Topological quantum phases of dipolar Fermi gases

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Motivation:

Experimental realizations of ultracold dipolar Fermi gases such as $^{40}$K$^{87}$Rb, $^{167}$Er, and $^{161}$Dy potentially provide a platform to study novel many-body phenomena.

Advantages:

(1) Anisotropy and long range of dipole-dipole interaction
(2) Manipulation by DC and microwave AC fields

Problems to overcome:

inelastic scatterings, cooling…
Outline

(1) Introduction-theoretical proposals

(2) $p_x + ip_y$ superfluid in 2D optical lattice

(3) Weyl superfluidity in 3D

(4) Fractional Chern states

(5) Summary
For dipoles fixed in \( z \)-direction

\[
V_{dd}(r) = \frac{d^2}{r^3} (1 - 3 \cos^2 \theta)
\]

Attraction in \( z \)-direction leads to superfluid gap

\[
\Delta(k) \propto \cos \theta
\]

$P_x$–wave superfluid in 2D

For dipoles trapped in x-y-plane

For dipoles trapped in x-y-plane

A microwave field is needed to turn the dipole interaction into an attraction

$$V_{dd}(r) \propto \frac{d^2}{r^3}$$

leading to superfluid phase

$$T_c \approx E_F \exp\left(-\frac{3\pi}{4k_F r^*}\right)$$

Rich phases in 2D optical lattice

Phase diagram of dipoles in DC field at half filling

$p$-wave bond order solid ($\text{BOS}_p$), $d$-wave bond order solid ($\text{BOS}_d$), checkerboard charge density wave ($\text{cb-CDW}$), and $p$-wave BCS superfluid ($\text{BCS}$)


(2) $p_x + ip_y$ superfluid in 2D optical lattice

Dipole interaction can be tuned by a rotating field.


In rotating electric field ($\theta = \pi/2$)

$$E(t) = E[\cos \varphi \hat{z} + \sin \varphi (\cos \Omega t \hat{x} + \sin \Omega t \hat{y})],$$

the effective interaction is time-averaged

$$V_{dd}(r) = \frac{d^2 (3 \cos^2 \varphi - 1)}{2r^3},$$

attractive with $\cos \varphi < \sqrt{1/3}$.
Effective model in a 2D optical lattice


\[ H = - \sum_{\langle ij \rangle} t(c_i^\dagger c_j + c_j^\dagger c_i) + \frac{1}{2} \sum_{i \neq j} V_{i-j} c_i^\dagger c_j c_j c_i, \]

\[ V_{i-j} = V_{dd}(|\mathbf{r}_i - \mathbf{r}_j|), \quad J \equiv |V_{dd}(a)| \]

Particle-hole symmetry:

Under the transformation \( c_i \rightarrow (-1)^i c_i^\dagger \),

\[ H \rightarrow H - \frac{1}{2} \sum_{i \neq i} V_{i-j}(n_i + n_j - 1) = H - V(0)(\hat{N}_f - 0.5N), \]

\[ |\psi(n)\rangle \rightarrow |\psi(1 - n)\rangle, \quad \mu(n) + \mu(1 - n) = V(0), \]

\[ V(0) = \sum_{m \neq 0} V_m = -9.02J \]
In superfluid phase, interaction energy $E_i$ includes:

Hartree energy 

$$E_h = \frac{1}{2} \sum_{i \neq j} V_{i-j} n^2,$$

Fock energy 

$$E_x = -\frac{1}{2} \sum_{i \neq j} V_{i-j} \langle c_i \dagger c_j \rangle \langle c_j \dagger c_i \rangle,$$

Pairing energy 

$$E_p = \frac{1}{2} \sum_{i \neq j} V_{i-j} \langle c_i \dagger c_j \rangle \langle c_j c_i \rangle.$$

The superfluid state may become unstable if the negative interaction energy dominates over the kinetic energy.
Mean-field Hamiltonian

\[ H' - \mu \hat{N}_f = \sum_k \left[ \xi_k c_k^\dagger c_k + \frac{\Delta_k^*}{2} c_{\mathbf{k}} c_{\mathbf{-k}} + \frac{\Delta_k}{2} c_k^\dagger c_{\mathbf{-k}}^\dagger \right] - E_I, \]

where \( \xi_k = \varepsilon_k + \Sigma_k - \mu \),

bare band \( \varepsilon_k = -2t (\cos k_x a + \cos k_y a) \),

Hartree-Fock self-energy \( \Sigma_k = V(0)n - \frac{1}{N} \sum_{k'} V(\mathbf{k} - \mathbf{k'}) n_{k'} \),

superfluid gap \( \Delta_k = \frac{1}{N} \sum_{k'} V(\mathbf{k} - \mathbf{k'}) g_{k'} \),

pair amplitude \( g_k = \left\langle c_{\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right\rangle \)
Quasi-particle energy

\[ E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} \]

HF self-energy equation

\[ \Sigma_k = V(0)n - \frac{1}{N} \sum_{k'} V(k - k') \frac{1}{2} \left[ 1 - \frac{\xi_{k'}}{E_{k'}} \tanh \left( \frac{\beta}{2} E_{k'} \right) \right], \]

gap equation

\[ \Delta_k = -\frac{1}{N} \sum_{k'} V(k - k') \frac{\Delta_{k'}}{2E_{k'}} \tanh \left( \frac{\beta}{2} E_{k'} \right), \]

density equation

\[ n = \frac{1}{2} \left[ 1 - \frac{1}{N} \sum_k \frac{\xi_k}{E_k} \tanh \left( \frac{\beta}{2} E_k \right) \right]. \]

The three equations can be solved together self-consistently.
$p_x + ip_y$ superfluid symmetry

$\Delta_k$ versus wavevector $k$ at $J/t = 0.6$, $n = 0.11$, and $T = 0$

(a) At $k_y a / \pi = 0$

Re($\Delta_k$)

(b) At $k_x a / \pi = 0$

Im($\Delta_k$)

$\Delta_k \propto \left( \sin k_x a + i \sin k_y a \right)$
Phase separation appears.

Phase separation is between $n=0.12$ and $n=0.88$. 

$\mu$ versus the filling factor $n$ at $J/t = 0.7$ and $T = 0$.
(1) Phase separation region increases with dipole interaction strength $J$.

(2) When $J>0.89t$, the system is only stable at $n=0$ or $n=1$. 
In 3D, with the same rotation scheme to tune the dipole-dipole interaction as, the effective interaction is attractive in \(x\)-\(y\) plane and repulsive in \(z\)-direction,

\[
V(r) = \frac{d^2(3\cos^2\varphi - 1)}{2r^3}(1 - 3\cos^2\theta) \equiv \frac{d^2}{r^3}(1 - 3\cos^2\theta),
\]

which at low temperatures leads to a \(p_x + ip_y\)–wave superfluid order

\[
\Delta_k \equiv \Delta(k_\rho, k_z)e^{i\varphi_k}
\]

Quasiparticle excitation energy \(E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}\) has two gapless points along \(z\)-axis at Fermi surface, which are Weyl points, \(k_\parallel \approx (0, 0, k_\parallel) = -k_\parallel\).
The mean-field Hamiltonian is consistent with 2D Dirac equation,

\[ H_{SF} = \sum_\mathbf{k} (\mathbf{c}_\mathbf{k}^\dagger, \mathbf{c}_{-\mathbf{k}}) \begin{pmatrix} \frac{\xi(\mathbf{k})}{2} & \frac{\Delta(\mathbf{k})}{2} \\ \frac{\Delta^*(\mathbf{k})}{2} & -\frac{\xi(\mathbf{k})}{2} \end{pmatrix} \begin{pmatrix} \mathbf{c}_\mathbf{k} \\ \mathbf{c}_{-\mathbf{k}}^\dagger \end{pmatrix} \]

\[ \equiv \sum_\mathbf{k} (\mathbf{c}_\mathbf{k}^\dagger, \mathbf{c}_{-\mathbf{k}}) \mathbf{d}(\mathbf{k}) \cdot \mathbf{\sigma} \begin{pmatrix} \mathbf{c}_\mathbf{k} \\ \mathbf{c}_{-\mathbf{k}}^\dagger \end{pmatrix}, \]
Properties

(a) Density of states with quadratic dependence near zero energy.
(b) Linear dispersion of quasiparticle energy.
(c) Hedgehoglike form of $\vec{d}(\mathbf{k})$ near Weyl points
Anisotropic superfluid gap

(a) Angle-dependence of the gap on the Fermi surface, with maximum in the x-y plane.
(b) In-plane k-dependence

\[ J \equiv \left| \frac{md^2}{\hbar^2} k_F \right| \]
The maximum mean-field $T_c$ is about $0.2 \, T_F$. As the interaction strength further increases, the system suffers mechanical collapse.
(4) Fractional Chern states


For dipoles in a Chern band where the Chern number is nontrivial, is dipolar interaction enough for producing Fractional Chern states at fractional fillings?
Checkerboard Model

\[ H_0 = \sum_{\langle r, r' \rangle} \left[ t_1 e^{i\phi(r, r')} a_r^\dagger b_{r'} + h.c. \right] + \sum_{\langle\langle r, r' \rangle\rangle} \left[ t_a(r, r') a_r^\dagger a_{r'} + t_b(r, r') b_{r'}^\dagger b_r \right] \]

\[ + \sum_{\langle\langle\langle r, r' \rangle\rangle\rangle} \left[ t_3 a_r^\dagger b_{r'} + h.c. \right] + \sum_r M(a_r^\dagger a_r - b_r^\dagger b_r), \]

\[ \phi(r, r') = \phi \times \text{Sign}[(x' - x)(y' - y)], \]

\[ t_a(r, r') = t_2(-1)^{y' - y} \]

\[ t_b(r, r') = t_2(-1)^{x' - x} \]
In pseudospin representation

\[
H_0 = \sum_k \psi_k^\dagger (h_k^0 + h_k^x \sigma_x + h_k^y \sigma_y + h_k^z \sigma_z) \psi_k, \quad \psi_k^\dagger = (a_k^\dagger, b_k^\dagger),
\]

\[
h_k^0 = -2t_3 [\cos(k_x + k_y) + \cos(k_x - k_y)], \quad h_k^z = M + 2t_2 (\cos k_x - \cos k_y),
\]

\[
h_k^x = 4t_1 \cos \phi \cos \frac{k_x}{2} \cos \frac{k_y}{2}, \quad h_k^y = 4t_1 \sin \phi \sin \frac{k_x}{2} \sin \frac{k_y}{2}
\]

Lowest Band \quad \[E_k = h_k^0 - \varepsilon_k \quad \varepsilon_k = \sqrt{(h_k^x)^2 + (h_k^y)^2 + (h_k^z)^2}\]

Nearly flat band at

\[
M = 0, t_2 = 0.3t_1,
\]

\[
t_3 = -0.2t_1, \quad \phi = \pi/4,
\]
Ground and excited state energies at fractional fillings with dipolar interaction by exact diagonalization

Hardcore bosons at fillings
(a) 1/2, (b) 1/4;

Fermions at fillings
(c) 1/3, (d) 1/5

Degenerate Fractional Chern states as ground states
The role of long-range interaction

Hardcore bosons at fillings
(a) 1/2, (b) 1/4 ;

Fermions at fillings
(c) 1/3, (d) 1/5

Long-range interaction is helpful to stabilize fractional Chern states
Phase transitions with dipolar interaction strength

Critical interaction strengths:

For fermions, $J=0.18$ at $1/3$, $J=0.47$ at $1/5$;

for bosons, $J=0$ at $1/2$, $J=0.24$ at $1/4$. 
With twisted boundary condition (TBC)

\[ \psi(r + N_x a) = e^{i\theta_x} \psi(r) , \psi(r + N_y a) = e^{i\theta_y} \psi(r) \]

flux insertion is equivalent to phase change in TBC.

**Energy flow with flux insertion**

Harcore bosons at fillings:
(a) 1/2, (b) 1/4;

Fermions at fillings:
(c) 1/3, (d) 1/5
Indication of crystalline order in 1D limit from density structure factors

Hardcore bosons at fillings
(a) 1/2, (b) 1/4 ;

Fermions at fillings
(c) 1/3, (d) 1/5
(5) Summary

- Novel quantum states, such as topological superfluids and fractional Chern states, may be created in a dipolar Fermi gas in optical lattices.
- There are experimental difficulties to be solved, including inelastic scattering and cooling.

Thank you!