Dynamic Monte Carlo Measurement of Critical Exponents

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Based on the scaling relation for the dynamics at the early time, a new method is proposed to measure both the static and dynamic critical exponents. The method is applied to the two-dimensional Ising model. The results are in good agreement with the existing results. Since the measurement is carried out in the initial stage of the relaxation process starting from independent initial configurations, our method is efficient.

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Numerical measurements of critical exponents are usually carried out from samples of configurations in the equilibrium generated with Monte Carlo algorithms. For the static exponents, Binder's method is one of the widely accepted [1,2]. A traditional way to obtain the dynamic exponent z is to measure the exponential decay of the time correlations for finite systems in the long-time regime [3,4]. At the critical point, except for some special algorithms as, e.g., the cluster algorithm [5,6], one suffers from critical slowing down. Recently, it has been suggested that z may be estimated from the power law decay of the magnetization in a relaxation process on a large enough lattice, in a sufficient large time but before the exponential decay starts [7,8].

In the past few years, better understanding has been achieved of the critical relaxation processes even up to the early time of the evolution. A representative example of these processes is that the Ising model initially in random states with a small magnetization is suddenly quenched to the critical temperature and then evolves with the dynamics of model A. It was shown by Janssen, Schaub, and Schmittmann [9] with ε expansion up to two-loop order that, besides the well-known long-time universal behavior, there exists another *universal* stage of the relaxation at earlier time, termed *"critical initial slip,"* which sets in right after the microscopic time scale and eventually crosses over to the long-time regime. For the critical initial slip the characteristic time scale is $t_0 \sim m_0^{-z/x_0}$ with x_0 being a new independent critical exponent and m_0 the initial magnetization. The scaling behavior including the increase of the order has been illustrated by a number of authors by Monte Carlo simulations [10–13], and also analytical calculations [14–17].

Of special interest here is the extension of the results in Ref. [9] to finite-size systems [13,14]. In accordance with the renormalization group analysis for finite-size systems, we expect a scaling relation to hold for the kth moment of the magnetization in the neighborhood of the critical point [9,18,19],

$$M^{(k)}(t,\tau,L,m_0) = b^{-k\beta/\nu}M^{(k)}(b^{-z}t, b^{1/\nu}\tau, b^{-1}L, b^{x_0}m_0), \tag{1}$$

where t is the evolution time, $\tau = (T - T_c)/T_c$ is the reduced temperature, L is the lattice size, and b is the spatial rescaling factor. It has been stressed in Ref. [9] that the initial states must have very *short* correlation lengths and the initial magnetization m_0 must be *sharply* prepared. The scaling relation for the three-dimensional Ising model has been tested by Monte Carlo (MC) simulation [13]. The numerical data fit into the scaling relation nicely, and x_0 has been determined with satisfactory precision. It was observed that the microscopic time scale is ignorably small; for example, for the heat-bath Monte Carlo algorithm it is smaller than one MC sweep. This clean behavior of the critical relaxation in its *early time* indicates a promising new way to measure both the *static* and *dynamic* critical exponents, which is similar to Binder's method in equilibrium. In the present Letter, we will illustrate this idea for the two-dimensional Ising model.

To make the computation simpler and more efficient, we set m_0 to its fixed point, $m_0 = 0$. Therefore the exponent x_0 will not enter the calculation. Furthermore, now the time scale $t_0 = m_0^{-z/x_0} \to \infty$, and the critical initial slip gets most prominent in time direction even though the magnetization itself will only fluctuate around zero. The choice of $m_0 = 0$ is essential in the calculation, which allows a more precise measurement of the critical exponents. After generating randomly an initial configuration the system is released to the evolution with the heat-bath algorithm at the critical temperature. We repeat this process with independent initial configurations. The average is taken over the initial configurations with $m_0 = 0$ and zero correlation length.

To determine z, we introduce a *time-dependent* Binder cumulant [2]

$$U(t,\tau,L) \equiv 1 - \frac{M^{(4)}}{3(M^{(2)})^2}. \tag{2}$$

Following the scaling relation in Eq. (1), cumulants measured in two different lattices have the simple equation

$$U(t,0,L_1) = U(b^{-z}t,0,L_2), \tag{3}$$

with $b = L_1/L_2$. The exponent z can easily be obtained through searching for a time rescaling factor b^{-z} such

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that the two curves represented by both sides in Eq. (3) collapse. In other words, the cumulant can be described by a scaling function

$$f(t/L^z) = U(t,0,L). \tag{4}$$

In the same way, it is easy to obtain the other two scaling functions

$$g^{(2)}(t/L^z) = L^{2\beta/\nu}M^{(2)}(t,0,L), \tag{5}$$

$$h(t/L^z) = L^{-1/\nu}\partial_\tau \ln M^{(2)}(t,\tau,L)|_{\tau=0}. \tag{6}$$

With z in hand, β/ν and ν can be determined by fitting respectively the scaling function $g^{(2)}(t/L^z)$ and $h(t/L^z)$ from different lattices, similar as in the determination of z. We adopt the derivative of $M^{(2)}(t,\tau,L)$ for the estimation of ν, rather than that of $U(t,\tau,L)$, since the measurement of $M^{(2)}$ is more stable for large L. To our knowledge, so far the measurement of the static and dynamic exponents from the dynamic process in the early time has not been investigated. One can easily realize that at time $t \rightarrow \infty$ the traditional way to calculate β/ν and ν is recovered [1]. [With a similar procedure used by Binder, the critical point can also be located from the behavior of $U(t,\tau,L)$ around $\tau = 0$.] Since the measurement can now be carried out already in the initial stage of the relaxation, the method becomes more efficient.

In Fig. 1, we plot the cumulants of the two-dimensional Ising model versus time. The curves corresponding to $L = 8$, 16, and 32, respectively, are labeled by +, ◇, and *. Each curve is compared with the time-rescaled cumulant of double lattice size given by solid lines. The exponents corresponding to the three pairs of best-fitted curves are $z = 2.0969$, 2.1493, and 2.1337, respectively. Our best value 2.1337 should be compared with the existing numerical results from [4], $z = 2.13(8)$, and, from [20], $z = 2.14(5)$, and also with that with the ϵ expansion from [21], $z = 2.126$, even though a relatively bigger value of

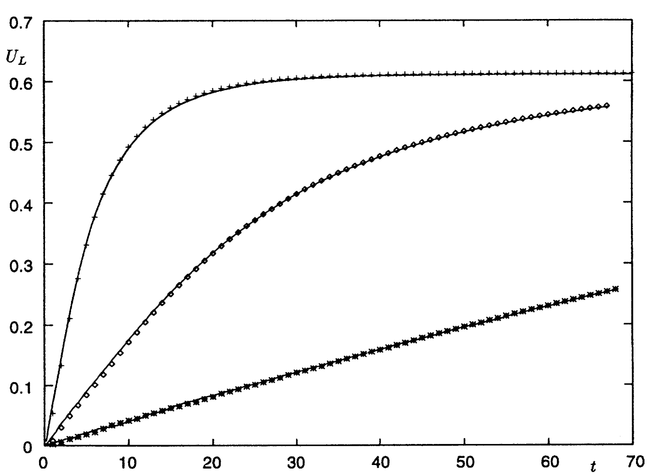

FIG. 1. The cumulants U_L for $L = 8$, 16, and 32 (labeled by +, ◇, and *, respectively) together with the rescaled cumulants of double lattice (solid lines).

z measured from the long-time behavior of the dynamic relaxation is also reported in some recent papers [7,8,22]. From the figure one can also see clearly that the scaling relation holds from the very beginning of the time evolution and how remarkably well the method is working.

In Table I, we have summarized the measured exponents z, $2\beta/\nu$, and $1/\nu$ for the two-dimensional Ising model. They are in good agreement with the exact results [23] and best values for z [4,20,21]. To obtain these results, we have averaged over 50 000 independent samples in each run and repeated 8 runs. Each run obtained for lattice size L was compared with each run for lattice size $2L$, and from these 64 values the average value and the error were estimated. The fluctuation as well as the finite-size effect are bigger for the measurement of $1/\nu$ than for that of z and $2\beta/\nu$, especially in the beginning of the time evolution. The measurement of $1/\nu$ from the exact value may be due to our way of estimating the mean value and the error. A more careful analysis, discarding the data in the first 10 time steps, yields $1/\nu = 0.997(40)$ for $L_1 = 32$ and $L_2 = 64$. The independent comparison of each of the 8 data sets for each lattice size may also underestimate the error. Details will be reported elsewhere [24].

Compared with traditional measurements in equilibrium, much less effort is needed with our *dynamic* Monte Carlo algorithm, at least for the dynamic exponent z, since we do not enter the long-time regime where critical slowing down is severe. For the measurement of static exponents, whether our measurement can be compatible with or superior to some special logarithms like, e.g., the cluster algorithm still needs further investigation. The efficiency of our method may more or less be traced back to the simple power law increase of the moments at the early time. For example, it is well known that $M^{(2)}(t,0,L) \sim t^{(d-2\beta/\nu)/z}$ when L is large enough [10,11]. Therefore the scaling relation implies $M^{(2)}(t,0,L) \sim L^{-d}t^{(d-2\beta/\nu)/z}$. Similar arguments exist for the cumulant $U(t,\tau,L)$.

In previous works on the critical initial slip [9,13,14,17], the increase of the order and the role of m_0 in the short- and long-time regimes are intensively discussed. We would like, however, to stress that even in the case of $m_0 = 0$ it is interesting and there exist fruitful applications, as reported in this Letter. Our results also provide a further confirmation of the scaling relation discovered by Janssen, Schaub, and Schmittmann [9]. Investigations for dynamics in other universality classes should be carried out.

TABLE I. Results for z, $2\beta/\nu$, and $1/\nu$, respectively, from the two-dimensional Ising model.

$L_1 \leftrightarrow L_2$	z	$2\beta/\nu$	$1/\nu$
$8 \leftrightarrow 16$	2.0969(20)	0.2480(02)	1.127(08)
$16 \leftrightarrow 32$	2.1493(18)	0.2494(06)	1.058(35)
$32 \leftrightarrow 64$	2.1337(41)	0.2504(29)	0.955(40)

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Note added.—After the present Letter was completed, we received a preprint of P. Grassberger (Wuppertal, FRG) where the critical initial slip related to damage spreading was investigated and the dynamic exponent z also measured with the Monte Carlo method in both two- and three-dimensional Ising models.

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