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A solvable (1 + 1)-dimensional gauge model and the corresponding lattice model

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Abstract. A $(1+1)$-dimensional gauge model is proposed and the spectrum with the energy eigenstates is solved exactly in Hamiltonian formalism. The exact solution of the corresponding lattice model is found. The results show that the lattice model has the correct continuum limit.

1. Introduction

Up to now, few gauge models have been solvable except for the Schwinger model. Even though the spectrum of the Schwinger model has been derived with a variety of methods [1], no one could find the energy eigenstates represented in the fermion operators. Our understanding about the structure of gauge theory is still very small.

On the other hand, it is still unclear how lattice gauge theory goes to the continuum limit although some work has been done in $(1+1)$-dimensional gauge theory [2]. In particular, no analytical results are available.

In this paper, some effort is made in the above directions in a very simple model. A solvable $(1+1)$-dimensional gauge model with non-relativistic fermions is proposed in section 2, in which both the spectrum and the energy eigenstates can be solved exactly. Furthermore, the exact solution of the corresponding lattice model is found and the continuum limit is discussed in section 3. Finally some discussions follow in section 4.

2. Gauge model

We consider a $(1+1)$-dimensional $SU(N)$ gauge model with non-relativistic fermions described by the Lagrangian

$$L=\int \mathrm{d}x\left(\tfrac{1}{2}F_{10}^{a}F_{10}^{a}-\mathrm{i}\psi^{+}D_{0}\psi+F\psi^{+}\gamma_{0}D_{x}^{2}\psi\right) \tag{2.1}$$

where

$$F_{10}^{a}=\partial_{x}A_{0}^{a}+\partial_{0}A_{x}^{a}-ef_{abc}A_{x}^{b}A_{0}^{c}, \tag{2.2}$$

$$D_{0}=\partial_{0}-\mathrm{i}eA_{0}\qquad D_{x}=\partial_{x}+\mathrm{i}eA_{x} \tag{2.3}$$

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and

$$A_0 = T^a A_0^a \qquad A_x = T^a A_x^a \tag{2.4}$$

f_{abc} is the structure constant of $SU(N)$ group, T^a is the generator satisfying

$$[T^a, T^b] = \mathrm{i} f_{abc} T^c \tag{2.5}$$

A_0^a and A_x^a are gauge fields, ψ and ψ^+ are fermion fields with two components, e is a real coupling constant and F is a real positive parameter. It is not too difficult to check that, besides $SU(N)$ gauge invariance, the Lagrangian also stays invariant under the two following transformations.

(i) Space reflection:

$$\begin{aligned} &x \to -x \\ &A_x \to -A_x \qquad A_0 \to A_0 \qquad \psi \to \psi \end{aligned} \tag{2.6}$$

where we notice that we do not have to demand $\psi \to \gamma_0 \psi$ as in the usual case.

(ii) Global γ_0 transformation:

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha\gamma_0}\psi \qquad A_0 \to A_0 \qquad A_x \to A_x \tag{2.7}$$

where α is a real constant.

Our model is a dynamical system with constraint since A_0 is not dynamical variable. For treating the constraint, we adopt the method developed by Dirac [3]. From the L in (2.1), it is easy to derive the canonical momenta of A_x^a and ψ

$$\begin{aligned} \pi_x^a &= \frac{\delta L}{\delta \dot{A}_x^a} = F_{10}^a = -E^a \\ \pi_\psi &= \frac{\delta L}{\delta \dot{\psi}} = -\mathrm{i}\psi^+ \end{aligned} \tag{2.8}$$

and the corresponding Hamiltonian

$$H = \int \mathrm{d}x \left(\tfrac{1}{2} E^a E^a - A_0^a Q^a - F\psi^+ \gamma_0 D_x^2 \psi\right) \tag{2.9}$$

where

$$Q^a = D_x E^a - e\psi^+ T^a \psi \tag{2.10}$$

and the equation of constraint

$$\frac{\delta L}{\delta A_0^a} = Q^a = 0. \tag{2.11}$$

Now we choose the temporal gauge

$$A_0 = 0 \tag{2.12}$$

and quantize the gauge fields by

$$[E^a(x), A_x^b(y)] = \mathrm{i}\delta_{a,b}\delta(x-y) \tag{2.13}$$

and consider the constraint (2.11) as a weak equation, which is imposed only on the states. In fact Q^a is the generator of the gauge transformation in the spatial direction, which is still an invariant transformation in the Hamiltonian after taking gauge condition (2.12). Therefore the constraint (2.11) is just a restriction on the states, that only the states with gauge invariance in the spatial direction are physical states.

We denote ψ and ψ^+ as

$$\psi=\begin{bmatrix}\xi\\ \eta^+\end{bmatrix} \qquad \psi^+=[\xi^+ \quad \eta] \tag{2.14}$$

where

$$\{\xi^+(x),\xi(y)\}=\delta(x-y) \qquad \{\eta^+(x),\eta(y)\}=\delta(x-y) \tag{2.15}$$

and choose

$$\gamma_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{2.16}$$

Then if we introduce the normal ordering on the fermion creation and destruction operators defined by (2.14), the Hamiltonian in (2.9) with gauge condition (2.12) can be written as

$$H=\int \mathrm{d}x(\tfrac{1}{2}E^aE^a - F\xi^+D_x^2\xi - F(D_x^2\eta^+)\eta). \tag{2.17}$$

We define the $|0\rangle$ state as

$$E^a\,|0\rangle=0 \qquad \xi\,|0\rangle=0 \qquad \eta\,|0\rangle=0. \tag{2.18}$$

Apparently,

$$H\,|0\rangle=0. \tag{2.19}$$

That is, $|0\rangle$ is an eigenstate of the Hamiltonian with zero energy. Neglecting the surface term in infinity, we can get

$$H=\int \mathrm{d}x(\tfrac{1}{2}E^aE^a + F(\xi^+\tilde{D}_x^+)(D_x\xi) + F(D_x\eta^+)(\eta\tilde{D}_x^+). \tag{2.20}$$

Since

$$(E^a)^+=E^a \qquad (D_x\xi)^+=\xi^+\tilde{D}_x^+ \qquad (\eta\tilde{D}_x^+)^+=D_x\eta^+ \tag{2.21}$$

H is positive definite. Therefore $|0\rangle$ is an exact ground state of H.

In order to solve the excited spectrum, we consider a translational invariant state $|\varepsilon\rangle$ with gauge invariance in the spatial direction, containing a fermion and antifermion pair, defined as

$$|\varepsilon\rangle=\int \mathrm{d}x\,\mathrm{d}x'f_\varepsilon(x-x')P\xi^+(x)\exp\left(\mathrm{i}e\int_{x'}^{x}A(\tau)\,\mathrm{d}\tau\right)\eta^+(x')\,|0\rangle \tag{2.22}$$

where P is the path-ordering operator. Now we demand $|\varepsilon\rangle$ satisfying the eigenvalue equation of H

$$H\,|\varepsilon\rangle=\varepsilon\,|\varepsilon\rangle. \tag{2.23}$$

It is easy to derive the equation for $f_\varepsilon(x)$

$$\tfrac{1}{2}C_Ne^2\,|x|\,f_\varepsilon(x)-2F\,\partial_x^2f_\varepsilon(x)=\varepsilon f_\varepsilon(x) \tag{2.24}$$

where C_N is the Casimir invariant of $SU(N)$ group. From (2.24), we can understand that the fermion and antifermion interact with the linear potential. Using the condition that $f_\varepsilon(x)$ is finite anywhere, we obtain

$$f_\varepsilon(x)=\begin{cases} C^{(+)}\phi\left[\left(\frac{C_Ne^2}{4F}\right)^{1/3}\left(x-\frac{2\varepsilon}{C_Ne^2}\right)\right] & (x>0) \\ C^{(-)}\phi\left[\left(\frac{C_Ne^2}{4F}\right)^{1/3}\left(-x-\frac{2\varepsilon}{C_Ne^2}\right)\right] & (x<0) \end{cases} \tag{2.25}$$

where $\phi[t]$ is an Airy function defined by [4]

$$\phi[t]=\frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}p\,\cos(\tfrac{1}{3}p^3+pt). \tag{2.26}$$

Taking into account the continuity of $f_\varepsilon(x)$ and $\partial_x f_\varepsilon(x)$ at $x=0$, for states with even parity

$$C^{(+)}=C^{(-)} \tag{2.27}$$

and the energy ε is determined by the equation

$$\phi'\left[-\left(\frac{C_Ne^2}{4F}\right)^{1/3}\frac{2\varepsilon}{C_Ne^2}\right]=0. \tag{2.28}$$

For states with odd parity

$$C^{(+)}=-C^{(-)} \tag{2.29}$$

and the energy ε is determined by the equation

$$\phi\left[-\left(\frac{C_Ne^2}{4F}\right)^{1/3}\frac{2\varepsilon}{C_Ne^2}\right]=0. \tag{2.30}$$

The first excited energy ε_1 and second excited energy ε_2 determined by (2.28) and (2.30) respectively are

$$\varepsilon_1=0.51C_Ne^2\left(\frac{4F}{C_Ne^2}\right)^{1/3} \qquad \varepsilon_2=1.17C_Ne^2\left(\frac{4F}{C_Ne^2}\right)^{1/3}. \tag{2.31}$$

When $t\to+\infty$, the Airy function $\phi[t]$ behaves as

$$\phi[t]\sim\tfrac{1}{2}t^{-1/4}\exp(-\tfrac{2}{3}t^{3/2}). \tag{2.32}$$

Therefore fermions are confined in our model.

3. Lattice gauge model

The Hamiltonian of the lattice gauge model corresponding to the Hamiltonian in (2.17) is

$$H=\tfrac{1}{2}e^2a\sum_x E^b(x)E^b(x) + \frac{F}{a^2}\sum_x :\psi^+(x)\gamma_0(2\psi(x)-U(x,1)\psi(x+1)-U(x,-1)\psi(x-1)): \tag{3.1}$$

where a is the lattice spacing, the lattice gauge fields $U(x, \pm 1)$ and electric field $E^b(x)$ satisfy

$$[U(x,1), E^b(x')] = \delta_{x,x'} T^b U(x,1)$$
$$[U(x,-1), E^b(x')] = -\delta_{x-1,x'} U(x,-1) T^b \tag{3.2}$$

the lattice fermion fields

$$\psi = \begin{bmatrix} \xi \\ \eta^+ \end{bmatrix} \qquad \psi^+ = [\xi^+ \eta] \tag{3.3}$$

satisfy

$$\{\xi^+(x), \xi(x')\} = \delta_{x,x'} \qquad \{\eta^+(x), \eta(x')\} = \delta_{x,x'} \tag{3.4}$$

and the normal ordering acts only on the fermion fields. Here we must note that the lattice fermion fields ψ and ψ^+ in (3.1) are different from those in the last section by a factor $\sqrt{a}$ where $a \to 0$. It is easy to realize that no doubling appears in our model since the fermions are non-relativistic.

We still define $|0\rangle$ as

$$E^b\,|0\rangle = 0 \qquad \xi\,|0\rangle = 0 \qquad \eta\,|0\rangle = 0. \tag{3.5}$$

Since H remains positive, $|0\rangle$ is an exact ground state. The state corresponding to that in (2.22) is

$$|\varepsilon\rangle = \sum_{x,n} f_\varepsilon(n) \xi^+(x) U(x,n) \eta^+(x+n)\,|0\rangle \tag{3.6}$$

where

$$U(x,n) = \begin{cases} P \prod_{i=0}^{n-1} U(x+i, 1) & (n > 0) \\ P \prod_{i=0}^{-(n-1)} U(x-i, -1) & (n < 0). \end{cases} \tag{3.7}$$

From

$$H\,|\varepsilon\rangle = \varepsilon\,|\varepsilon\rangle \tag{3.8}$$

we derive the equation

$$f_\varepsilon(n+1) + f_\varepsilon(n-1) = \frac{2(|n| + 8F/(C_N e^2 a^3) - 2\varepsilon/(C_N e^2 a))}{8F/(C_N e^2 a^3)}. \tag{3.9}$$

Comparing (3.9) with the recurrence formula of the Bessel function [5]

$$B_{\nu+1}(z) + B_{\nu-1}(z) = \frac{2\nu}{z} B_\nu(z) \tag{3.10}$$

we obtain the general solution of (3.9)

$$f_\varepsilon(n) = \begin{cases} C_1^{(+)} J_{\nu_n}(z) + C_2^{(+)} Y_{\nu_n}(z) & (n \geqslant 0) \\ C_1^{(-)} J_{\nu_{-n}}(z) + C_2^{(-)} Y_{\nu_{-n}}(z) & (n \leqslant 0) \end{cases} \tag{3.11}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions, $C_i^{(\pm)}$ is independent of n and

$$z = 8F/(C_N e^2 a^3) \qquad \nu_{\pm n} = \pm n + z - 2\varepsilon/(C_N e^2 a). \tag{3.12}$$

Since $f_\varepsilon(n)$ must be finite when $n \to \pm\infty$

$$C_2^{(\pm)} = 0. \tag{3.13}$$

That is

$$f_\varepsilon(n) = \begin{cases} C_1^{(+)} J_{\nu_n}(z) & (n \geqslant 0) \\ C_1^{(-)} J_{\nu_{-n}}(z) & (n \leqslant 0). \end{cases} \tag{3.14}$$

For the states with even parity

$$C_1^{(+)} = C_1^{(-)} \tag{3.15}$$

and the energy ε is determined by

$$\partial_z J_{\nu_0}(z) = 0. \tag{3.16}$$

For the states with odd parity,

$$C_1^{(+)} = -C_1^{(-)} \tag{3.17}$$

and the energy ε is determined by

$$J_{\nu_0}(z) = 0. \tag{3.18}$$

Now we discuss the continuum limit. In fact, when $a \to 0$, the asymptotic behaviour of $J_{\nu_{\pm n}}(z)$ is

$$J_{\nu_{\pm n}}(z) \sim \phi\left[\left(\frac{2}{z}\right)^{1/3} (\nu_{\pm n} - z)\right] \tag{3.19}$$

if na is kept to be finite. Therefore according to (3.12)

$$f_\varepsilon(n) \sim \begin{cases} C_1^{(+)} \phi\left[\left(\frac{C_N e^2}{4F}\right)^{1/3}\left(na - \frac{2\varepsilon}{C_N e^2}\right)\right] & (n \geqslant 0) \\ C_1^{(-)} \phi\left[\left(\frac{C_N e^2}{4F}\right)^{1/3}\left(-na - \frac{2\varepsilon}{C_N e^2}\right)\right] & (n \leqslant 0) \end{cases} \tag{3.20}$$

and it coincides with $f_\varepsilon(x)$ in (2.25). Correspondingly, (3.16) and (3.18) can be rewritten as (2.28) and (2.30) respectively. That is, the lattice model has the correct continuum limit.

4. Discussion

(i) A $(1+1)$-dimensional $SU(N)$ gauge model is proposed and both the spectrum and energy eigenstates are solved exactly in the Hamiltonian formalism. Although the model is non-relativistic, there exist fermions and antifermions. This concept is widely used in solid state physics. In our solution non-Abelian structure does not make too much trouble, as it happens, in working out the ground state of $(1+1)$-dimensional $SU(N)$ gauge theory with relativistic fermions [6].

(ii) The exact solution of the corresponding lattice model is found. The results show that the lattice model has the correct continuum limit. This is a unique example which demonstrates how lattice gauge theory goes to its continuum limit although the model is too simple to give us much information about real QCD.

(iii) The gauge model and the corresponding lattice model remain solvable even when the fermion mass term $\psi^+\gamma_0\psi$ is introduced. The model with the fermion mass term might be the heavy fermion limit of the usual $(1+1)D$ $SU(N)$ gauge theory with relativistic fermions. It may be possible to go further in this direction by adding higher-order fermion kinetic energy terms to the Lagrangian.

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