Drift velocity of rotating spiral waves in the weak deformation approximation

Hong Zhang
Department of Physics and Centre for Nonlinear Studies, Hong Kong Baptist University, Hong Kong, China

Bambi Hu
Department of Physics and Centre for Nonlinear Studies, Hong Kong Baptist University, Hong Kong, China and Department of Physics, University of Houston, Houston, Texas 77204-5005

Gang Hu
Department of Physics, Beijing Normal University, Beijing 100875, China

Jinghua Xiao
Department of Physics and Centre for Nonlinear Studies, Hong Kong Baptist University, Hong Kong, China

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The drift velocities of spiral waves driven by a periodic mechanic deformation or a constant or periodic electric field are obtained under the weak deformation approximation around the spiral wave tip. An approximate formula is derived for these drift velocities and some significant results, such as the drift of spiral waves induced by a mechanical deformation with \( \omega = 3 \omega_0 \), are predicted. Numerical simulations are performed demonstrating qualitative agreement with the analytical results. © 2003 American Institute of Physics. [DOI: 10.1063/1.1592791]

Spiral waves are one of the most striking patterns in excitable and oscillatory media. They have been the subject of extensive research in a large variety of systems, which include the oxidation of CO on platinum, nerve impulse, reacting chemical systems like the Belousov–Zhabotinsky (BZ) reaction, and cardiac muscle. In particular, it is generally believed that the spirals in cardiac muscle play an essential role in heart diseases such as arrhythmia and fibrillation, the latter being the leading cause of death in the industrialized world. Therefore, the control of spirals is of crucial importance. An extremely interesting as well as practical point in this regard is that spiral waves are often subjected to some form of external forcing which might occur, for example, in the daily variation of sunlight. Thus, it is important to understand the role such external driving will play in these systems.

In the study of the control of spiral waves, an active field of recent investigation concerns the resonant drift of the spiral core induced by applying an external field, such as a dc or an ac electric field. For rotating waves, resonance corresponds to a net drift of the rotation centers along a straight line. To address the problem of how cardiac muscle contraction affects the dynamics of rotating spiral waves, Muñuzuri et al. designed an elastic excitable medium by incorporating the BZ reaction into a polyacrylamide-silica gel to investigate the effect of mechanical deformation on spiral waves. They reported that for equal frequencies of deformation and spiral rotation, spirals will drift. Moreover, the direction of the drift does not coincide with the stretching direction and can be varied by changing the phase shift between the deformation and the spiral rotation. In their paper, the authors also suggested a simple kinematical model providing an intuitive understanding of the main features observed. However, this kinematical model is not based on the original partial-differential equation and so far no theoretical results have been obtained. Therefore, analytical derivations directly from the reaction-diffusion equation and its spiral wave solution are of crucial importance for a deeper and more comprehensive understanding of the mechanism of the drift and for a wider application of spiral wave control.

In this paper, we will introduce a new approach to compute the velocity of the spiral tip based on the characteristics of spiral waves. We start with a general reaction-diffusion equation and its spiral wave solution. By applying an approximation of a weak spiral wave deformation around the tip, we derive an approximate formula for the drift velocities of the spiral tip induced by an arbitrary weak perturbation field that includes the mechanical deformation perturbation and the dc and ac electric field perturbations as special cases. This approximate formula can only explain a variety of phenomena experimentally observed, but also predict some new phenomena. For instance, for a mechanical deformation we find a new phenomenon: the spiral can drift when the frequency of mechanical deformation is equal to the triple frequency of the spiral rotation. For the dc driven system, the drift velocity parallel to the electric field is identified simply to \(-0.5ME\) (where \(M\) is the ion mobility and \(E\) is the intensity of the dc field), independent of the particular models generating spiral waves. Numerical computations are carried out as well, and they agree with the main qualitative conclusions of our analytical results.

Let us begin with a general two-variable reaction-diffusion system

\[
\frac{du}{dt} = f(u,v) + D_v \nabla^2 u, \quad \frac{dv}{dt} = g(u,v) + D_v \nabla^2 v. \tag{1}
\]

Here variables \(u\) and \(v\) represent the concentrations of the reagents or the temperature, the electric potential, etc.; \(D_u\) and
and $D_v$ are the diffusion coefficients of both variables; $\nabla^2 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the Laplacian operator; $f(u,v)$ and $g(u,v)$ are the reaction terms. For certain parameter values the system (1) has a dense and rigidly rotating spiral wave solution, and in the neighborhood of the tip the solution can be generally expressed as

$$
\dot{u} = u_0 + a_1 r \cos(-\sigma \theta + \omega_0 t - \beta) + b_1 r^2 \\
\times \cos[2(-\sigma \theta + \alpha_0 t - \delta)] + \tilde{b}_1 r^2 + \cdots,
$$

$$
\dot{v} = v_0 + a_2 r \cos(-\sigma \theta + \omega_0 t - \alpha) + b_2 r^2 \\
\times \cos[2(-\sigma \theta + \alpha_0 t - \psi)] + \tilde{b}_2 r^2 + \cdots,
$$

(2)

where $u_0$, $v_0$, $a_1$, $a_2$, $b_1$, $b_2$, $\tilde{b}_1$, and $\tilde{b}_2$ are eight coefficients and $\alpha$, $\beta$, $\delta$, and $\psi$ are four phase shifts; $\sigma = \pm 1$ is the chirality of the spiral ($\sigma = 1$ for counterclockwise rotating spiral, while $\sigma = -1$ for clockwise one); $\omega_0$ is the rotating frequency; $x - \tilde{x}_0(t) = r \cos \theta$, $y - \tilde{y}_0(t) = r \sin \theta$, where $(\tilde{x}_0(t), \tilde{y}_0(t))$ is the position of the possibly moving tip. In following discussions, we will not calculate the values of these coefficients, but we will apply the general form (2) to the derivation of the spiral drift problems.

Now we consider the influence of two arbitrary external fields $\Gamma_1(x,y,t)$ and $\Gamma_2(x,y,t)$ on the system motion,

$$
u_i = f(u,v) + D_u \nabla^2 u + \Gamma_1(x,y,t),
$$

$$v_i = g(u,v) + D_v \nabla^2 v + \Gamma_2(x,y,t).
$$

(3)

Supposing $\Gamma_1$ and $\Gamma_2$ induce a drift of the spiral wave tip with velocity $V(t) = [V_1(t), V_2(t)]$, we can then rewrite the perturbed Eq. (3) in the moving frame of $V$ as

$$
u_i = f(u,v) + D_u \nabla^2 u + \nabla V \cdot \nabla u + \Gamma_1,
$$

$$v_i = g(u,v) + D_v \nabla^2 v + \nabla V \cdot \nabla v + \Gamma_2.
$$

(4)

We assume that for small $\Gamma_1$ and $\Gamma_2$, the deformation of the spiral solution around the tip is weak, and all the values of $u$, $v$ and their derivatives at the tip in the moving frame remain unchanged compared with those of the unperturbed spiral (2) at the tip that is, in Eq. (4), $[u_i = \tilde{u}_i]_{\text{tip}}$, $[f(u,v) = f(\tilde{u}, \tilde{v})]_{\text{tip}}$, $[D_u \nabla^2 u = D_u \nabla^2 \tilde{u}]_{\text{tip}}$, $[\nabla V \cdot \nabla u = \nabla \tilde{V} \cdot \nabla \tilde{u}]_{\text{tip}}$, $[v_i = \tilde{v}_i]_{\text{tip}}$, $[g(u,v) = g(\tilde{u}, \tilde{v})]_{\text{tip}}$, $[D_v \nabla^2 v = D_v \nabla^2 \tilde{v}]_{\text{tip}}$, and $[\nabla V \cdot \nabla v = \nabla \tilde{V} \cdot \nabla \tilde{v}]_{\text{tip}}$. Submitting these results into Eq. (4) and remembering $\tilde{u}_i = f(\tilde{u}, \tilde{v}) + D_u \nabla^2 \tilde{u}]_{\text{tip}}$, and $\tilde{v}_i = g(\tilde{u}, \tilde{v}) + D_v \nabla^2 \tilde{v}]_{\text{tip}}$, we obtain the formula of the drift velocity of the tip:

$V_i = \frac{-C(\sigma \cos(\omega_0 t - \phi))}{\cos(\omega_0 t - \Phi)}
\left[2 \tilde{b}_1 \sin(\omega_0 t - \alpha) - b_1 \cos(3 \omega_0 t - 2 \delta + \alpha)
\right]

(8)

where $C = 2A/\{a_1 \sin(\alpha - \beta)\}$. Thus an approximate formula is obtained directly from the original partial differential equation for the drift of mechanical deformation where the effects of the driving phase and the chirality of the spiral are given explicitly.

To get a net drift velocity, $\omega$ should take the resonant frequency $\omega_0$ or $3 \omega_0$. For $\omega = \omega_0$, the constant (or average) drift velocity reads

$V_i = \frac{\sigma \cdot C \cos(\omega_0 t - \phi)}{2 \tilde{b}_1 \cos(\omega_0 t - \alpha)}
\left[2 \tilde{b}_1 \sin(3 \omega_0 t - 2 \delta - \alpha)
\right]

(9)

where the oscillatory parts in Eq. (8) are averaged out. It is clear that there exists a drift velocity $\tilde{V}_i$ orthogonal to the stretching direction, and the direction of the drift can be adjusted by changing the phase shift $\phi$ of the stretching pertur-
In Figs. 1(a) and 1(b), we present the tip motion by numerical simulation of Eq. (7) for counterclockwise ($\sigma = +1$) and clockwise ($\sigma = -1$) spirals at $\omega = \omega_0$ for different phase shifts. The formula (9) agrees with the numerical observations and explains the experimental observations mentioned in the introduction.

A significant feature predicted in Eq. (8) is that the constant drift velocity can also appear at triple frequency of the mechanical deformation, i.e., for $\omega = 3 \omega_0$. The net drift velocity for $\omega = 3 \omega_0$ is

$$\bar{V}_1 = 0.5 C b_1 \sin(\alpha - \phi + 2 \delta),$$
$$\bar{V}_\perp = \sigma \cdot 0.5 C b_1 \cos(\alpha - \phi + 2 \delta).$$  (10)

In Figs. 1(c) and 1(d), we show the numerical results of Eq. (7) for the drift of counterclockwise ($\sigma = +1$) and clockwise ($\sigma = -1$) spirals at $\omega = 3 \omega_0$ for various phase shifts. They confirm the results of Eq. (10). In our simulations, a constant drift has not been observed for $\omega = 2 \omega_0$, $4 \omega_0$, $5 \omega_0$, $6 \omega_0$, and so on.

From Eqs. (9) and (10), some relations can be computed without knowing the particular constants of Eq. (2). These relations are interesting because they are independent of the special models and should be of general significance. Defining the constant drift velocity amplitude $|\bar{V}|$ and the drift angle as

$$|\bar{V}| = \sqrt{\bar{V}_1^2 + \bar{V}_\perp^2}, \quad \tan \Theta = \bar{V}_1 / \bar{V}_\perp$$  (11)

we find in Eq. (10) for the triple-frequency resonance that $|\bar{V}| = 0.5 C b_1$, which is independent of $\phi$, and $\Theta$ is linear in $\phi$, i.e., $\Theta = \sigma \cdot [\phi + (\pi/2 - \alpha - 2 \delta)]$. But, these simple relations definitely do not hold for the $\omega = \omega_0$ resonance according to Eq. (9), and such predictions are confirmed in Fig. 2.

The general result of Eq. (5) can also be applied to the case of the spiral wave drift of a homogeneous electric field, which has been a topic of extensive investigation. In this case Eq. (1) becomes

$$u_i = f(u,v) + \nabla^2 u + M E u_i, \quad v_i = g(u,v),$$  (12)
where $M$ is the ion mobility and $E$ the electric field strength. For $|ME| \ll 1$, we can insert $\Gamma_1 = ME \omega_1$, $\Gamma_2 = 0$, and Eq. (2) to Eq. (5) and then obtain an approximate formula for the drift velocity:

$$V_i(t) = -0.5ME \cdot 0.5ME \csc(\alpha - \beta) \times \sin(2\omega_0t - \beta - \alpha),$$

(13)

A clear conclusion in Eq. (13) is that the spiral wave undergoes a directional drift when the electric field frequency is zero (dc field) or twice the spiral frequency. When a dc electric field is applied, only the first components in Eq. (13) contribute to the constant drift of the spiral tip, and $\bar{V}_i$

FIG. 2. $\sigma = +1$. The quantities $|\bar{V}|$ [(a), (c)] and $\Theta$ [(b), (d)] defined in Eq. (11) vs $\phi$. $\omega = 3\omega_0$ for (a) and (b), and $\omega = \omega_1$ for (c) and (d). These numerical results confirm the results of Eqs. (9) and (10) [also see the discussions after Eq. (11)].

FIG. 3. The numerical results of Eq. (12) for counterclockwise ($\sigma = +1$) spiral wave. The dc electric field is along the $x$ direction, $ME = 0.002$. (a) $V_y$, $b$ is fixed at 0.07; (b) $V_y$, $a$ is fixed at 0.84. The dashed lines in (a) and (b) show the values of the theoretical prediction of Eq. (13) for dc electric field. The ac electric field $ME_0 = 0.02$, with $\omega = 2\omega_0$. (c) $|\bar{V}|$ vs $\phi$; (d) $\Theta$ vs $\phi$. These numerical results agree with the conclusions of Eq. (13) for the drift induced by electric fields. Parameters in Eq. (12) are $a = 0.84$, $b = 0.07$, $\epsilon = 0.02$. 

$V_i(t) = -\sigma \cdot 0.5ME \cot(\alpha - \beta) - \sigma \cdot 0.5ME \times \csc(\alpha - \beta) \cos(2\omega_0t - \beta - \alpha)$.
of the excitability of an excitable medium, which has been well investigated by Mikhailov, Davydov, and Zykov, Mantel and Barkley, and Hakim and Karma. We will discuss this kind of drift phenomenon and compare our work with other works in detail in another paper.

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2 Waves and Patterns in Chemical and Biological Media, edited by H. L. Swinney and V. I. Krinsky (Elsevier, Amsterdam, 1991).
9 The idea of neglecting spiral deformation comes from Mikhailov, Davydov, and Zykov in studying the resonant drift of spiral under periodic modulation of the excitability of an excitable medium, where spiral waves are modeled by moving curves (see Sec. 6 in Ref. 14). The difference is that we start directly from the original partial differential equations and its spiral wave solution. Numerical results show that this approximation is good for resonantly drifting spiral wave that was dense and rigidly rotating.
