OFFPRINT

Circular-interface selected wave patterns in the complex Ginzburg-Landau equation

Bing-Wei Li, Xiang Gao, Zhi-Gang Deng, He-Ping Ying and Hong Zhang

EPL, 91 (2010) 34001

Please visit the new website
www.epljournal.org
TARGET YOUR RESEARCH WITH EPL

Sign up to receive the free EPL table of contents alert.

www.epljournal.org/alerts
Circular-interface selected wave patterns in the complex Ginzburg-Landau equation

BING-WEI LI¹,²(a), XIANG GAO², ZHI-GANG DENG², HE-PING YING² and HONG ZHANG²(b)

¹ Department of Physics, Hangzhou Normal University - Hangzhou 310036, China
² Zhejiang Institute of Modern Physics and Department of Physics, Zhejiang University - Hangzhou 310027, China

received 11 June 2010; accepted in final form 20 July 2010
published online 16 August 2010

PACS 47.54.-r – Pattern selection; pattern formation
PACS 82.40.Ck – Pattern formation in reactions with diffusion, flow and heat transfer
PACS 89.75.Kd – Patterns

Abstract – The selection of wave patterns in the complex Ginzburg-Landau equation in the presence of a circular interface was investigated. Depending on the initial conditions, the circular interface could select different wave patterns: concentric and spiral waves. Unlike the previously found targets or spirals, we found that the source of these interface selected waves is located at the circular interface instead of the center such as the “pacemaker” or spiral core; the coexistence of the opposite phase velocities in the same wave pattern is observed. What is more, the frequencies (or wave numbers) are identical in two domains with totally different control parameters, and it looks as if the interface selected waves propagate in a homogeneous medium without experiencing any difference across the interface. For the interface selected spiral pattern, we argue that its formation was attributed to the wave competition and topological constraints.

Introduction. – Spatially extended active systems are able to sustain a wide class of wave patterns [1,2]. Target waves and spiral waves are two paradigmatic examples that have been found in physical, chemical or biological systems. In most cases, these waves propagate out from the source (pacemaker or spiral core) and are thus called normal waves (NWs). Recently, waves propagating toward the wave source, termed antiwaves (AWs) are also reported in experiments [3,4] and in simulations [5]. An interesting result related to AWs is that their pacemaker or spiral core still acts as the wave source [6,7], like NWs.

The spatial heterogeneity is one of the important ways that can strongly affect the selection of wave patterns, which has attracted great interest during the last decades [8–12]. For example, a meandering spiral wave could be forced to rigidly rotate when a small unexcited inhomogeneity is introduced close to the spiral core [8]; coherent wave patterns could be created from spiral turbulence around a small inhomogeneity in both excitable and oscillatory systems [9,10]. Furthermore, the heterogeneity can also select new wave patterns [11,12]. The authors in ref. [11] observed the formation of a sink-source pair of spiral waves and nonuniformity of excitability was believed to play a key role in its formation. We reported the sinklike (and “dense-sparse”) spirals in nonuniform oscillatory media [12]. It is found that the core of the sinklike spiral acts as the wave sink instead of the wave source.

Currently, considerable attention has been paid to a particular kind of nonuniform system that is in the presence of an interface [13–16]. For instance, it has been reported that negative refraction could occur if chemical waves propagated across the interface in nonlinear oscillatory systems [14]. What is more, the authors in ref. [15] observed the interface selected traveling waves, which propagated in two different media with the same frequency and same wave number. The role of interface selected waves in wave competition in two-dimensional (2D) systems was discussed, however the studies in ref. [15] mainly focused on the 1D case. The knowledge on 2D interface selected waves (e.g., there exists a topological defect) is still insufficient and is thus worth further investigating.

In this paper, we study the selection of wave patterns in the 2D complex Ginzburg-Landau equation in the presence of a circular interface. Our findings indicate that the interface can select two types of wave patterns, e.g., the concentric and the spiral pattern, depending...
Numerical model and results. – We use the complex Ginzburg-Landau equation (CGLE) as our numerical model throughout this paper. The CGLE provides a universal description of spatially extended oscillatory systems close to the supercritical Hopf bifurcation, which reads [17,18]

$$\frac{\partial Z}{\partial t} = Z - (1 + i\alpha)|Z|^2 Z + (1 + i\beta)\nabla^2 Z,$$  \hspace{1cm} (1)

where $Z(x,y,t)$ is a complex variable describing the amplitude of the pattern modulations. $\alpha$ and $\beta$ are real parameters, representing the nonlinear frequency shift and the dissipative coefficient, respectively. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the 2D Laplacian operator.

In polar coordinates, eq. (1) admits a solution in the following form [19]:

$$Z(r,\theta,t) = \rho(r)e^{i\psi(r)} = \rho(r)e^{i(-\omega_{\perp}t + m\theta + \psi(r))},$$  \hspace{1cm} (2)

where $\rho(r)$ and $\psi(r)$ are real functions, and $d\psi/dr \rightarrow k$ and $\rho \rightarrow \sqrt{1 - k^2}$ as $r \rightarrow \infty$. $m$ is defined by

$$m = \frac{1}{2\pi} \int_L \nabla \phi dl.$$  \hspace{1cm} (3)

If $m$ is zero, the solution (2) denotes the target waves. And if $m$ is nonzero, e.g., $m = \pm 1$, the solution (2) describes the one-armed spiral waves. In the latter case, $m$ is called the topological charge.

For large $r$, the solution (2) is asymptotic to the plane-wave solution

$$Z(r,t) = \sqrt{1 - k^2} e^{i(-\omega_{\perp}t + k r)}.$$  \hspace{1cm} (4)

Here, the asymptotic frequency $\omega_k$ and the wave number $k$ are related via the nonlinear dispersion relation

$$\omega_k = \alpha + (\beta - \alpha)k^2.$$  \hspace{1cm} (5)

From eq. (5), the group velocity and the phase velocity can be analytically derived, i.e., $v_{ph} = d\omega_k/dk = 2(\beta - \alpha)k$ and $v_{ph}/k = \omega_k/k$. In terms of these quantities, for NWs, $v_{ph} > 0$ and for AWs, $v_{ph} < 0$; while for normal spirals (NSs) or antispirals (ASs), $v_{ph} > 0$ still holds. For the CGLE, the parameter regimes under which NSs or ASs exist are clear [6,7]. This is one of reasons why we choose this equation for our study.

To model the system with a circular interface (denoted by $M$), we vary the parameters $\alpha$ and $\beta$ in eq. (1) in two domains separated by the interface located at $R$. More specifically,

$$\alpha(r), \beta(r) = \begin{cases} \alpha_1, \beta_1, & r < R, \\ \alpha_2, \beta_2, & r \geq R. \end{cases}$$  \hspace{1cm} (6)

$R = \sqrt{(x-x_c)^2 + (y-y_c)^2}$ and $(x_c, y_c)$ is the center of the system. We call the region $(r < R)$ domain I and the other $(r \geq R)$ domain II for simplicity of the description in the following paragraphs.

We start our work from the uniform initial condition (e.g., we set $Z = 0.5$ initially) for the CGLE along with eq. (6). For a set of appropriate parameters $\alpha_1$, $\beta_1$, $\alpha_2$ and $\beta_2$, concentric waves arise, as shown in figs. 1(a) and (b). In fig. 1(a), we take $\alpha_1 = -0.5$, $\beta_1 = 1.0$, $\alpha_2 = 0.04$ and $\beta_2 = -1.4$. For these parameters, domain I generally supports waves traveling toward the interface and domain II supports waves propagating away from the interface. In this sense, the whole concentric waves seem to travel outward as seen in fig. 1(c), the space-time plot of fig. 1(a). In fig. 1(b), we take $\alpha_1 = 0.01$, $\beta_1 = -1.4$, $\alpha_2 = -0.5$ and $\beta_2 = 1.4$. In contrast to fig. 1(a), waves travel away from the interface in domain I and propagate.
toward the interface in domain II for these parameters. As a result, the whole concentric waves seem to move inward as seen from fig. 1(d), the space-time plot of fig. 1(b).

As a further step, we are now ready to consider a more complex case that a topological defect is enclosed by the circular interface. To create the topological defect approximately at the center of the system (see the snapshot in fig. 2(a1)), we employ the cross-gradient initial condition [17]. In this case, one finds that the circular interface selected wave patterns, are new and not negative, for the same type of waves.

The first point worth being mentioned is on the wave source. It is well known that for the previously found target or spiral, its center (pacemaker or spiral core) acts as the wave source. However, for the interface selected concentric or the spiral pattern, its source is located at the interface instead of the center. To show this point, we apply artificial perturbations in small regions in both domains. From figs. 2(b2) and (c2), one can see evidently that the perturbation transports toward the spiral core in domain I and moves away from the spiral core in domain II. This fact is also true for the interface selected concentric waves, see figs. 1(c) and (d). These evidences strongly suggest that the circular interface but not the center (e.g., spiral core) acts as the wave source for these circular-interface selected wave patterns. As a comparison, we present the transport direction of the perturbation in fig. 2(d2), the corresponding space-time plot of the AS in fig. 2(d1). In this case, the perturbation is always away from the core, independent of the location where it is applied.

Another point is on the phase velocity. It is found that the phase velocities of the interface selected wave patterns in two domains are opposite. For example, in fig. 2(b1), we find \( v_{ph}^I < 0 \) in domain I and \( v_{ph}^II > 0 \) in domain II. Waves in domain I moving away from the center actually propagate toward the source (please note that the origin is now located at the interface but not at the center for the present case) and thus \( v_{ph}^I < 0 \); while waves in domain II moving away from the center still propagate away from the source (or interface) and so \( v_{ph}^{II} > 0 \). In this sense, the phase velocities with opposite directions coexist in an “integrated” wave pattern. This feature differs from the one of the previously found target waves or spiral waves in that there is only one phase velocity, either positive or negative, for the same type of waves.

The above-mentioned features, particularly for the interface selected spiral patterns, are new and not explored so far. To tell them from previously found targets or spirals and considering the crucial roles played by the interface, we call the concentric waves in fig. 1(a) or (b) the interface selected concentric waves (ISCWs) and call spiral patterns in fig. 2(b1) or (c1) the interface selected spiral patterns (ISSPs).
Fig. 3: (Color online) Typical examples illustrating the dependence of the frequency (a) and the wave number (b) of ISSPs on $\beta_2$. The open circle (or square in (b)) and cross denote the absolute value of the frequency and the wave number of ISSPs in domain I and II, respectively. The solid lines in (a) and (b) are calculated according to eq. (8) and eq. (9), respectively. To get these plots, we fix $\alpha_1 = -0.5$, $\beta_1 = 1.0$ and $\alpha_2 = 0.2$.

The third point we are concerned with is about the frequency and the wave number. The control parameters in the two domains are totally different, however we find that the frequencies (or the wave numbers) in the two domains are identical. Figure 3(a) shows a typical dependence of the frequency $\omega_{k_1,k_2}$ on $\beta_2$ for ISSPs ($\omega_{k_1,k_2}$ and $k_{1,2}$ denote the frequency and the wave number in domain I and II, respectively). The overlap of the open circle and the cross suggests that $\omega_{k_1} = \omega_{k_2}$, as seen in fig. 3(a). In this sense, two domains with totally different control parameters are synchronized by the wave propagation. The dependence of the frequency on $\beta_2$ also reveals that varying parameters in domain II (e.g., $\beta_2$) can effectively change the frequency of the ISSPs. In other words, properties (e.g., frequency) of the ISSPs are not only determined by the parameters in the domain I, where the core of the ISSPs is located, but also by the parameters in domain II.

The wave numbers $k_{1,2}$ in two domains as $\beta_2$ varies, as an example for ISSPs, are also measured (see fig. 3(b)). As we find from this figure, the wave numbers are also identical in the two domains, like the frequencies. Due to the same wave numbers in the two domains, it looks as if ISSPs propagate in a homogeneous medium without experiencing any difference across the interface (see figs. 2(b1)–(c1) and (b2)–(c2)), similar to the 1D case [15]. The above facts are also true for ISCWs.

The dependence of the numerical values of the frequency and the wave number, as we note, is consistent with the one from the theoretical prediction if we assume that

$$\omega_{k_1} = \omega_{k_2}, \quad k_1 = k_2. \quad (7)$$

Substituting eq. (7) into eq. (5), one can easily get

$$\omega_{k_1} = \omega_{k_2} = \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{(\beta_2 - \beta_1) - (\alpha_2 - \alpha_1)} \quad (8)$$

and

$$k_1^2 = k_2^2 = \frac{\alpha_1 - \alpha_2}{(\beta_2 - \beta_1) - (\alpha_2 - \alpha_1)}. \quad (9)$$

The theoretical dependence of $|\omega_k|$ and $|k|$ on $\beta_2$ for ISSPs is illustrated in figs. 3(a) and (b) (see the solid line), respectively. One can see that the numerical results in fig. 3 are well fitted by the curve from eq. (8) and eq. (9).

The formation of ISSPs in figs. 2(b1) and (c1) is closely associated with ISCWs. To be concrete, ISSPs (small full circles) emerge only in the parameter regimes where ISCWs (large open circles) are found, as seen from fig. 4. Additionally, to create ISSPs, there is a necessary frequency condition between ISCWs and spirals in a uniform CGLE with the same parameters of ISCWs in domain I. For example, in fig. 5(a), we show the dependence of the frequency of ISCWs with $\alpha_1 = -0.5$, $\beta_1 = 1.0$.
and $\alpha_2 = 0.0$ (open circles) on $\beta_2$ and the frequency of spirals (solid line) in a uniform CGLE with $\alpha = \alpha_1 = \alpha_2 = -0.5$, $\beta = \beta_1 = \beta_2 = 1.0$, the same parameters of ISCWs in domain I. In this case, only if the frequency of ISCWs is slower than that of spirals (actually ASs in this case), ISSPs can arise (full circles). Similarly, in the case of $\alpha_1 = 0.01$, $\beta_1 = -1.4$ and $\alpha_2 = -0.2$ (see fig. 5(b)), only if the frequency of ISCWs is faster than that of spirals (actually NSs in this case), ISSPs can arise (full circles). These facts are in accordance with the results in figs. 4(a) and (b) as well.

The above facts can be well explained by the wave competition, the underlying mechanism of the formation of ISSPs. Explicitly, on one hand, the topological defect created by the cross-gradient condition initially in domain I tends to self-evolve to a spiral wave (see the wave denoted by “1” in the snapshot in fig. 2(a2)); on the other hand, the circular interface selects rotating waves instead of concentric waves (see the wave denoted by “2” in the snapshot in fig. 2(a2)) because of the cross-gradient condition. As a result, two kinds of rotating waves (i.e., the “1” wave and the “2” wave) coexist initially and they could compete with each other as time progresses. It is their frequencies that determine which wave survives. For example, in the case of fig. 2(b1), we get the frequency of the “1” wave $|\omega_1| \approx 0.236$ and the frequency of the “2” wave $|\omega_2| \approx 0.221$. Please bear in mind that both waves in this case (in domain I) propagate toward the source (i.e., the “1” wave propagates toward the spiral core and the “2” wave propagates toward interface). In other words, these waves are ingoing. According to the competition rule that the slower one wins if ingoing waves compete with ingoing waves [15], for fig. 2(b1) $|\omega_1| > |\omega_2|$, so the “2” wave suppresses the “1” wave in domain I and ISSPs emerge finally. Similarly, for fig. 2(c1), we get $|\omega_1| \approx 0.993$ and $|\omega_2| \approx 0.212$. In the present case, both waves in domain I propagate outwardly. According to the competition rule that the faster one wins if outgoing waves compete with outgoing waves [15], the “2” wave still suppresses the “1” wave due to $|\omega_1| < |\omega_2|$, so ISSPs also emerge.

It is worth mentioning that the frequency of the interface selected waves is independent of the type of waves. In other words, the frequencies of ISCWs and ISSPs are the same for the same control parameters. Due to this reason, that the frequency of ISCWs is smaller (larger) than the “1” wave means that the frequency of ISSPs (i.e., the “2” wave) is smaller (larger) than the “1” wave. This is the reason why ISSPs arise only if the frequency of ISCWs is slower (or larger) than that of spirals (ASs in fig. 5(a) and NSs in fig. 5(b)). Additionally, compared to ISSPs, the formation of ISCWs is not from the wave competition and needs no additional frequency requirement. Therefore, the parameter regimes for ISCWs are much broader than the ones for ISSPs, see fig. 4.

Finally, we would like to point out that the topological charges are conserved during the formation of ISSPs. The formation of ISSPs, as we discussed, is via the wave competition between two types of rotating waves in the domain I: one is from the topological defect and the other is selected by the circular interface. It has been well known that during the process of wave collisions, only pairs of topological charges with opposite signs can be destroyed or created, and the net topological charges are conserved [2,11,20]. During the formation process of ISSPs, no wave breaks and no topology defects are swept through the interface, so the topological charges of the ISSPs should be the same as the initial ones. The law of topology conservation seems a hidden reason accounting for that the circular interface selects spiral shaped waves instead of other types of waves when a topological defect is present in the system.

**Conclusion.** – In summary, we have investigated the selection of wave patterns in the CGLE with a circular interface. The interface can select concentric or spiral wave patterns depending on the initial conditions. These interface selected waves seem similar to the previously found target waves or spiral waves at first glance, however they possess some new features including: the source of these wave patterns hides at the interface and thus artificial perturbation would transport in opposite directions; as a direct consequence, the phase velocities in the “integrated” wave pattern are opposite; interestingly, the frequencies and the wave numbers in two domains with totally different control parameters are identical and thus it looks as if the waves propagate in a homogeneous medium without experiencing any difference in domain I and II. For the interface selected spiral pattern,
we have argued that the wave competition and topological constraints are responsible for the emergence of this new spiral pattern.

Our present studies deal with the chemical systems, however they may lead to interesting consequences in non-chemical systems, such as cardiac tissue. It is reported that inward propagation of spiral waves can only occur in oscillatory systems and thus cannot occur in the heart, in which the tissue is excitable [21]. However, in some particular cases, the cardiac tissue can be regarded as mixtures of excitable and oscillatory cells [22]. In this case, if the interior region composed of excitable cells is arranged to be circularly enclosed by the exterior region with oscillatory cells supporting ingoing waves, it may be possible to observe inward propagation of spiral waves. These inward-propagating waves may be related to the leading-circle hypothesis, another mechanism of functional reentry [21]. Additionally, investigations of the interface problems in chemical systems may broaden our understanding of chemical wave propagation in other heterogeneous media and provide potential implications in diverse fields such as optical materials [13], condensed matters [23], granular media [24] and so on.

***

This work was supported by the National Nature Science Foundation of China (Grants No. 10947021, 10975117 and 10774130).

REFERENCES