{2m} + \frac{2gn}{\hbar} \right)},$$

1. At low k the eigenmodes are phonons with \( \omega_k = ck, c = \sqrt{gn / m} \)

Superfluid behavior follows from the order parameter instead of the LRO. At high k we have free particle modes

$$\hbar \omega_k = \frac{\hbar^2 k^2}{2m} + gn$$

Cross-over happens at \( k \sim 1/ \xi = \sqrt{\tilde{g}n} \)

2. The relative importance of phase and density fluctuations can be calculated by using the virial theorem for harmonic oscillators

$$\left\langle \frac{m \omega^2 x^2}{2} \right\rangle = \left\langle \frac{p^2}{2m} \right\rangle,$$

which leads to

$$\frac{\langle |d_k|^2 \rangle}{\langle |c_k|^2 \rangle} = \frac{\frac{\hbar^2 k^2}{2m}}{\frac{\hbar^2 k^2}{2m} + 2gn}.$$

We conclude: long wave length phonons involve only phase fluctuations while high momentum phonons involve both density and phase fluctuations in equal weight.
3. Density fluctuations are not “soft” Goldstone modes, and therefore do not lead to divergent effects, while the phase fluctuations do not cost energy for $k \to 0$. Density of states at low $k$ leads to divergent effect destructing the LRO.

4. Temperature dependence of density fluctuation

$$n^2(\vec{r}, t) = n^2(1 + 2\eta(\vec{r}, t))^2, \Delta n^2 \equiv \mathbf{\langle n^2(\vec{r}, t) \rangle} - n^2, \frac{\Delta n^2}{n^2} = 4\mathbf{\langle \eta^2 \rangle}$$

To calculate, we simplify the Bose-Einstein distribution by using the **equipartition law** that in thermal equilibrium each degree of freedom has an energy $k_B T / 2$

$$nL^2 \left( \frac{\hbar^2 k^2}{2m} + 2gn \right) \langle |d_k|^2 \rangle = \begin{cases} \frac{k_B T}{2} & \text{if } \hbar \omega_k < k_B T, \\ 0 & \text{if } \hbar \omega_k > k_B T. \end{cases}$$
Here we introduced a cut-off, (otherwise divergence would arise). Physically this is justified that higher modes are rarely populated. We obtain

\[ \frac{\Delta n^2}{n^2} = 4 \sum_k \langle|d_k|^2\rangle \approx \frac{L^2}{4\pi^2} \int_{k<k_T} \frac{4}{nL^2} \frac{k_B T/2}{h^2 k^2/2m + 2gn} d^2k \]

\[ \frac{\Delta n^2}{n^2} \approx \frac{2}{n\lambda^2} \ln \left( \frac{k_B T}{2gn} \right) \]

For realistic values of \( T \) and \( g \), we conclude that the density fluctuation is suppressed at \( n\lambda^2 >> 1 \).

4. Algebraic decay of correlation function

We study the correlation function at low temperature for \( r >> \xi, \lambda \) (density fluctuation neglected) which depends essentially on the phonon modes \( k << 1/\xi \) \( \psi(r) = \sqrt{n_s} e^{i\theta(r)}. \)

For low \( k \) \( \hbar \omega_k << k_B T \) the B-E distribution can be simplified

\[ \langle \exp(\beta\hbar \omega_k) - 1 \rangle^{-1} \rightarrow \frac{k_B T}{(\hbar \omega_k)} \]

i.e. equipartition:

\[ n_s L^2 \frac{\hbar^2 k^2}{2m} \langle|c_k|^2\rangle = \frac{k_B T}{2} \]
$c_k = c_k' + ic_k''$, $c_k'$ and $c_k''$ are independently fluctuating variables

$$\langle |c_k'|^2 \rangle = \langle |c_k''|^2 \rangle = \frac{\pi}{n_s \lambda^2 L^2 k^2}.$$

$$\langle c_k' c_k'' \rangle = 0.$$

$$g_1(r) = \langle \psi^*(r) \psi(0) \rangle = n_s \left\langle e^{i(\theta(r) - \theta(0))} \right\rangle$$

$$\theta(r) - \theta(0) = \sum c_k' (\cos(k \cdot r) - 1) - c_k'' \sin(k \cdot r)$$

$$\theta(\vec{r}) = \sum c_k e^{ik \cdot \vec{r}}$$

For each independent Gaussian variable $u$, $\langle e^{iu} \rangle = e^{-\frac{1}{2} \langle u^2 \rangle}$.

$$g_1(r) = n_s \exp \left(-\frac{1}{2\pi n_s \lambda^2} \int \frac{1 - \cos(k \cdot r)}{k^2} d^2k \right).$$

The integral in the exponent has significant contributions only from modes $k > 1/r$ so that $1 - \cos(k \cdot r) \sim 1$. $k_{\text{max}} = 1/\xi$.

$$\int_{1/r}^{1/\xi} \frac{d^2k}{k^2} = 2\pi \ln \frac{r}{\xi}$$

$g_1(r) = n_s \left( \frac{\xi}{r} \right)^{1/(n_s \lambda^2)}$

Mermin-Wagner Theorem

More exactly

$$\nabla^2 \int (1 - \cos(k \cdot r)) k^{-2} d^2k = (2\pi)^2 \delta(r)$$

2D Poisson eq. of a point charge

$$\int \frac{1 - \cos(k \cdot r)}{k^2} d^2k = 2\pi \ln \left( \frac{r}{\xi} \right)$$
The BKT phase transition

A simple picture: the vortices drive the BKT phase transition. We begin from a vortex in the superfluid.

Vortex energy
\[ E = \int_\varepsilon^R \frac{1}{2} n_s \left( \frac{\hbar}{mr} \right)^2 d^2 r = \frac{\hbar^2 \pi}{m} n_s \ln \left( \frac{R}{\xi} \right) \]

Entropy
\[ S = k_B \ln \left( \frac{R^2 \pi}{\xi^2 \pi} \right) = 2k_B \ln \left( \frac{R}{\xi} \right) \]

Free energy
\[ F = \frac{k_B T}{2} \left( n_s \lambda^2 - 4 \right) \ln \frac{R}{\xi} \]

Critical point \( n_s \lambda^2 = 4 \)

For \( n_s \lambda^2 > 4 \), \( F \) is large and positive, the superfluid is stable against proliferation of vortices.

For \( n_s \lambda^2 < 4 \), \( F \) is negative, and the superfluid becomes unstable.
In 2D superfluid density cannot have any value between $4/\lambda^2$ and 0. Under the BKT transition vortex pairs play the dominant role. Energy of pair is finite, so that the free energy is always negative. At any finite $T$ pairs can be continuously created and annihilated by thermal fluctuations.

At the critical temperature $T_{KT}$, $n_s \lambda^2 = n_s 2\pi / mT_{KT} = 4$

$$\rho_s = mn_s = T_{KT} \frac{2m}{\pi^2}$$ universal jump (Nelson-Kosterlitz)
As $T$ is increased, the density of pairs grows, and the size of pairs also grows. As $T$ approaches the BKT temperature, pairs begin to dissociate, and free vortices proliferates eventually destructing the superfluidity. The universal jump result is elegant, it is independent of the interaction strength. But it does predict the value of $T_{KT}$.

The microscopic theory (Prokofiev, Ruebenbackner, Svistunov) gives

$$D_c = \left( n \lambda^2 \right)_c = \ln \frac{C}{\tilde{g}}, \quad C \approx 380$$

This relation holds for weak interaction. We can set one obvious bound to its validity: Since $n \geq n_s$, we can set $D_c > 4$, which leads to $\tilde{g} \leq 7$. 
It also gives that when $T$ is approached from above the correlation length is

$$l_c = \lambda \exp\left(\frac{\sqrt{aT_{BKT}}}{\sqrt{T - T_{BKT}}}\right) \quad a \approx 1$$

which diverges logarithmically when $T$ decreases toward $T_{KT}$ while 2\textsuperscript{nd} order transition in 3D has coherent lengths diverging in a polynomial way. The critical region where $l$ becomes larger than $\lambda$ as a precursor of the BKT transition is therefore very broad in the case of finite size systems.

Diverging correlation length is typical in phase transitions. When the system has a finite size, certain fraction of condensate can appear when $l$ becomes larger than the size.
Critical behavior of 3D trapped interacting Bose gas  Esslinger group ETH Science 315, 1556, 2007

Correlation function for $T \gg T_c$: Gaussian, with correlation $\sim$ de Broglie wave length closer to $T_c$, correlation length $> \text{de Broglie wave length}$, $g_s(r) \sim \frac{1}{r} e^{-\frac{r}{\xi}}$

At $T_c$ the correlation diverges, the correlation function becomes algebraic $g_s(r) \sim \frac{1}{r}$

\[ \xi \propto \left| \frac{T_c}{T - T_c} \right|^\nu \quad \nu — \text{critical exponent} \]
• III 2D Bose Gas in a Finite Box

Finite size effects can play a significant role in phase transition, especially in 2D case, in which thermodynamical limit can be achieved only in marginal cases$
\sim \ln R / \xi$

1. Ideal gas   There are 3 regimes

Non-degenerate regime  $D \ll 1$,  \quad $g_1(r) = n e^{-2 \pi r^2 / \lambda^2}$,  \quad for $\lambda \ll L$

Degenerate regime  for $l = \frac{\lambda}{\sqrt{4 \pi}} e^{D/2} < L$,  \quad i.e. $D < \ln \frac{4 \pi L^2}{\lambda^2}$

\quad the finite size has no appreciable effect  \quad $g_1(r) = n e^{-r/l}$

Condensate regime for $D \geq \ln \frac{4 \pi L^2}{\lambda^2}$  \quad i.e. $l > L$  \quad significant phase coherence exists between any two points in the gas
2. Interacting gas  The finite size effect and the effect of interaction co-exist and compete
We study the case with repulsive interactions to see what can be expected in the vicinity of BKT transition. The size of the sample is $L$, and we assume that the point of BKT transition is well before BEC $D_c \ll \ln \frac{4\pi L^2}{\lambda^2}$. When $D$ approaches $D_c$, BKT transition occurs, and $g_1(r)$ acquires an algebraically long range order $g_1(r) \approx n_s (\xi / r)^{\alpha}$, and $\alpha = 1/n_s \lambda^2 \leq 4$.

In sharp contrast with the infinite sample case, the BKT transition is accompanied by a significant fraction of condensate fraction. This fraction is associated with the largest eigenvalue $\Pi_0$ of the single particle density matrix

$$g_1(\vec{r}) = N \sum_j \Pi_j \phi_j^*(0) \phi_j(\vec{r})$$

The condensate fraction $\Pi_0$ is associated with the state $\phi_0 = 1/L$. Integrating the LHS over a circle of radius $L/\pi$ we get
\[
\int g_1(r) \, d^2r = 2\pi n_s \int_\xi^R (\xi/r)^\alpha r \, dr \simeq L^2 \frac{2}{2-\alpha} \pi^{\alpha/2} g_1(L).
\]

and we integrate the RHS over a square of side L. For \( j=0 \), \( \phi_0(0) = \phi_0(L) = 1/L \), and non-zero \( j \) states are integrated to 0. The result is then \( g_1(L)/n \sim \Pi_0 \).

For cold atom systems, the typical values of \( \lambda \) and \( \xi \) are in the range 0.1 to 1 micrometer, whereas the maximal size of the gas is \( L \sim 100 \, \mu\text{m} \). Once the algebraic decay regime is reached, we have \( g_1(L)/n_s \sim (10^{-3})^{1/4} \) to \( (10^{-2})^{1/4} \sim 0.2 \) to 0.3, hence a significant condensed fraction (since \( n \) and \( n_s \) are comparable).

In order to observe a BKT transition with no significant BEC, one would need to consider unrealistically large samples.

“With a magnetization at the BKT critical point smaller than 0.01 as a reasonable estimate for the thermodynamic limit, the sample would need to be bigger than the state of Texas for the Mermin–Wagner theorem to be relevant!”
Width of the critical region and crossover

\[ \ln \frac{l}{\lambda} = \left( \frac{aT_{BKT}}{(T - T_{BKT})} \right)^{1/2} \]

\[ \frac{\Delta T}{T_{BKT}} = \frac{\Delta D}{D_c} \sim \frac{a}{(\ln(L/\lambda))^2} \]

\( a \approx 1 \), width of critical region \( \sim T_{BKT} \)

for \( L / \lambda \approx 10 - 100 \), \( \Delta T / T_{BKT} \approx 20\% - 5\% \)

* What comes first: BEC or BKT?

A subtle question

Large system \( D_c = \ln 380 / \tilde{g} \ll \ln 4\pi L^2 / \lambda^2 \) We can either increase \( D \) under constant \( \tilde{g} \) toward \( D_c \) or increase \( \tilde{g} \) under constant \( D \) so that \( D_c \) decreases toward \( D \), the first relevant mechanism is BKT. Appreciable condensed fraction appears when BKT is approached. The first case is called the BKT
driven condensation, and the second case is called the interaction enhanced. In the square box, these variants are equivalent.

* Small system \( D_c > \ln \frac{4\pi L^2}{\lambda^2} \)

When \( D \) is increased, it reaches the BEC point first, the condensation is conventional.

In the intermediate cases it is hard to distinguish the two regimes experimentally, even if the criteria are separated by more than the crossover region.
• IV 2D Bose Gas in a Harmonic Trap

We find dramatic change of the properties of the system through the change of density of states.

1. The ideal case

\[ V(r) = m\omega^2 r^2 / 2; \quad E_j = (j + 1) \hbar \omega \]

Levels with quantum number \( j \) have \( j+1 \) fold degeneracy: \( j \) is distributed among the two dimensions: \( j = n_x + n_y \) hence the degeneracy.

Putting \( \mu \) equal to the energy of the ground state, we get the number of atoms on all excited states

\[ N_c^{(id)}(T) = \sum_{j=1}^{+\infty} \frac{g_j}{e^{\zeta j} - 1} , \quad \zeta = \hbar \omega / k_B T \]
• For $\xi << 1$ the sum is replaced by integral, and
\[
\frac{1}{\xi^2} \int_0^\infty x dx e^x - 1 = \frac{1}{\xi^2} \frac{\pi^2}{6}
\]
\[
N_c^{(id)}(T) \approx \frac{\pi^2}{6} \left( \frac{k_BT}{\hbar\omega} \right)^2
\]
For $N > N_c^{id}$ atoms begin to accumulate on the ground state
\[
k_B T_c = \frac{\sqrt{6}}{\pi} \hbar \omega \sqrt{N}
\]
These results can be obtained from the ideal case by using the local density approximation
\[
n = -\lambda^{-2} \ln(1 - e^{\beta\mu}) \Rightarrow N = -\lambda^{-2} \int \ln(1 - e^{\beta(\mu - V(r))}) 2\pi r dr
\]
\[
\mu \to \mu - V(r)
\]
\[
R = r/r_T \text{ with } r_T^2 = k_B T/m\omega^2.
\]
For $\mu = 0 (Z = 1), \int = -\pi^2 / 6$ the above result is recovered.
In the ideal gas case,
\[
\mu = 0 \to n(r = 0) \text{ diverges in BEC transition}
\]
2. Interacting case, LDA

In mean field Hartree-Fock theory the interaction energy when no condensate exists is $2gn$ (direct + exchange),

$$D(r) = -\ln \{1 - Z \exp[-\beta V(r) - \tilde{g}D(r)/\pi]\}$$

$$-\beta 2gn = -\tilde{g}D(r)/\pi$$

The number of atoms on all excited states is

$$\frac{N}{N_{c^{(id)}}} = \frac{6}{\pi^2} \int_0^{+\infty} D(R) R dR$$

where $D$ is the solution of

$$D(R) = -\ln \{1 - Z \exp [-R^2/2 - \tilde{g}D(R)/\pi]\}$$

and it depends only on $Z$ and $\tilde{g}$, $\omega$ and $T$ do not enter: scaling of the atom number with $\omega$ and $T$ at fixed $Z$ is the same as in the ideal case. For any $\omega$ and $T$ and non-zero $g$, $N$ can be arbitrarily large by properly choosing $Z$. The condensation does not occur. For ideal gas the saturation on excited states occur when $N(0)$ diverges, while repulsive interaction removes the singularity, and the mean field theory provides
a solution for any value of $N$.

Interactions, when treated at the mean-field level, dramatically change the nature of the solution of eqs. For a given trapping frequency $\omega$ and temperature $T$, and for any non-zero $g$, the atom number $N$ obtained can be made arbitrarily large by choosing properly the fugacity $Z$. The condensation that was obtained in the ideal gas case does not occur anymore. This can be understood qualitatively. For an ideal gas, saturation of the atom number occurs when the central density in the trap becomes infinite. In presence of repulsive interactions, this singular point cannot be reached and the mean-field treatment provides a solution for any atom number. We give the prediction of the mean-field approach for the central phase space density $D(0)$ as a function of the total number of atoms in the trap, for various values of $g$. 
Given $\tilde{g}$ and $N$, we solve the implicit equation

$$D(R) = -\ln \left\{ 1 - Z \exp \left[ -R^2/2 - \tilde{g}D(R)/\pi \right] \right\}$$

together with $\int 1^{\lambda^{-2}} D(R) 2\pi R dR = N$. This determines $D(R)$ as a function of $\tilde{g}$ and $N$, plotted below.

Black squares indicate the values of $N$ at which BKT criterion is met at the center of the trap.

The mean field theory cannot provide an accurate description of the transition. Indeed the mean-field expression $2gn(r)$ for the interaction energy can only be valid at relatively low density, where the density fluctuations are important, so that
$g_2(0) = 2(bunching)$ and $\langle n^2 \rangle = 2n^2$. When the density increases and/or the temperature decreases, density fluctuations are reduced and one eventually reaches a situation at very low temperature where those fluctuations are frozen out $g_2(0) = 1 \text{ and } \langle n^2 \rangle = n^2$. At zero temperature, one expects a quasi-pure condensate in the trap with a density profile given by the Thomas-Fermi law $g n(r) = \mu-V(r)$ (whereas the Hartree-Fock approximation would lead to replacing $g$ by $2g$ in this equation). To capture the reduction of density fluctuations as the phase space density increases, we now use the numerical results of Prokofiev and Svistunov (classical field Monte-Carlo)
Holzmann et al (mean field Hartree-Fock)

\[
\frac{N_c^{(mf)}}{N_c^{(id)}} = 1 + \frac{3\tilde{g}}{\pi^3}D_c^2
\]

For a given trap at a given temperature the BKT threshold in presence of interaction requires a larger atom number than the BEC of ideal gas. For a given atom number the superfluid transition temperature in the presence of interaction is lower than the ideal condensation temperature.

3. What comes first
This question is more subtle than the box case because of the change of the density of states, and depends on what we keep constant and what we vary in an experiment.

i. For non-zero \( g \) increasing \( D \) until it reaches \( D_c \), BKT comes first, with induced condensation, conventional BEC does not occur in the mean field theory.
ii. For fixed $N$, increasing $g$ further increases $N_{c}^{BKT}$ and interaction enhanced condensation never occurs. These two processes are not equivalent in the harmonic trap case. Note that it is not inconsistent that the BKT transition occurs at a lower critical density but higher critical number than ideal gas BEC, because in a harmonic trap with a fixed $N$ and $T$ the peak (phase space) density in a repulsively interacting gas is lower than in an ideal gas. Also note that if we work within the BKT theory and then Formally take the $\sim g \to 0$ limit, we exactly recover the criterion for ideal gas condensation, which is usually derived from a conceptually completely different viewpoint of the saturation of single-particle excited states.
iii. BEC transition can be considered as a special non-interacting limit of the more general BKT theory in the case of a harmonically trapped gas. We briefly comment on the case of a realistic harmonic trap, where the spacing of the single-particle energy levels is non-zero. The results for the critical atom numbers for the BKT and the ideal gas BEC transition are essentially unaffected by the finite level-spacing. It therefore remains true that interaction-enhanced condensation is impossible. However, the ideal gas in this case occurs at a finite phase space density $D_{BEC}$ in the trap center, which can in principle be lower than $Dc$ for some values of $\sim g$. In this case BEC would come first independent of paths. But the value of $g$ would be unrealistically small, this regime is essentially indistinguishable from the $\sim g \rightarrow 0$ limit, where the BKT and the BEC transition are no longer distinct.
V. Probing 2D Atomic Gases

1. Interaction in a 2D gas

Petrov, Holzmann, Shlyapnikov

3D: $k \to 0$, $f(k) \to a$; 2D energy dependence even in low energy

Consider instead the 3D scattering in confined geometry (quasi-2D), the authors obtained

$$f(k) \approx \frac{4\pi}{-\ln(k^2a_2^2) + i\pi}$$

and establishing the equivalence, they got

$$a_2 = a_z \sqrt{\kappa} \exp \left( -\sqrt{\frac{\pi}{2}} \frac{a_z}{a_g} \right)$$

When the log term is negligible compared to the leading term

$$f(k) \equiv \tilde{g} \approx \sqrt{8\pi a_g/a_z}$$

$$\psi_{sc} = \varphi_0(z) e^{ik \cdot \tilde{\rho}}$$

$\kappa \approx 3.5$
2. In situ density distribution

There is no loss of information due to the integration along the line of sight (vs 3D). In the local density approximation the local density can be obtained from that of homogeneous gas \( n\lambda^2 = F(\mu, k_B T, a_2) \) where \( F \) at this stage is an unknown dimensionless function. LDA and dimensional analysis suggests \( \mu \to \mu - V(r) \quad m\omega^2 r_T^2 = k_B T \quad r_T \) is the length scale

\[
D = n(r)\lambda^2 = G\left(\alpha - r^2 / 2r_T^2, \tilde{g}\right) (*), \quad \alpha = \mu / k_B T
\]

This expression clearly shows a scale invariance for a given interaction strength \( \sim g \). Suppose that different density profiles \( n(r) \) are recorded for various temperatures \( T \) and various atom numbers \( N \) (hence different chemical potentials). According to eq. (*) the profiles can all be superimposed on the same curve \( G \), provided they are plotted as a function of \( r^2 / r_T^2 \) and translated along the x axis by \( \alpha \). (checked MC)
For $\tilde{g} \ll 1$, various asymptotic forms of the function $G(\alpha; \tilde{g})$ have been given earlier. When interactions can be neglected, the equation of state is $D = -\ln(1 - e^\alpha)$.

In presence of interactions and for small phase space densities, the mean-field Hartree-Fock method amounts to replace $\mu$ by $\mu - 2gn$ into the ideal gas result, which leads to the implicit equation $D = -\ln(1 - e^{\alpha - \tilde{g}D/\pi})$, from which one can extract $D$ as a function of $\alpha$ and $\sim g$. $\frac{2gn}{k_BT} = \frac{\tilde{g}D}{\pi}$

In the strongly degenerate limit, where $\mu \gg k_BT, D \gg 1$, density fluctuations are strongly reduced and one expects $\mu = gn$, which can be written as $D = \frac{2\pi\alpha}{\tilde{g}}$.

In the intermediate regime, in particular close to the BKT transition point, one can use the results of the classical field Monte-Carlo analysis.
The results are represented below

Homogeneous system

\[ F(\alpha, \tilde{g}) \] (a)

\[ \tilde{g} = 0.15 \]

system in trap

\[ D(r / r_T) \] (b)

Remarkable: \( F \) is featureless at the BKT transition! The transition is an infinite order one.

**Fig. 3.** (a): Phase space density as a function of \( \alpha = \mu/k_B T \) for \( \tilde{g} = 0.15 \). Continuous line: Total phase space density \( D = n\lambda^2 \); dashed line: superfluid phase space density \( n_s\lambda^2 \). The dotted and dash-dotted lines represent the asymptotic regimes for low and high phase space densities, respectively. (b) *In situ* density profiles in a trap deduced from the left panel using the local density approximation. The plot is made for \( \mu/k_B T = 0.5 \) so that the Thomas-Fermi radius \( r_{TF} = \sqrt{2\mu/m\omega^2} \) is equal to \( r_T \). \( \text{LDA: } \alpha = \left( \mu - V(r) \right) / k_B T \)
2. Time of flight expansion  Generally speaking, a Time-of-Flight (TOF) procedure consists in switching off abruptly the potential confining the atoms, letting the cloud expand for an adjustable time, and then measuring the density profile. If the role of interactions is negligible during the expansion, the density profile after a long TOF is proportional to the in-trap momentum distribution. For a two-dimensional system, two types of TOF can be considered.

2D TOF switching off at t=0 the confining potential in xy plane and keep the axial confinement. A Bogoliubov analysis (Kagan et al PRA 54 R1753 1966) predicts that the density at time t is given by the scaling law

\[ n(r, t) = \eta_t^2 n_{eq}(\eta_t r) \quad \eta_t = (1 + \omega^2 t^2)^{-1/2} \]

This means that the global form of the spatial distribution is preserved during the TOF. As the expansion proceeds, the
interaction energy that was initially present in the gas is converted into kinetic energy in such a way that the density profile at time \( t \) is obtained using a scaling transform of the initial one.

**3D TOF** Confining potentials are switched off simultaneously. The axial and radial expansions have very different time scales \( \omega_z^{-1} \ll \omega_r^{-1} \) and the expansion has two phases. During the first phase, whose duration is a few \( \omega_z^{-1} \) (typically 1 ms if \( \omega_z / 2\pi = 3 \text{ kHz} \)), the thickness of the gas along \( z \) increases by a factor much larger than 1, but the xy spatial distribution is nearly not modified. At the end of this first phase, the interactions between atoms have become negligible. During the subsequent phase the expansion in the xy plane becomes significant, but on a much longer time scale. It corresponds to the expansion of an ideal gas, whose initial state is equal to
the state of the system in the $xy$ plane before the beginning of the TOF.

We now focus on the evolution of the $xy$ degrees of freedom during the second phase, which is essentially governed by single particle physics. The evolution of the density distribution in the $xy$ plane can be determined from the initial one-body density matrix $g_1(r, r') = \langle r | \rho^{(1)} | r' \rangle$, or from its Fourier transform $\Pi(p)$ with respect to the variable $r - r'$, which represents the momentum distribution in the $xy$ plane.

In absence of any extended coherence in the gas, $g_1(r, r')$ tends to zero when $|r - r'|$ increases, with a characteristic decay length given by the thermal wavelength $\lambda$. The corresponding momentum width is $\Delta p \sim \hbar/\lambda$ and the spatial distribution after TOF will reflect the initial momentum distribution if the TOF duration $t$ is such that $\Delta p \, t/m \gg r_T$, where $r_T$ is the initial size of the gas. For a harmonic confinement in the $xy$ plane, this “far field” regime corresponds to $\omega t \gg 1$. Taking $\omega/2\pi = 30$ Hz as a typical value, the far field regime (say $\omega t > 3$) is reached for $t > 15$ ms. This corresponds to a typical value for TOF experiments, which thus give access to the momentum distribution in this non (strongly) degenerate regime.

The situation is very different if a significant condensed fraction is present in the gas, as expected in the vicinity and below the BKT transition temperature. In this case we have seen in § 5.4 that the size $r_c$ of the coherent region of the cloud is $r_c \sim r_T$. The momentum width $\Delta p_c = \hbar/r_c$ of this coherent component is then extremely narrow,
and it would require a very long TOF to reach the ‘far field’ regime for this coherent component. Taking $r_c = r_T$ as a typical value, we find that the time $t$ required for a significant expansion of this component, i.e. $\Delta p_c t/m = r_c$, is such that $\omega t = k_B T/\hbar \omega$. For $\omega/2\pi = 30$ Hz and $T = 100$ nK, this gives $t > 300$ ms, which is too long in practice for a TOF.

Therefore in the regime where a relatively strong coherence of the gas is present, a 3d TOF of a realistic duration gives access to a hybrid information. The high energy fraction of the gas is in the far field regime and the wings of the density profile after TOF give access to the large momentum part of the initial state. On the contrary the central feature corresponding to the condensed, superfluid fraction, has not yet undergone a significant expansion. The detailed study of the border between these two components is still a matter of debate.
3. Interference between different planes

Time-of-flight experiment harmonic oscillator ground state

\[ u_0(z) = \frac{1}{\pi^{1/4}\sqrt{\sigma}} e^{-z^2/2\sigma^2}, \quad \sigma = \sqrt{\frac{\hbar}{m\omega_z}} \]

Z-confinement \( \Delta p_z = \hbar / \sigma = \sqrt{\hbar m\omega_z}, \quad \Delta v_z = \sqrt{\hbar \omega_z / m} \)

For time interval \( \Delta t \) the wave function expands freely along the z direction for a distance \( d_z = \sigma \)

The expansion in xy plane is much longer

\[ \sqrt{\frac{\hbar}{m\omega_z}} = \sqrt{\hbar \omega_z / m} \Delta t \Rightarrow \Delta t = \omega_z^{-1} \quad \omega^{-1} \gg \omega_z^{-1} \]

Therefore it is a good approximation to decompose a 3d TOF into two phases. During the first phase, whose duration is a few \( \omega_z^{-1} \) (typically 1 ms if \( \omega_z/2\pi = 3 \) kHz), the thickness of the gas along \( z \) increases by a factor much larger than 1, but the xy spatial distribution is nearly not modified. At the end of this first phase, the interactions between atoms have become negligible. During the subsequent phase the expansion in the xy plane becomes significant, but on a much longer time scale. It corresponds to the expansion of an ideal gas, whose initial state is equal to the state of the system in the xy plane before the beginning of the TOF.
thermal gas $r_c \sim \lambda$, $\Delta p \sim \hbar / \lambda$, TOF far field
superfluid $r_c \sim r_T$, $\Delta p \sim \hbar / r_T$, central region


Not aiming at studying phase difference between planes, but
1. To characterize the normal and superfluid regime
   (according to the behavior of $g_1$)
2. To demonstrate the BKT transition
   representing a new direction of probing the physics of quantum correlations, a property of quantum many-body systems.
The planes are prepared in identical conditions (same temperature and density). 3D TOF. From the interference pattern the behavior of $g_1$ can be extracted. Polkovnikov et al suggested to characterize both normal and superfluid regime using a single experiment.

The state of the planes are described by $\psi_a$ and $\psi_b$ distance $d_z$ Ketterle(97) : the relative speed at any point after time $t$ is $d_z / t$. The fringe period is the de Broglie wave length associated with the relative motion of atoms $\lambda = \hbar / m v$, $D_z = \hbar t / m d_z$

The spatial density is

$$n \propto |\psi_a|^2 + |\psi_b|^2 + \left( \psi_a \psi_b^* e^{i2\pi z/D_z} + \text{c.c.} \right)$$
The local contrast (complex) is proportional to $\psi_a^*(z)\psi_b(z)$ if one performs absorption imaging along the $y$ axis, the image involves an integration of the local contrast $\psi_a^*\psi_b$ along the $y$ direction. Averaging the result of this contrast measurements one can define the average contrast $C(A)$:

$$C^2(A) = \frac{1}{A^2} \langle \left| \int_A \psi_a(r)\psi_b^*(r)\,d^2r \right|^2 \rangle$$

$$\int = \int \psi_a(\vec{r})\psi_b^*(\vec{r})\psi_a^*(\vec{r}')\psi_b(\vec{r}')d^2rd^2r'$$

this quantity is independent of the phase difference between the two planes, but is determined by the phase fluctuation within each condensate.

The correlation functions are the same, but the two planes fluctuate independently, therefore $\langle A \int |g_1(r)|^2\,d^2r \rangle$
In the two extreme limits:

**Normal** \[ g_1(r) \propto e^{-r/L}, \quad C^2(A) \propto A^{-1} \]

**Superfluid** \[ g_1(r) \propto r^{-\alpha}, \quad C^2(A) \propto A^{-2\alpha} \]

\( \alpha < 1/4 \)

This corresponds to a decay always slower than \( A^{-1/2} \). This method is very appealing in the sense that the measurement of a single number, i.e., the exponent \( \eta \) characterizing the variation of \( C \propto A^{-\eta} \), is sufficient to identify the two possible regimes of a 2d Bose gas, and obtain the value of \( n_s \lambda^2 = 1/\eta \) in the superfluid case.

If a single isolated vortex is present in one of the planes, the pattern exhibits a sharp dislocation at the coordinate of the vortex core.
• Observation of a 2D Bose Gas: From Thermal to Quasi-condensate to Superfluid  
  W. D. Phillips group PRL 102, 170401, 2009 by measuring the profile of the quasi-2D system after time-of-flight

Na atoms (weakly interacting with $g \approx 0.02$) in optical dipole trap, cooled to $\sim 100\text{nK}$. The temperature is determined by the trap depth. Absorption imaging along the vertical direction is carried out after turning off the trap and waiting for a TOF period, and the 2D density profile of the released atomic cloud is obtained. During the experiment the temperature is fixed while the number of atoms $N$ is varied.

Prokofiev et al did MC calculations for (homogeneous) 2D weakly interacting system and obtained $n_{BKT} \lambda^2 = \ln(C / \tilde{g})$
Prokofiev et al predicted the presence of a non-superfluid quasi-condensate (low phase coherence). The present authors applied LDA to the MC results and calculated the total density, quasi-condensate density and the density of superfluid, as functions of $N/N_c$, with $N_c = \pi^2 / 6 \left[ k_B T / \hbar \omega_\perp \right]^2$, the critical number for BEC of a 2D ideal gas. It is seen that at low densities the profile is a Gaussian, and at higher densities the (red) line is bimodal: a superposition of two Gaussians. After 5 $\mu$s of TOF time, the QC and the superfluid components cannot be resolved. For 10 $\mu$s a tri-modal profile can be observed. $N > N_{BKT}$.
4. Interfering a single plane with itself: alternative for g_1

Philips group PRL 102, 170401, 2009

its “self-interference”, using a Ramsey-like method [28]. The gas is initially prepared in internal state $|1\rangle$. Half of the atoms are coherently transferred into another internal state $|2\rangle$ by a stimulated laser Raman process ($\pi/2$ pulse) that also provides a momentum kick $k_0$ to the atoms. After this process, the part of the cloud in $|1\rangle$ is still globally at rest and the part in $|2\rangle$ moves with the global velocity $v_0 = \hbar k_0/m$. After an adjustable time $t$ a second $\pi/2$ Raman pulse remixes the amplitudes of $|1\rangle$ and $|2\rangle$ and provides a momentum kick $k_0 - k_1$. Immediately after this second Raman pulse, one measures the spatial density distribution in $|2\rangle$. This distribution exhibits a modulation along the direction $k_1$, resulting from the interference between the initial state of the cloud and the state displaced by the distance $R = v_0 t$:

$$n(r) \propto |\psi(r)|^2 + |\psi(r - R)|^2 + (\psi(r)\psi^*(r - R)e^{ik_1\cdot R} + c.c.)$$

B: point of observation, receiving waves of $F=2$ from point A, where the atom results from the 1st Raman beams at O acquiring velocity $v_0$ and moving to A, and remaining in the state $F=2$, and from the point O, where the atom remaining in $F=1$ at the 1st Raman beams and becoming $F=2$ by the 2nd Raman beams

phase difference = momentum difference $\cdot R = k_1 \cdot R$
The modulated density profile of the above equation gives a direct access to the function $g_1$. It can be observed with an imaging beam along the $z$ direction, so that its measurement does not involve any line-of-sight integration. This method can then reveal finer details than the one presented earlier. In particular, the present authors could observe a gradual increase of the coherence length $l$ of the cloud, 

$$l = \lambda \exp \left( \frac{a_{T_{KT}}}{(T - T_{KT})} \right).$$

For small phase space densities the measurement gives $l \sim \lambda$, and $l$ increases to much larger values when the temperature decreases towards the critical temperature $T_{BKT}$. When $T < T_{BKT}$ a significant interference contrast is observed for all values of $R$ within the size of the central superfluid region.

Right: Images of atoms in the $F = 2$ state after the interferometer sequence for two different delays between Raman pulses (upper: $\tau = 7.5 \mu s$, lower: $\tau = 17.5 \mu s$). For the longer delay, the interference of the thermal cloud is washed out and there are only fringes from the superfluid region, which has a longer coherence length.

Upside down!
• The contrast as a function of $R$ is measured for different densities. For small density, the correlation ends at $1 \mu m$, the thermal wave length $\lambda$ for $100 nK$. For intermediate densities the correlation drops, but then remains corresponding to the quasi-condensate up to $10 \mu m$, and for still larger densities ($>N_{BKT}$), the superfluid component appears, and the correlation exists at $20 \mu m$, corresponding to algebraic decay of coherence.
• V. Outlook
• 1. Higher order correlation functions

The matter-wave interference between two statistically similar, but independent quasi-condensates can reveal a wealth of information on the correlations within each individual 2d gas. So far only a fraction of this information has been harnessed, with the study of the average contrast of the interference pattern integrated over some area of interest. A convenient tool for extracting more complete information on $g_1$ as well as higher-order correlation functions is the full statistical distribution of interference contrasts. Two limiting cases can easily be characterized: (i) If the two independent fluids are fully condensed, each image shows a 100% contrast, with the position of the fringes fluctuating randomly from shot to shot. (ii) If each cloud exhibits only short-ranged correlations, the observed interference results from many uncorrelated fringe patterns along the light of sight, and the distribution of contrasts is an exponential function. For 1d gases, it is possible to describe quantitatively the transition between these two limiting cases.
In the 2d case, the evolution of the contrast distribution through the BKT transition is still an open problem.

The 1D interacting Bose liquid can be described by bosonizing the Luttinger liquid theory (Cazalilla).

Quasi 1D Luttinger liquid theory: $T=0$, quasi LRO, finite $T$, exp decay.

1D: no need to integrate along the line of sight. Average over a tunable distance along the longitudinal direction.

We probe more deeply in the quantum correlations in the many-body problem system.

Shot-to-shot record enables to measure the statistical distribution of the local contrast, the higher order moments of which is associated with the higher order correlation functions of the 1D boson gas.
Demler et al
Nphys 2006, theory
Hofferberth, Demler et al
Expt nphys 2008
NB $z$ and $y$ Inverted!

Local contrast averaged over distance $L$

\[ A_Q(L) = \int_{-L/2}^{L/2} dz \, a_1^\dagger(z) a_2(z) \]

quantity $\langle |A_Q(L)|^2 \rangle$. This quantity is independent of the overall phase difference but is strongly affected by phase twisting within each condensate.

\[ \langle |A_Q(L)|^2 \rangle = \int_{-L/2}^{L/2} dz_1 \int_{-L/2}^{L/2} dz_2 \langle a_1^\dagger(z_1) a_1(z_2) \rangle \langle a_2^\dagger(z_2) a_2(z_1) \rangle \]
In the case of ideal non-fluctuating condensates we expect to find perfect contrast for any size of the system. This implies $\langle |A_Q(L)|^2 \rangle \propto L^2$. In the opposite regime of short-range phase correlations with finite correlation length $\xi_\phi$, the net interference pattern comes from adding up fringes in $L/\xi_\phi$ uncorrelated domains. In this case the net interference pattern is strongly suppressed and appears only as a square root fluctuation, $\langle |A_Q(L)|^2 \rangle \propto L^{\xi_\phi}$.

Luttinger liquid theory gives

$$A_Q(L; (n_1, g, \xi_\phi))$$

$$\langle |A_Q(L)|^2 \rangle = n_{1D}^2 L^2 \left( \frac{\xi_h}{L} \right)^{1/K} f \left( \frac{\xi_\phi(T)}{KL}, K \right)$$

$K = \pi \hbar \sqrt{n_{1D}/gm}$ is the Luttinger parameter

g = 2\hbar v_\perp a_s$ the effective 1D coupling constant and $a_s$ the s-wave scattering length. $\xi_h = \hbar/\sqrt{mg n_{1D}}$ is the healing length and $\xi_\phi(T) = \hbar^2 n_{1D}\pi/mk_B T$ is the thermal correlation length of the 1D condensates (for the weakly interacting regime). The function $f(x, K)$ is given by

$$f(x, K) = \int_0^1 \int_0^1 \text{d}u \text{d}v \left( \frac{\pi}{x \sinh \left( \frac{\pi |u-v|}{x} \right)} \right)^{1/K}. \quad (3)$$
Temperature and L dependence

1. Low temperature/small size $L/\xi_\phi(T) \ll 1$ quantum fluctuation (arising from interaction between atoms) dominates
   ideal gas $\rightarrow$ no phase fluctuation $\left\langle |A_Q(L)|^2 \right\rangle \propto L^2$
   strong interaction (Tonks-Girardeau) $\left\langle |A_Q(L)|^2 \right\rangle \propto L$
   intermediate (new) $\left\langle |A_Q(L)|^2 \right\rangle \propto L^{2-1/K}$

2. Finite temperature: thermal fluctuation results in $\xi_\phi(T)$
   for $L/\xi_\phi(T) \ll 1$ thermal fluctuation dominates $\left\langle |A_Q(L)|^2 \right\rangle \propto L\xi_\phi(T)$

Shot-to-shot individual measurement of fluctuations enables to find higher moments $\left\langle |A_Q|^{2n} \right\rangle$ and the distribution with the normalized fluctuation $\alpha(L) = |A_Q(L)|^2/\left\langle |A_Q(L)|^2 \right\rangle$
For the distribution of averaged contrast $W(\alpha(L))$, we obtain

$L < \xi_\phi(T)$: quantum fluctuation dominates
for ideal gas $W$ is a delta function
for weak interaction $W$ has a narrow peak, width $1/K$
for finite interaction $W$ approaches a Gumbel distribution

$L \sim \xi_\phi(T)$: both quantum and thermal fluctuations are important
a double peak situation

$L > \xi_\phi(T)$: thermal fluctuation dominates, $W$ approaches a
Poisson distribution as a sum over interference from many
uncorrelated domains

![Graphs and histograms showing the distribution of averaged contrast for different temperatures and sizes.](image)
2. Out-of-equilibrium dynamical effects

Burkov et al. [112] have studied the dynamics of decoherence between two planar Bose gases, assuming that their local phases are initially locked together, and then the two gases are allowed to evolve independently. This can be achieved experimentally by having a weak potential barrier and hence large tunnel coupling between the two planes for $t < 0$, and then suddenly raising the barrier at $t = 0$. The contrast of the interference between the two gases gives access to the evolution of the phase distribution under the influence of thermal fluctuations.

3. Transition from 2D to 3D behavior

*Transition from 2d to 3d behavior.* The possibility to vary the tunnel coupling between two or more planar gases can also be used to study the so-called “deconfinement transition” [114], corresponding to a gradual evolution from 2d to 3d behavior. The phase coherence between the planes will build up as the strength of the coupling is increased, a large number of parallel planes, the deconfinement transition should give rise to a true Bose–Einstein condensate [114]. The two-plane situation is also very interesting, and can lead to the observation of the Kibble-Zurek mechanism [115]: the superfluid transition temperature is higher for two coupled planes than for a single one, so that sudden switching on of the coupling between the planes (initially in the normal state but close to the single-plane critical temperature) constitutes a quench of the system, and one could observe the subsequent dynamical apparition of a macroscopic quantum (quasi-)coherence.
4. Tunable Interactions

In the weak coupling regime ($\tilde{g} < 10^{-1}$) we expect a gradual change from the BKT-dominated to the BEC-dominated behavior, as discussed in Sections 4 and 5. Further, it would be very interesting to explore the strong-coupling regime ($\tilde{g} > 1$), which is closer to liquid helium films. This regime, which is outside the domain of validity of the Monte-Carlo results [29, 43], corresponds to the case where the scattering length $a_s$ becomes comparable the thickness of the sample along the kinematically frozen direction, $a_z$ (see 6.2). There the very nature of two-body interactions is expected to change from 3d to 2d [82, 83, 116, 117]. Therefore, experimentally reaching the condition $a_s \geq a_z$ would correspond to producing a “truly 2d” as opposed to a quasi-2d Bose gas.

5. Superfluid Density

*Superfluid density.* Generally speaking, studies of coherence and correlation functions in a 2d fluid, which are well suited to experimental tools of atomic physics, are a natural complement to the “traditional” studies of superfluidity based on transport measurements, which are well suited to other physical systems such as liquid helium films [2].

2D system: both transport and coherence measurements can be performed, allowing experimental scrutiny of the theory
Two promising schemes for a direct measurement of the superfluid density (as traditionally defined through transport properties [Leggett]) in an atomic gas have recently been proposed. The first scheme [Ho and Zhou] is based on extracting the superfluid density from the in-situ density profiles of a rotating 2d gas. The second scheme [Cooper and Hadzibabic] is based on using a vector potential generated by Raman laser beams to simulate slow rotation of a gas, and allows direct spectroscopic measurement of the superfluid density.
Linear gauge potential
Nature 462, 628, 2009

\[ \{ |1, 2k_L\rangle, |0, 0\rangle, |1, -2k_L\rangle \} \]

3 states mixed by Raman beams
Proposal by N. Cooper and Z. Hadzibabic PRL 104, 030401 Azimuthal gauge potential
To generate an azimuthal vector potential, we consider two Laguerre-Gauss (LG) beams [16] with different orbital angular momenta, copropagating in the direction perpendicular to the toroidal trap. In this way, a two-photon transition imparts negligible linear momentum to the atoms, but a nonzero angular momentum, \( \pm \Delta \ell \), where \( \Delta \ell \) is the difference in the orbital angular momenta of the two beams. For a three-level system \([14, 15]\) this leads to an effective Hamiltonian

\[
\begin{pmatrix}
\left( \frac{\hbar}{2MR^2} (\ell + \Delta \ell)^2 - \delta \right) & \Omega_R/2 & 0 \\
\Omega_R/2 & \frac{\hbar}{2MR^2} \ell^2 - \epsilon & \Omega_R/2 \\
0 & \Omega_R/2 & \left( \frac{\hbar}{2MR^2} (\ell - \Delta \ell)^2 + \delta \right)
\end{pmatrix}
\]

which is a matrix in the hyperfine states \( m = -1, 0, +1 \).

A shift in the min is equivalent to an azimuthal vector potential

**Significance of \( l^* \):** it is as if the optical field causes the laboratory frame to behave as a frame of reference that is rotating with angular frequency \( \omega_{\text{eff}} = \frac{\hbar \ell^*}{M^*R^2} \).

when the normal fluid comes to equilibrium at a nonzero \( \ell^* \), it is at rest in the laboratory frame. However, the superfluid is rotating, as follows from (8). The azimuthal vector potential causes a steady superfluid flow around the ring-shaped trap.
The wave function in the lowest band is a linear superposition of the three hyperfine levels $|\psi\rangle \equiv \sum_{m=-1,0,1} \psi_m |m\rangle$ with amplitudes $\{\psi_m\}$ which vary with $\ell$. A perturbative analysis shows that there are equal and opposite corrections to $|\psi_{\pm 1}|^2$ which depend linearly on $\ell$. Thus, $\langle L\rangle$ can be obtained from a measurement of the difference in the number of particles in the states $m = \pm 1$.

\[
\Delta p \equiv \frac{N_{-1} - N_1}{N}
\]

**FIG. 2.** Angular momentum $\ell^*$ at the bottom of the band (dashed line) and change in particle imbalance $\Delta p - \Delta p_0$ (solid line) as a function of $\delta/\Omega_k$ for a normal fluid (i.e., centered on $\ell^*$) for $\Omega_R = 1000 \hbar / MR^2$, $\Delta \ell = 10$, $\epsilon = 0$. This illustrates the precision required to distinguish a normal fluid (here $\Delta p - \Delta p_0 \sim 3\%$) from a perfect superfluid ($\Delta p - \Delta p_0 = 0$).