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Networking effects on cooperation in evolutionary snowdrift game

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Abstract. – The effects of networking on the extent of cooperation emerging in a competitive setting are studied. The evolutionary snowdrift game, which represents a realistic alternative to the well-known Prisoner's Dilemma game, is studied in the Watts-Strogatz network that spans the regular, small-world, and random networks through random re-wiring. Over a wide range of payoffs, a re-wired network is found to suppress cooperation when compared with a well-mixed system. Two extinction payoffs, that characterize the emergence of a homogeneous steady state, are identified. It is found that, unlike in the Prisoner's Dilemma game, the standard deviation of the degree distribution is the dominant network property that governs the extinction payoffs.

Evolutionary game theory has become an important tool for investigating cooperative or altruistic behavior in systems consisting of competitive entities. These systems may be biological, ecological, social, and political in nature, and scientists have found the emergence of cooperative behavior fascinating. The ground-breaking work on repeated games based on the Prisoner's Dilemma (PD) game by Axelrod [1] has led to much effort in exploring the determining factors of cooperative behavior in evolutionary games [2–9], with a recent emphasis on the effects of spatial structures such as regular lattices [4,5,10–12] and networks [7,13–18]. For example, cooperation can be induced in a repeated PD by cleverly designed strategies, and spatial structures are found to promote cooperative behavior in the evolutionary PD. The physics of networks [19] and emergent phenomena are among the most rapidly growing branches in physics.

Our work was motivated by the recent concerns on using the PD as the sole model for studying emerging cooperative phenomena [5]. Due to practical difficulties in quantifying the payoffs, the snowdrift game (SG) has been proposed as a possible alternative to the PD game. Previous work on the SG have focused on the effects of structures such as lattices [5] and fully connected networks [5,20]. Here, we investigate the networking effects on an evolutionary

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snowdrift game [5,21] within the Watts-Strogatz (WS) [19,22] model of small world constructed by re-wiring a regular lattice. Starting with a random mixture of competing nodes of opposite characters, we found that i) the population in the long-time limit may consist of only one kind of nodes or a mixture of nodes of different characters, depending on the degree K in a regular world before re-wiring and the extent of re-wiring q, ii) for a wide range of payoffs, a re-wired network *suppresses* the fraction of cooperative nodes, when compared with the well-mixed case, and iii) the critical payoffs for the extinction of one kind of nodes depends sensitively on the standard deviation of the degree distribution induced by re-wiring.

The snowdrift game, also called the hawk-dove or chicken game [21], is best described using the following scenario [5]. Consider two drivers heading home in opposite directions on a road blocked by a snowdrift. Similar to the PD game, each driver has two possible actions —to shovel the snowdrift (cooperate (C)) or not to do anything (not-to-cooperate or "defect" (D)). If the two drivers cooperate, they could be back home and each will get a reward of b. Shovelling is a laborious job with a total cost of c. Thus, each driver gets a net reward of $R = b - c/2$. If both take action D, they get stuck, and each gets a reward of $P = 0$. If only one driver takes action C and shovels the snowdrift, then both drivers get through. The driver taking action D (not to shovel) gets home without doing anything and gets a payoff $T = b$, while the driver taking action C gets a "sucker" payoff of $S = b - c$. The SG refers to the case of $b > c > 0$, leading to $T > R > S > P$. This ordering of the payoffs *defines* the SG. Without loss of generality, we assign $R = 1$ so that the payoffs can be characterized by a single parameter $r = c/2 = c/(2b - c)$ for the cost-to-reward ratio. In terms of $0 < r < 1$, we have $T = 1 + r$, $R = 1$, $S = 1 - r$, and $P = 0$. For a player, the best action is: to take D if the opponent takes C, otherwise take C. The SG becomes the PD game when the cost c is high such that $2b > c > b > 0$, which amounts to $T > R > P > S$ [23]. Therefore, the SG and PD differ only by the ordering of P and S. Due to the difficulty in measuring payoffs, the SD represents a possible alternative in studying emerging cooperative phenomena [5, 20].

Evolutionary SG (ESG) amounts to letting the character of the players in a connected inhomogeneous population evolve, based on their performance [5]. Consider a network of N players (nodes). Initially, the nodes are randomly assigned a character: C or D. The character of each node is updated every time step simultaneously. At each time step, every node i interacts with all its connected k_i neighbours and gets a payoff per neighbour $\overline{V}_i = V_i/k_i$, where V_i is the sum of k_i payoffs after comparing characters with its neighbours. Every node i then randomly selects a neighbour j for possible updating or evolution. To compare performance of nodes i and j, we construct

$$w_{ij} = (\overline{V_j} - \overline{V_i})/(T - P) = (\overline{V_j} - \overline{V_i})/(1 + r). \quad (1)$$

If $w_{ij} > 0$, the character of node i is *replaced* by the character of node j with *probability* w_{ij}, and thus node i becomes an offspring of node j. If $w_{ij} < 0$, the character of node i remains unchanged. In the long-time limit, a state will be attained with possible coexistence of characters. The fraction of C-nodes f_C, also called the frequency of cooperators [5], measures the level of cooperation and is determined by the network structure and the payoff parameter r. For a fully connected network, $f_C = 1 - r$ [5,20]. In two-dimensional lattices, the spatial structure tends to lower f_C [5], when compared with a fully connected network. In contrast, spatial connections are found to enhance f_C in evolutionary PD games [10]. While different updating rules have been studied [5,10,18], they tend to give qualitatively similar results. Here, we follow the replicator dynamics in ref. [5] and use a probabilistic updating rule based on w_{ij}. For inhomogeneous networks, the average payoffs $\overline{V_i}$ used in defining w_{ij} make sure that $w_{ij} < 1$ and tend to suppress the dominance of the nodes with higher degrees in the comparison of performance.

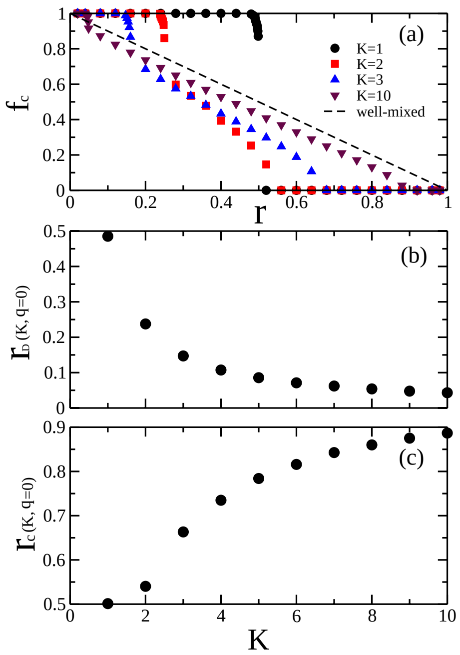

Fig. 1 – (a) The frequency of cooperators f_C as a function of the payoff parameter r in a Watts-Strogatz network of size $N = 10^3$ nodes with $K = 1, 2, 3, 10$ before re-wiring ($q = 0$). The run time is 10^4 steps. The dashed line gives the result $f_C = 1-r$ for a fully connected network. (b) The extinction payoff $r_D(K, q = 0)$ as a function of K. (c) The extinction payoff $r_C(K, q = 0)$ as a function of K.

Here, the nodes in the ESG are connected in the form of the Watts-Strogatz [22] model. Starting with a one-dimensional regular world consisting of a circular chain of N nodes with each node connected to its $2K$ nearest neighbours, each of the K links to one side of a node is cut and re-wired to a randomly chosen node with a probability q. Double and multiple links connecting two nodes are forbidden, and configurations with disconnected clusters are discarded. This model gives the small-world effect [22], which refers to the phenomena of small separations between two nodes observed in many real-life networks [19], for $q \sim 0.1$. The parameter q thus takes the network from a regular world through the small world to a random world, with a fixed mean degree $\langle k \rangle = 2K$. Here, we aim at understanding how the frequency of cooperators f_C behaves as the structural parameters K and q change, and as the payoff parameter r governing the evolutionary dynamics changes.

Figure 1(a) shows f_C as a function of r in *regular lattices* ($q = 0$) of $N = 10^3$ nodes with different values of K. The key features are: i) there exists a value $r_D(K, q = 0)$ so that $f_C = 1$ for $r < r_D$, *i.e.*, the extinction of D-nodes resulting in an all-C population; ii) for $r < r_D$, f_C is enhanced when compared with the well-mixed case (dashed line); iii) for a wide range of r, f_C drops below that in the well-mixed case; iv) there exists a value $r_C(K, q = 0)$ so that $f_C = 0$ for $r > r_C$, *i.e.*, the extinction of cooperative nodes [24]. These extinction payoffs thus characterize a transition between a homogeneous and an inhomogeneous population, as the payoff r is varied [25]. The extinction payoff for defectors (cooperators) r_D (r_C) decreases (increases) monotonically with K in regular lattices (fig. 1(b)-(c)). For $K = 1, 2$, $r_D(K, q = 0)$ is close to $1/2\,K$; while for $K \geq 3$, $r_D(K, q = 0)$ is closer to $1/(2K+1)$. In addition, we observe that for $K \geq 3$, the relation $2r_D + r_C \approx 1$ is satisfied. The feature iii) is analogous to that

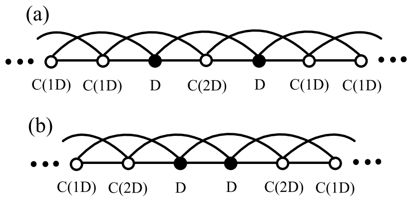

Fig. 2 – Two last surviving patterns in $K = 2$ regular lattices. These patterns consist of two connected D-nodes in a background of C-nodes. The C-nodes connected to one D-node (labelled C(1D)) and two D-nodes (labelled C(2D)) have different average payoffs, and r_D is found by equating the average payoffs $\overline{V_{C(1D)}}$ and $\overline{V_D}$.

observed in two-dimensional lattices [5]. We have also found that r_D and r_C are independent of the number of nodes N, as long as $N \gg 1$. An important quantity in the network is the shortest path $L = N/4K$, which scales with N in regular lattices. The results here imply that the K-dependence of the extinction payoffs does not come from L.

Analytically, we can estimate r_D by considering how patterns with a few D-nodes can be replaced. A single D-node connected to C-neighbours always survives, as the average payoff of the D-node is higher than the C-nodes for $0 < r < 1$. This leads us to consider patterns involving D-nodes, called the last surviving patterns, that can be replaced by C-nodes in *one* time step. Figure 2 shows two such patterns for $K = 2$. These patterns show two types of neighbouring C-nodes. For C-nodes connecting to only one D-node and two D-nodes, the average payoffs are $\overline{V_{C(1D)}} = 1 - r/2K$ and $\overline{V_{C(2D)}} = 1 - r/K$, respectively. In general, $\overline{V_{C(1D)}} > \overline{V_{C(2D)}}$. For the D-nodes, $\overline{V_D} = (1 + r)(1 - 1/2K)$. We estimate r_D by equating $\overline{V_D}$ and $\overline{V_{C(1D)}}$. Thus, *in principle*, $r_D(K, q = 0) = 1/2K$. This is observed for $K = 1$ and 2. For $K \geq 3$, $r_D \approx 1/(2K + 1)$. It is because for $1/(2K + 1) < r < 1/(2K)$, we have $\overline{V_{C(1D)}} > \overline{V_D} > \overline{V_{C(2D)}}$ and it becomes highly unlikely that the last surviving patterns become all-C. For $r < 1/(2K+1)$, $\overline{V_{C(1D)}} > \overline{V_{C(2D)}} > \overline{V_D}$ and replacing the D-nodes in *one* time step can be realized in reasonably long run times. We have tried longer run times for $K \geq 3$, *e.g.*, 3×10^5 time steps, and r_D only tends to approach $1/2K$ very slowly.

Figure 3(a) shows $f_C(r)$ in WS networks of $K = 3$ and $N = 10^3$ nodes with $q = 0.01$, 0.3 and 1.0. The results indicate that the effect of increasing q for fixed K is similar to increasing K for fixed $q = 0$ (regular lattices). As q increases, f_C is higher than that in a regular lattice with the same K for a large range of r. For fixed K, the extinction payoff r_D (r_C) drops (rises) with the extent of re-wiring q (see figs. 3(b) and (c)). We have checked that while the shortest path $L(q)$ changes sensitively with q, it is however not the determining factor for the extinction payoffs. For regular lattices, the clustering coefficient C increases with K and saturates at large K following $C(0) = 3(2K-2)/[4(2K-1)]$ [19,22]. If the drop in r_D with K in regular lattices were attributed to the increase in C, one would have expected r_D to *increase* with q as $C(q) \simeq C(0)(1 - q)^3$ [19, 22] in WS networks. This is, however, *not* what is observed in numerical results. The clustering coefficient C is thus not the determining factor for r_D and r_C. These observations are in sharp contrast to the networking effects in evolutionary PD games [13–17]. For WS networks, $\langle k \rangle = 2K$ with or without re-wiring, and r_D and r_C cannot be determined by $\langle k \rangle$. Thus, r_D and r_C are not determined by the commonly studied quantities in networks.

Figures 1 and 3 revealed that $r_D(K, q)$ and $r_C(K, q)$ depend on the structural parameters K and q. To pin down the geometrical property of a re-wired network that determines r_D and r_C, we focus on analyzing $r_D(K, q)$, as $r_C \approx 1 - 2r_D$. Figure 4 shows collectively the

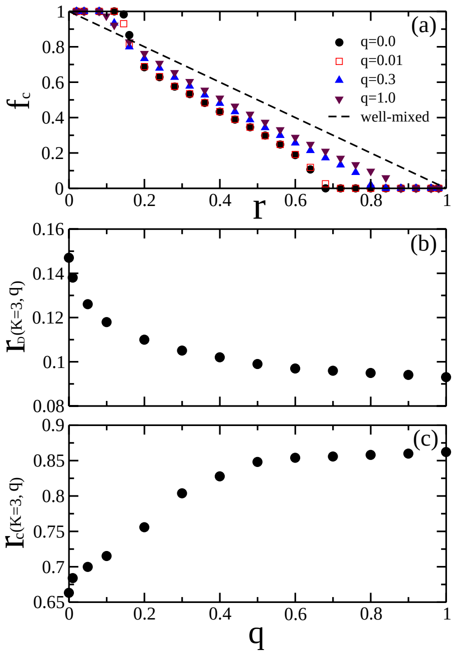

Fig. 3 – (a) The frequency of cooperators f_C as a function of the payoff parameter r in Watts-Strogatz networks of $K = 3$ for different values of the re-wiring probabilities. The run time is 10^4 time steps. The dashed line indicates the result $f_C = 1-r$ for a fully connected network. (b) The extinction payoff $r_D(K,q)$ as a function of q for $K = 3$. (c) The extinction payoff $r_C(K,q)$ as a function of q for $K = 3$.

results of $r_D(K,q)$ for different values of K and q. Interestingly, we found that by plotting $(1/r_D(K,q) - 1/r_D(K,q=0))$ as a function of q, the data follow the same functional form (see fig. 5). The choice of the y-axis avoids the difficulty in getting the expected value of $r_D(K,q=0)$ even for long run times [26]. Examining the q-dependence, the behaviour is

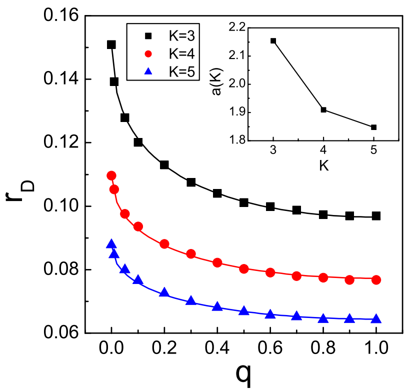

Fig. 4 – The extinction payoff $r_D(K,q)$ is plotted against q for different values of $K = 3$, 4, 5. The results are obtained by runs of 3×10^5 time steps. The lines are obtained by fitting to the form given by eq. (2). The inset gives fitting parameter $a(K)$.

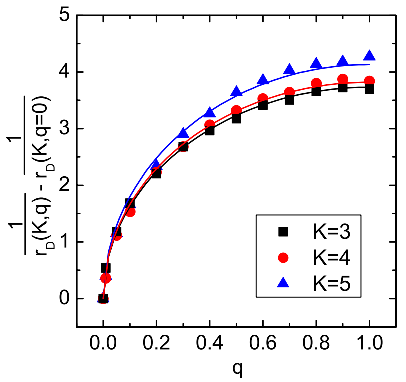

Fig. 5 – By suitably choosing the quantity on the y-axis, the scattered data in fig. 4 exhibit a similar q-dependence. The dependence is found to follow that of $\sqrt{2q-q^2}$, which is the q-dependence in the standard deviation of the degree distribution in WS networks after re-wiring for fixed K. The lines are obtained by using the fitting parameters obtained in fig. 4

similar to that of the *standard deviation* σ of the degree distribution $P(k)$ in WS networks. Re-wiring changes $P(k)$ gradually from a delta-function at $k = 2K$ for $q = 0$ to one that has a lower cutoff at $k = K$ and a finite $\sigma(K, q)$ that increases with q [19]. More specifically, $\sigma(K, q) = \sqrt{K} \cdot \sqrt{2q - q^2}$. Motivated by this observation, we perform a fit to the results in fig. 4 to the functional form

$$r_D(K, q) = \frac{1}{\frac{1}{r_D(K,0)} + a(K)\sigma(K, q)}, \tag{2}$$

where $a(K)$ is a fitting parameter. The fitted lines, which describe the data accurately, are also shown in fig. 4. This implies that the q-dependence in r_D follows that in $\sigma(K, q)$, *i.e.*, $\sim \sqrt{2q - q^2}$. The fitting parameters are found to be $a(3) = 2.154$, $a(4) = 1.910$, and $a(5) = 1.847$. Note that $a(K)$ drops with K (see the inset of fig. 4). Since $\sigma(K, q)$ goes like $\sqrt{K}$, the combination $b(K) = a(K)\sqrt{K}$ becomes quite insensitive to K, with b increasing from 3.73 for $K = 3$ to 4.13 for $K = 5$. As a consistency check, we also plot the fitted lines in fig. 5, using the same coefficients $a(K)$ as obtained in fig. 4. Note that even for $q = 1$, the spread in the data (and fitted lines) is small, reflecting the weak K-dependence in the coefficients $b(K)$. The extinction payoff is thus found to be well described by

$$\frac{1}{r_D(K, q)} = \frac{1}{r_D(K, q = 0)} + b(K)\sqrt{2q - q^2}, \tag{3}$$

where b carries only a weak K-dependence. Our results show that in WS networks, the dependence of r_D on K is dominated by that in $r_D(K, q = 0)$ and the q-dependence is dominated by that in $\sigma(K, q)$ of the degree distribution. This finding suggests that it is the existence of nodes with higher degrees due to random re-wiring that plays a dominant role in determining the extinction payoffs.

In summary, we studied the extent of cooperation of the ESG in a Watts-Strogatz network. Comparing to the well-mixed case, regular lattices suppress f_C over a wide range of the payoff r and re-wiring lowers the suppression. We identified two extinction payoffs r_D and r_C and sorted out how $r_D(K, q)$ depends on K and q. Re-wiring increases the standard deviation of the degree distribution, which is found to govern the q-dependence of the extinction payoffs.

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[24] This is the case except for a few isolated values of $r > r_C$ for which the chain can be evolved into a configuration with a majority of D-nodes with isolated C-nodes distributed at equal distances.
[25] The results $f_C(r)$ do not depend sensitively on the 50%-50% initial distribution of C and D nodes. The results in fig. 1(a) and fig. 3(a) remain valid for a large range of initial distributions. Only for invasion problems with one or a few initially isolated C-nodes, r_D becomes different.
[26] The tiny probability of replacing the last surviving patterns (see discussion after fig. 2) by C nodes implies the numerical $r_D(K, q = 0)$ deviates from $1/2K$, even for such long runs of 3×10^5 time steps for $K = 3$. The difficulty becomes more severe as K increases. Our choice of y-axis amounts to using the numerical $r_D(K, q = 0)$, and allows us to focus on the q-dependence.