Localized motion in random matrix decomposition of complex financial systems

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\textbf{HIGHLIGHTS}

- The impacts of business sectors are identified from the localized motion.
- Localized motion induces different characteristics of the market index and stocks.
- Return-volatility correlations of eigenmodes are reproduced with a two-factor model.

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\textbf{ABSTRACT}

With the random matrix theory, we decompose the multi-dimensional time series of complex financial systems into a set of orthogonal eigenmode functions, which are classified into the market mode, sector mode, and random mode. In particular, the localized motion generated by the business sectors, plays an important role in financial systems. Both the business sectors and their impact on the stock market are identified from the localized motion. We clarify that the localized motion induces different characteristics of the time correlations for the stock-market index and individual stocks. With a variation of a two-factor model, we reproduce the return-volatility correlations of the eigenmodes.

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1. Introduction

Financial markets, as typical dynamic systems with highly complex interactions, share common features in various aspects with those in traditional physics. With large amounts of historical data, it becomes possible to analyze the complex financial systems based on concepts and methods in statistical physics [1–8]. Price movements of financial markets are characterized by the collective behaviors, such as the business-sector structure [9–11], leverage effect [2,12], extreme-volatility dynamics [3,13], and systemic risk [14–16].

The random matrix theory (RMT) is an important method to investigate the collective behaviors of complex systems, including financial markets. Based on the RMT theory, we develop the random matrix decomposition, with which the cross-correlation matrix can be decomposed into the mode cross-correlation matrices [17]. To further understand spatiotemporal
structures of complex financial systems, we decompose the multi-dimensional time series into eigenmode functions. In particular, we investigate two typical spatiotemporal structures, i.e., the business-sector structure and return-volatility correlation.

The business-sector structure is not only crucial for theoretically understanding the complex structure of financial systems, but also important for optimal portfolio choice [18–20]. The RMT theory and network technique are two main approaches to identify the business sectors in the financial markets [17,21–28]. Besides, the emergence of collective behavior may also be described by the network synchronization [29,30]. However, both the RMT theory and network method cannot characterize the impact of business sector on the stock market, which is valuable for investors to hedge risk.

Time correlation functions describe how one variable is statistically influenced by another. For example, a negative return-volatility correlation, i.e. the leverage effect, is detected in almost all stock markets in the world [2,31]. Meanwhile, a positive return-volatility correlation, i.e. the anti-leverage effect, is observed in the Chinese stock market before the year 2000 [12,32,33]. However, this correlation is moderate for the individual stocks, while it is much stronger for the stock-market indices [2]. Therefore, the second motivation is to explore the complicated behavior of individual stocks in financial markets with the random matrix decomposition.

In this paper, with the RMT theory, we decompose the multi-dimensional timeseries of complex financial systems into a set of orthogonal eigenmode functions, which are classified into the market mode, sector mode and random mode. These three kinds of eigenmodes represent the global motion, localized motion, and quasi-random motion, respectively. The nontrivial interactions of financial systems are mainly dominated by the market mode and sector mode. Business sectors can be identified from the expansion coefficients of the individual stocks in the localized motion. More importantly, the positive or negative sign of the business sector describes the positive or negative impact of the business-sector motion on the stock market. In particular, the different magnitudes of the return-volatility correlations for the market index and individual stocks are rooted in the localized motion. We reveal that the leverage effect of the market index mainly originates from the global motion. Finally, a variation of a two-factor model is introduced to generate the return-volatility correlations of the three modes.

2. Method and basics

We define the logarithmic price return of the $i$th stock over an one-day time interval as

$$ R_i(t) \equiv \ln P_i(t+1) - \ln P_i(t) , $$

where $P_i(t)$ is the daily closed price of the $i$th stock at time $t$. To ensure different stocks with the same weight, the price return is normalized to

$$ r_i(t) = \frac{R_i(t) - \langle R_i(t) \rangle}{\sigma_i} , $$

where $\langle \cdot \cdot \cdot \rangle$ is the average over time $t$, and $\sigma_i = \sqrt{\langle R_i^2(t) \rangle - \langle R_i(t) \rangle^2}$ denotes the standard deviation of $R_i(t)$. Then, elements of the cross-correlation matrix $C$ is defined by

$$ C_{ij} \equiv \langle r_i(t) r_j(t) \rangle . $$

Assuming $N$ stocks with a total time length $T$, and in the large-$N$ and large-$T$ limit with $Q \equiv T/N \geq 1$, the probability distribution $P_{rm}(\lambda)$ of the eigenvalue $\lambda$ for the Wishart matrix is given by

$$ P_{rm}(\lambda) = \frac{Q}{2\pi} \sqrt{\frac{\lambda_+ - \lambda_\pm}{\lambda}} \frac{\lambda_\pm}{\lambda} , $$

where $\lambda_{\pm}$ are the upper and lower bounds given by

$$ \lambda_{\pm} = \left[ 1 \pm \left( 1/\sqrt{Q} \right) \right] . $$

We define the eigenmode function $k_\alpha(t)$ as

$$ k_\alpha(t) = \sum_{i=1}^{N} u_{\alpha i}^* r_i(t) , $$

where $u_{\alpha i}$ is the $i$th component in the $\alpha$th eigenvector of the matrix $C$. The eigenmode functions form a set of orthogonal bases. For example, the market index may be expanded according to the orthogonal bases,

$$ \phi(t) = \sum_{\alpha} B_\alpha k_\alpha(t) . $$

The expansion coefficient $B_\alpha$ is given by

$$ B_\alpha = \langle k_\alpha(t) \phi(t) \rangle . $$
Fig. 1. Probability distributions of returns for the market mode, sector mode and random mode.

where $B_\alpha$ measures the contribution of the $\alpha$th eigenmode to the market index. Note that the sign of an eigenmode function $k_\alpha(t)$ cannot be determined with Eq. (6), since the sign of an eigenvectors of $C$ is arbitrary. To identify the sign of $k_\alpha(t)$, we assume that the contribution of an eigenmode function to the market index is positive. Therefore, the sign of $k_\alpha(t)$ is fixed to ensure $B_\alpha > 0$. In other words, $k_\alpha(t)$ is set in the same moving direction of the market index.

We arrange the eigenvalues of the matrix $C$ in the order of $\lambda_0 > \lambda_{\alpha+1}$, and $\lambda_0$ is the largest one. The eigenmode functions are classified into three types of modes in terms of their eigenvalues: (i) the market mode, is the eigenmode function of the largest eigenvalue $\lambda_0$; (ii) the sector mode, is the sum of the eigenmode functions satisfying $\lambda_{\alpha} \leq \lambda_\alpha < \lambda_0$; (iii) the random mode, includes the eigenmodes belonging to $\lambda_\alpha < \lambda_{\alpha}$. The market mode describes the global motion, rooted in the global interaction of the entire market. The sector mode depicts the localized motion, generated by local interactions in the business sectors. The random mode is the quasi-random motion.

We have gathered the daily data of three stock markets, i.e., the New York Stock Exchange (NYSE) from Jan. 1990 to Dec. 2006, the Hong Kong Exchange (HKE) from Jan. 2003 to Sep. 2011, and the Shanghai Stock Exchange (SSE) from Jan. 2003 to Jul. 2011. We select 259 weighted stock in the three markets, respectively. The data of the three market indices in the same time periods are also collected, i.e., the S&P 500 Index for the NYSE market, the HSI Index for the HKE market and the SSE Index for the SSE market. If the price of a stock is absent on a particular day, we set the price to be the same as the preceding day [34]. It has been pointed out that the missing data do not result in artifacts [35].

As shown in Fig. 1, the probability distributions of price returns for the three modes are similar in the NYSE. The results in the SSE and HKE are not displayed, since they are almost the same as those in the NYSE. Therefore, the probability distributions of price returns may not precisely quantify the difference between these three modes. The approximate entropy, $S$, is a good measure to estimate the regularity of the stock markets and foreign exchange markets [36–38]. The approximate entropy is small for the time series with a high degree of regularity. Following Refs. [37,38], we set the embedding dimension $m = 2$, and the distance $d = 20\%$ of the standard deviation of the time series.

The approximate entropy of the market mode, sector mode and random mode are denoted by $S_m$, $S_s$ and $S_r$, respectively. All these values for the SSE and HKE are smaller than those for the NYSE. It implies that the market efficiency of the three modes is higher for the NYSE. This result is consistent with the observation that the efficiency for the mature foreign exchange market is higher than that for the emergent one [38]. Furthermore, it is observed that $S_m < S_s < S_r$ for all three markets. In other words, the degree of regularity for the market mode is the highest, and that for the random mode is the lowest. The regularity of the financial market is significantly dominated by the market mode and sector mode. The random mode is a quasi-random process. Therefore, the random matrix decomposition provides us a new sight into the financial systems, since the higher degree of the regularity indicates the higher possibility of predictability.

3. Impact of business-sector motion

To quantify the contribution of the eigenmode functions to the market index, we compute the expansion coefficients of the market index in the eigenmode functions with Eq. (8). The expansion coefficient $B_\alpha$ is the probability amplitude, measuring the co-movements between the eigenmode function and the market index. The largest coefficient $B_{\text{max}}$, corresponding to the market mode, is almost the same for different markets, i.e., 0.89 for the NYSE and SSE, and 0.88 for the HKE. The large values of $B_{\text{max}}$ indicate that the motion of the market mode is very close to that of the market index. Most of the other larger $B_\alpha$ describe the sector mode, and their values range from 0.10 to 0.25.

To understand the role of the individual stocks in the localized motion, we compute the expansion coefficients of stocks in eigenmode functions, i.e., $b_\alpha^i = \langle r_i(t) k_\alpha(t) \rangle$. The coefficient $b_\alpha^i$ represents the contribution of the $i$th stock on the $\alpha$th eigenmode function. The probability distributions of $b_\alpha^i$ for four typical stocks in the NYSE are displayed in Fig. 2, i.e., Walmart Stores, British Petroleum (BP), Lockheed Martin, and Panasonic, which belong to different business sectors. In particular,
Fig. 2. Probability distributions of the expansion coefficients for four stocks. The eigenmodes corresponding to the largest five $|\hat{b}_i|$ are labeled with $\lambda$. 

Table 1
Business sectors classified from $|\hat{b}_\alpha|$ in the NYSE. $\langle |\hat{b}_\alpha| \rangle$ is the average value over stock $\mu$ in a business sector. The positive or negative sign of a business sector represents the positive or negative impact of the business-sector motion on the stock market. Japan-IT denotes IT-related Japanese companies.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>12</th>
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<tr>
<td>Sign</td>
<td>--</td>
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<td>+</td>
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<td>+</td>
</tr>
<tr>
<td>Sector</td>
<td>Utility</td>
<td>Oil</td>
<td>Gold</td>
<td>Japan-IT</td>
<td>Defense</td>
</tr>
<tr>
<td>$\langle</td>
<td>\hat{b}_\alpha</td>
<td>\rangle$</td>
<td>0.48</td>
<td>0.45</td>
<td>0.72</td>
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</tbody>
</table>

those large $|\hat{b}_\alpha|$ for different stocks may correspond to different eigenmode functions. It implies that the business sector impacts on the individual stocks differently.

The expansion coefficient with the largest absolute value for the $i$th stock is denoted by $|\hat{b}_i|$. It is interesting that $|\hat{b}_i|$ do not always correspond to the market mode. In other words, the influence of the business sector on a stock can be more important than that of the market mode. If $|\hat{b}^{\alpha}_\mu|$ does not correspond to the market mode, we replace the subscript $i$ by $\mu$, and there are 66 $|\hat{b}^{\alpha}_\mu|$ in the NYSE. Here, those stocks of $|\hat{b}^{\alpha}_\mu|$ in a specific eigenmode may dominate this eigenmode, and they can be identified as the $\mu$th business sector. In addition, the sign of the business sector is represented by the sign of $\hat{b}^{\alpha}_\mu$ in that business sector.

The business-sector structure of the NYSE is listed in Table 1 as an example. The 23 stocks of negative $\hat{b}^{\alpha}_\mu$ for $\alpha = 1$ form a Utility sector. There are 17 stocks of negative $\hat{b}^{\alpha}_\mu$ for $\alpha = 2$, and they are defined as the Oil sector. The other three sectors are the Gold in $\alpha = 5$, the Japan-IT in $\alpha = 10$, and the Defense in $\alpha = 12$. The sign of the Japan-IT sector is negative, while the signs of the Gold and the Defense are positive. To quantitatively measure the significance of $|\hat{b}^{\alpha}_\mu|$, we compute $\langle |\hat{b}^{\alpha}_\mu| \rangle$, where $\langle \cdots \rangle$ is the average over stock $\mu$ in the $\alpha$th business sector. $\langle |\hat{b}^{\alpha}_\mu| \rangle$ of the business sectors are shown in Table 1. For comparison, the average values of $|\hat{b}^{\alpha}_\mu|$ and $|\hat{b}^{\alpha}_i|$ are 0.4 and 0.04, respectively. Therefore, the dominance of the localized motion in those business sectors is more significant than that of the global motion.

What is the physical meaning of the signs of the business sectors? $b^{\alpha}_\mu$ measures the contribution of the $i$th stock to the $\alpha$th eigenmode, which is dominated with the business sector. Meanwhile, the direction of the motion for the eigenmode is set to be the same for the market index. Therefore, the positive or negative sign of the business sector represents the positive or negative correlation between the business sector and the market index. It further implies that the positive or negative impact of the business sector on the stock market. As a business sector is a group of stocks belonging to an industry, the sign indicates the positive or negative impact of the economic cycle of the industry on the stock market.

Let us examine a typical example of the negative sign in the eigenmode of $\alpha = 2$, which represents the localized motion of the stock market mainly driven by the Oil sector. The negative sign directly indicates that the motion of the Oil sector has a negative impact on the stock market. Indeed, it is consistent with fact that crude oil prices have a negative impact on the stock markets [39–41]. Such a kind of negative sectors is a new finding, which may be important for deeper understanding of the localized motion of the financial systems, and for improving the risk management of asset portfolio.

As a further application, we analyze the dynamic system of global market indices (GMI), which contains the daily data of 57 market indices, from Sep. 1997 to Oct. 2008, in total 2669 days. The GMI is controlled by the regional sectors and subsectors [27]. In this study, we also observe this region-dominated sector structure from $|\hat{b}^{\alpha}_\mu|$, such as the North American, European, and Chinese sectors. Both the signs of the North American sector and European sector are positive. In contrast, the sign of the Chinese sector is negative, which suggests that the price motion of the Chinese stock market
anti-correlates with that of other stock markets in the world. In other words, the economy of China has a negative impact on the world economy to some extent, since the stock market reflects the economy of the country. In fact, dynamic correlations at the business cycle frequencies are negative between China and OECD (Organization for Economic Cooperation and Development) countries [42], which supports our observations in this paper.

4. Return-volatility correlation

For further understanding the dynamic behavior of financial systems, it is necessary to investigate the temporal structure of the eigenmodes. For example, the return-volatility correlation is an important higher-order time-correlation, which is defined as

\[
L(t') = \frac{\langle r(t) r(t+t') \rangle}{Z},
\]

with \(Z = \langle |r(t+t')|^2 \rangle\). This correlation measures how future volatilities are affected by past returns. \(L(t')\) with a negative value is called the leverage effect, which indicates that a positive return may lead to stable prices and a negative return induces high volatilities. On the other hand, a positive \(L(t')\) is the so-called anti-leverage effect.

We compute the return-volatility correlation for the market index \(\phi(t)\), denoted as \(L_{\phi}(t')\). For comparison, the return-volatility correlation averaged over all individual stocks, denoted by \(L_0(t')\), is also calculated. As shown in Fig. 3, for both the NYSE and HKE, \(L_{\phi}(t')\) shows a strong leverage effect, while \(L_0(t')\) displays a moderate one. The results for the SSE are similar to those for the HKE. In other words, there is a difference in magnitude between \(L_{\phi}(t')\) and \(L_0(t')\). It indicates that the leverage effect of the market index is not a simple average of individual stocks. This phenomenon is consistent with the relevant observation reported in Ref. [2].

To understand how the leverage effect of the market index emerges from the individual stocks, we compute the return-volatility correlation of the \(\alpha\)th eigenmode function, denoted by \(L_\alpha(t')\). The return-volatility correlation of the sector mode is \(L_m(t') = L_0(t')\). The average return-volatility correlation of the sector mode is defined as

\[
L_\alpha(t') = \frac{1}{N_\alpha} \sum_{\alpha \in \alpha} L_\alpha(t'),
\]

where \(N_\alpha\) is the number of the eigenmodes belonging to the sector mode. In the same way, we calculate average return-volatility correlation of the random mode, \(L_r(t')\), which fluctuates around zero for all three markets.

The return-volatility correlations, \(L_m(t'), L_\alpha(t')\) and \(L_r(t')\), corresponding to the market mode, sector mode and random mode, are displayed for the NYSE and HKE in Fig. 3. For both markets, we observe similar behaviors of \(L_m(t')\) and \(L_\alpha(t')\), showing the leverage effect. However, \(L_r(t')\) exhibits the anti-leverage effect for the HKE, and it fluctuates around zero for the NYSE. Moreover, the magnitude of \(L_r(t')\) is weaker than that of \(L_{\phi}(t')\). Therefore, it implies that the leverage effect of the market indices mainly originates from that of the market mode.

Let us come back to the difference in magnitude between \(L_{\phi}(t')\) and \(L_0(t')\). Returns of a stock are composed of the market-mode factor, the sector-mode factor, and the quasi-random factor [21,35]. Roughly speaking, \(L_{\phi}(t')\) is the average effect of \(L_m(t'), L_\alpha(t')\), and \(L_r(t')\). Since \(L_r(t') = 0\) for all markets, we will ignore it. As shown in Fig. 3, \(L_m(t')\) is negative, while \(L_\alpha(t')\) is weakly positive or around zero. Thus, \(L_{\phi}(t')\) shows a moderate leverage effect. In addition, due to positive
$L_i \left( t' \right)$ for the HKE and zero for the NYSE, the difference in magnitude between $L_{\varphi} \left( t' \right)$ and $L_{\varphi} \left( t' \right)$ for the former is qualitatively bigger than that in the latter. Therefore, the difference in magnitude between $L_{\varphi} \left( t' \right)$ and $L_{\varphi} \left( t' \right)$ is mainly contributed from the localized motion $L_i \left( t' \right)$. Moreover, the return-volatility correlation is investigated for the GMI. As shown in Fig. 4, $L_i \left( t' \right)$ is negative, which results in a weaker difference in magnitude $L_{\varphi} \left( t' \right)$ and $L_{\varphi} \left( t' \right)$.

5. Simulation

To better understand the dynamic mechanism of the return-volatility correlations in financial markets, we introduce a variation of the two-factor model [21,35]. The normalized return of the $i$th stock from the $k$th business sector can be decomposed into (i) a market-mode factor $r_m \left( t \right)$, common for all stocks; (ii) a sector-mode factor $r_s^k \left( t \right)$, representing local dynamic effects of the $k$th sector; (iii) a random factor $\eta_i \left( t \right)$. Thus,

$$r_i^k \left( t \right) = \beta_i r_m \left( t \right) + \gamma_i^k r_s^k \left( t \right) + \sigma_i \eta_i \left( t \right),$$

where the constants $\beta_i$, $\gamma_i^k$, and $\sigma_i$ represent the relative strengths of the three terms. The time series $r_m \left( t \right)$ shows the leverage effect, while $r_s^k \left( t \right)$ may exhibit the leverage or anti-leverage effect. Both of them are generated with an agent-based model [43]. $\eta_i \left( t \right)$ is a Gaussian noise. $r_m \left( t \right)$, $r_s^k \left( t \right)$ and $\eta_i \left( t \right)$ are normalized to zero mean and unit variance. The unit variance of $r_s^k \left( t \right)$ requires

$$\beta_i^2 + \left( \gamma_i^k \right)^2 + \sigma_i^2 = 1.$$  \hspace{1cm} (12)

For each stock, we can independently assign $\sigma_i$ and $\gamma_i^k$, and obtain $\beta_i$ from Eq. (12). We practically choose $\sigma_i$ and $\gamma_i^k$ from a uniform distribution with a width $\delta$ and centered about the mean value $\gamma$ and $\sigma$, respectively.

With the above model, we generate $N$ times series with length $T$ for returns $r_s^k \left( t \right)$. The $K$ business sectors are composed of $n_1$, $n_2$, $\cdots$, $n_K$ stocks. Practically, we take $N = 200$ and $T = 100$, and assign 4 business sectors, each with 50 stocks. The business sectors having the leverage and anti-leverage effects are denoted as $r_s^{k-} \left( t \right)$ and $r_s^{k+} \left( t \right)$, respectively. Therefore, there are five simulation cases, denoted by $m r_s^k \left( t \right)$, $m = 0, \ldots, 4$, where $m$ represents the number of business sectors having the leverage effect. We choose $\gamma_i^k = 0.4, \sigma = 0.15$, and $\delta = 0.05$. With these fixed parameters, one calculates $\beta = 0.58$.

We obtain $L_m \left( t' \right)$, $L_s \left( t' \right)$, $L_t \left( t' \right)$, and $L_0 \left( t' \right)$ from the simulated $r_s^k \left( t \right)$, with the random matrix decomposition. As examples, the results of two typical simulation cases of $1r_s^{k-} \left( t \right)$ and $2r_s^{k-} \left( t \right)$ are shown in Fig. 5. In both two simulation cases, $L_m \left( t' \right)$ exhibit the leverage effect and $L_t \left( t' \right)$ fluctuate around zero, in agreement with those for the empirical data. However, the behaviors of $L_s \left( t' \right)$ are quite different in these two simulation cases. $L_s \left( t' \right)$ shows the anti-leverage effect in the case of $1r_s^{k-} \left( t \right)$, while it is zero in the case of $2r_s^{k-} \left( t \right)$. In addition, the difference in magnitude between $L_m \left( t' \right)$ and $L_0 \left( t' \right)$ will be larger or smaller, when $L_s \left( t' \right)$ is positive or negative. Similar to the empirical data, the magnitude of $L_0 \left( t' \right)$ for the simulations is also intervenient between the ones of $L_m \left( t' \right)$ and $L_s \left( t' \right)$. Similar results are obtained in the other three cases.

6. Conclusion

In this article, the multi-dimensional time series in complex financial systems are decomposed into a set of orthogonal eigenmode functions with the RMT theory. The eigenmode functions are classified into the market mode, sector mode and
random mode, which are dominated by the global motion of the entire market, localized motion generated by the business sectors, and the quasi-random factor, respectively. The non-trivial interactions of the financial system mainly come from the market mode and sector mode.

The business sectors are identified from the expansion coefficients of the individual stocks in the localized motion. In particular, the sign of a business sector indicates the positive or negative influence of the business sector on the stock market. We clarify that the localized motion generates different characteristics of the time correlations for the market index and individual stocks, i.e., different magnitudes for $L_\psi(t')$ and $L_\theta(t')$. On the other hand, the leverage effect of the market index mainly originates from the global motion. Finally, with a variation of a two-factor model, we reproduce the return-volatility correlations for financial systems.

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