Thermodynamics and Statistical Physics

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Chapter 1

Thermodynamics

Chapter 2

Classical statistical mechanics

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Canonical ensemble and grand canonical ensemble

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Quantum statistical mechanics

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Applications of quantum statistical mechanics

5.1 Photons

A "black body cavity":
eletromagnetic radiation enclosed in a volume V at temperature T
Experimental realization:
make a cavity in any material
evacuate the cavity
heat up the material to temperature T

If V is sufficiently large, the material is irrelevant, therefore, any boundary conditions can be imposed on the system.
It is a free field
Photons are massless bosons

A photon corresponds to a plane wave of electric field

$$\vec{E}(\vec{r}) = \vec{\epsilon} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

and

$$\begin{aligned} \textbf{energy} &= \hbar\omega \\ \textbf{momentum} &= \hbar\vec{k}, \qquad |\vec{k}| = \frac{\omega}{c} \\ \textbf{polarization} &= \vec{\epsilon}, \qquad |\vec{\epsilon}| = 1, \vec{k}\cdot\vec{\epsilon} = 0 \end{aligned}$$

$\therefore$ only two independent directions
Question: why?
In a box $V = L^3$

$$\vec{k} = \frac{2\pi\vec{m}}{L} \qquad (\vec{p} = \hbar\vec{k})$$

$$m_x = 0, \pm 1, \pm 2, \cdots \qquad m_y = \cdots \quad m_z = \cdots$$

$$\sum_{\vec{m}} f(|\vec{k}|) = \frac{V}{(2\pi)^3} 4\pi \int \mathrm{d}k\, k^2 f(|\vec{k}|) \quad k = |\vec{k}|$$

For convenience, we denote the sum over $\vec{m}$ as the sum over $\vec{k}$.

Suppose $n_{\vec{k},\vec{\epsilon}}$ is the occupation number of momentum $\vec{k}$ and polarization $\vec{\epsilon}$. Total energy

$$E\{n_{\vec{k},\vec{\epsilon}}\} = \sum_{\vec{k},\vec{\epsilon}} \hbar\,\omega n_{\vec{k},\vec{\epsilon}}$$

An important feature of the system is that the number of the photons is not conserved. This is not the same as in a grand canonical ensemble, where a photon may move to the "environment" which is still a part of the "whole"system of a microcanonical ensemble.

In the cavity, a photon can just disappear, i.e. absorbed by the material which is not described by statistical mechanics.

From the view of a grand canonical ensemble, the chemical potential

$$\mu = 0$$

since one does not need any energy to remove a photon. Therefore, the partition function

$$Z = \sum_{\{n_{\vec{k},\vec{\epsilon}}\}} e^{-\beta E\{n_{\vec{k},\vec{\epsilon}}\}}$$

with no upper bound for $n_{\vec{k},\vec{\epsilon}}$**,**

$$\begin{aligned} Z &= \prod_{\vec{k},\vec{\epsilon}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \prod_{\vec{k},\vec{\epsilon}} \frac{1}{1 - e^{-\beta\hbar\omega}} \\ \log Z &= -2 \sum_{\vec{k}} \log(1 - e^{-\beta\hbar\omega}) \end{aligned}$$

the factor 2 is from $\sum_{\vec{\epsilon}}$**.**
Then, the average occupation number, regardless polarization,

$$< n_{\vec{k}} >= -\frac{\partial}{\partial(\beta\hbar\omega)} \log Z = \frac{2}{e^{\beta\hbar\omega} - 1}$$

the internal energy

$$U = \sum_{\vec{k}} \hbar\omega < n_{\vec{k}} >$$

Since $\mu = 0$**, the system looks like a canonical ensemble.**

Question: why?

E.g., one may think that N **is a microscopic degree of freedom.**

$$
\begin{aligned}
\because \quad P &= -\frac{\partial A}{\partial V} \\
\therefore \quad \frac{P}{kT} &= \frac{\partial}{\partial V} \log Z(V,T)|_T \\
\because \quad \omega &= c|\vec{k}| \sim V^{-1/3} \\
\therefore \quad P &= \frac{1}{3V} \sum_{\vec{k}} \hbar\omega < n_{\vec{k}} >
\end{aligned}
$$

Thus we have

$$PV = \frac{1}{3}U.$$

Noting that U is a function the temperature T, this is just the equation of state.

Question: is this equation of state consistent with $PV/kT = \log Z(V,T)$?

E.g., one may prove that $\log Z(V,T) = U/3kT$ from $U \sim (kT)^4 V$.

In the limit $V \to \infty$,

$$
\begin{aligned}
U &= \frac{2V}{(2\pi)^3} \int_0^\infty \mathrm{d}k\, 4\pi k^2 \frac{\hbar c k}{e^{\beta\hbar c k} - 1} \\
&= \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \mathrm{d}\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1}
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{U}{V} &= \int_0^\infty \mathrm{d}\omega\, u(\omega,T) \\
u(\omega,T) &= \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}
\end{aligned}
$$

$u(\omega, T)$ **can be verified experimentally, which is one of the facts leading to** $Quantum Mechanics$

$$\begin{aligned}\frac{U}{V} &\sim \int \mathrm{d}\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \\ &\sim (kT)^4 \int \mathrm{d}(\beta\omega) \frac{(\beta\omega)^3}{e^{\beta\hbar\omega} - 1} \\ &\sim (kT)^4\end{aligned}$$

This behavior and $PV = \frac{1}{3}U$ **can be also derived in classical physics.**

5.2 Phonons in solids

Phonons are quanta of sound waves in a macroscopic body.

Mathematically, phonons are similar to photons.

For low-lying excitations, the Hamiltonian for a solid may be approximated by a sum of harmonic oscillators. In quantum theory, these oscillators give rise to quanta called phonons. Therefore, at low temperature, a solid can be regarded as a gas of free phonons.

The sound wave function

$$\vec{\epsilon} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$|\vec{k}| = \frac{\omega}{c}, \qquad c: \quad \text{the velocity}$$

$\vec{\epsilon}$: **polarization, 3 independent directions**

Phonons are massless bosons, and the number of phonons is not conserved.

If a solid has N atoms, it has $3N$ normal modes, which are just the total degrees of freedom of vibrations. The frequencies $\{\omega_i\}$ of the vibrations correspond to the energy of phonons.

In the case of photons, $\{\omega_i\}$ may take any values according to the dispersion relation $\{\omega_i\} = c|\vec{k}|$. In the case of phonons, $\{\omega_i\}$ take only one value in the Einstein model of a lattice.

Debye theory.

One may consider the solid as an elastic continuum of volume V. $\{\omega_i\}$ then take the lowest $3N$ normal frequencies of such a system.

Number of normal modes with frequencies between ω and $\omega + \mathrm{d}\omega$

$$\begin{aligned} &\equiv f(\omega)\mathrm{d}\omega \\ &= \frac{3V}{(2\pi)^3} 4\pi k^2 \mathrm{d}k \end{aligned}$$

The counting of the number of the states is exactly the same as in the preceding section, and the factor 3 is from the three directions of the polarization.

The maximum frequency ω_m is obtained by

$$\int_0^{\omega_m} f(\omega)\mathrm{d}\omega = 3N$$

$$\therefore \quad \omega_m = c\left(\frac{6\pi^2 N}{V}\right)^{1/3}$$

The minimum wavelength

$$\lambda_m = \frac{2\pi c}{\omega_m}$$
$$\simeq \text{interparticle distance}$$

In other words, the interparticle distance, the lattice spacing, leads to the maximum frequency ω_m. This is different from the case of photons.

The total energy

$$E\{n_i\} = \sum_{i=1}^{3N} n_i \hbar\omega_i$$

with $\{n_i\}$ being the occupation number of frequencies $\{\omega_i\}$. The the sum over i is actually the sum over all normal modes.

Similar to the case in the preceding section

$$Z = \sum_{\{n_i\}} e^{-\beta E\{n_i\}} = \prod_{i=1}^{3N} \frac{1}{1 - e^{-\beta\hbar\omega_i}}$$
$$\log Z = -\sum_{i=1}^{3N} \log(1 - e^{-\beta\hbar\omega_i})$$

Note: the number of ω_i is $3N$, but the number of phonons are not conserved.

$$< n_i > = -\frac{\partial}{\partial\beta\hbar\omega_i} \log Z = \frac{1}{e^{\beta\hbar\omega_i} - 1}$$
$$U = \sum_{i=1}^{3N} \hbar\omega_i < n_i >$$

In the limit $V \to \infty$

$$\frac{U}{V} = \frac{3}{2\pi^2 c^3} \int_0^{\omega_m} \mathrm{d}\omega\, \omega^2 \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

$$\text{or} \quad \frac{U}{N} = \frac{9(kT)^4}{(\hbar\omega_m)^3} \int_0^{\beta\hbar\omega_m} \mathrm{d}t \frac{t^3}{e^t - 1}$$

We define the Debye function

$$D(x) \equiv \frac{3}{x^3} \int_0^x \mathrm{d}t \frac{t^3}{e^t - 1}$$

$$= \begin{cases} 1 - \frac{3}{8}x + \cdots & x \ll 1 \\ \frac{\pi^4}{5x^3} + O(e^{-x}) & x \gg 1 \end{cases}$$

and the Debye temperature T_D

$$kT_D \equiv \hbar\omega_m$$

then

$$\frac{U}{N} = 3kT \cdot D\left(\frac{T_D}{T}\right)$$

$$= \begin{cases} 3kT(1 - \frac{3}{8}\frac{T_D}{T} + \cdots) & T \gg T_D \\ 3kT\left(\frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^3 + \cdots\right) & T \ll T_D \end{cases}$$

$$\therefore \quad \frac{C_V}{Nk} = 3D\left(\frac{T_D}{T}\right) + 3T\frac{\mathrm{d}D\left(\frac{T_D}{T}\right)}{\mathrm{d}T}$$

$$\frac{C_V}{Nk} = \begin{cases} 3\left[1 - \frac{1}{20}\left(\frac{T_D}{T}\right)^2 + \cdots\right] & T \gg T_D \\ \frac{12\pi^4}{5}\left(\frac{T}{T_D}\right)^3 + \cdots & T \ll T_D \end{cases}$$

This is quite in agreement with experiments.
At high temperatures the system behaves classically.

At low temperature, $C_V \sim T^3$, verifying the third law.

Actually, all calculationsin this section are similar to those in the preceding section, but with a maximum frequency.

5.3 Bose-Einstein condensation

Define a class of functions

$$g_n(z) \equiv \sum_{l=1}^{\infty} \frac{z^l}{l^n}$$

then, the equation of state may be obtained from

$$\begin{aligned}
\frac{P}{kT} &= \frac{1}{\lambda^3} g_{5/2}(e^{\beta\mu}) - \frac{1}{V} \log(1 - e^{\beta\mu}) \\
\frac{N}{V} &= \frac{1}{\lambda^3} g_{3/2}(e^{\beta\mu}) + \frac{1}{V} \frac{e^{\beta\mu}}{1 - e^{\beta\mu}} \\
\lambda &= \sqrt{2\pi\hbar^2/mkT}
\end{aligned}$$

by eliminating $e^{\beta\mu}$. Here we must keep in mind that N denotes $\bar{N}$, for convenience.

From the above second equation, or from

$$< n_0 >= \frac{e^{\beta\mu}}{1 - e^{\beta\mu}}$$

it is necessary

$$0 \le e^{\beta\mu} \le 1, \qquad \textbf{i.e.,} \qquad \mu \le 0$$

$\because \qquad$ if $\qquad e^{\beta\mu} < 0$

$$< n_0 >= \frac{e^{\beta\mu}}{1 - e^{\beta\mu}} < 0$$

if $\quad e^{\beta\mu} > 1$

$$< n_0 >= \frac{e^{\beta\mu}}{1 - e^{\beta\mu}} < 0$$

For $0 \leq z \leq 1, g_{3/2}(z)$ **is**

- **bounded**
- **positive**
- **monotonically increasing function of** z

$$\begin{aligned} g_{3/2}(z) &= z + \frac{z^2}{z^{3/2}} + \cdots \\ g_{3/2}(1) &= 2.612\cdots \\ g_{3/2}(z) &\leq 2.612\cdots \end{aligned}$$

In other words

$$\lambda^3 \frac{< n_0 >}{V} = \frac{\lambda^3 N}{V} - g_{3/2}(e^{\beta\mu})$$

Usually, $< n_0 >$ **is finite. Therefore,** $< n_0 > /V \to 0$ **in the limit** $V \to \infty$**. i.e.,**

$$\frac{1}{v} - \frac{g_{3/2}(e^{\beta\mu})}{(2\pi\hbar^2/mkT)^{3/2}} = 0, \qquad v = \frac{V}{N}$$

Suppose v **is a constant,** μ **must increase (i.e., approach zero) as** T **decreases, if** $< n_0 >$ **is kept finite.**

If at a temperature T_C**,**

$$\mu = 0$$

This implies that

$$\frac{< n_0 >}{V} = \frac{1}{v} - \frac{g_{3/2}(1)}{\lambda^3} > 0 \quad \textbf{for} \quad T < T_c$$

This means that a finite fraction of particles occupies the energy level with $\vec{p} = 0$. This is the so-called Bose-Eistein condensation, and it is the condensations in momentums space.

In general, v is not constant, then the condition for the BE condensation is

$$\frac{\lambda^3}{v} > g_{3/2}(1)$$

Therefore, a phase transition occurs at the surface

$$\frac{\lambda^3}{v} = g_{3/2}(1)$$

in the $P - v - T$ space.

If v is given

$$kT_c = \frac{2\pi\hbar^2/m}{\left[vg_{3/2}(1)\right]^{2/3}}$$

If T is given

$$v_c = \frac{\lambda^3}{g_{3/2}(1)}$$

λ is called the thermal wavelength because it is of the order of the Broglie wavelength of a particle with the energy kT. At the phase transition, the thermal wavelength is of the same order of magnitude as the average interparticle distance. In the region of condensation, the thermal wavelength is larger than the average interparticle distance.

We should note that condensation occurs only at the energy level with $\vec{p} = 0$. For example, one may also separate $< n_1 > /V$ from $g_{3/2}(z)$. But one can easily prove that $< n_1 > /V$ goes to zero in the thermal dynamic limit.

The phase transition of the BE condensation is of first order. Actually, we discern no difference between the BE condensation and the ordinary gas-liquid condensation. For example, we may consider the particles at the state with $\vec{p} = 0$ as the condensed phase, and other particles as the gas phase. When the two phases coexist, the gas phase has the specific volume v_c, whereas the condensed phase has the specific volume $v = 0$. Therefore v can be considered to be the order parameter.

5.4 Ideal Fermi gases

5.4.1 Equation of state

The equation of state is obtained from

$$\begin{aligned} \frac{P}{kT} &= \frac{1}{\lambda^3} f_{5/2}(e^{\beta\mu}) \\ \frac{N}{V} &= \frac{1}{\lambda^3} f_{3/2}(e^{\beta\mu}) \end{aligned}$$

by eliminating $e^{\beta\mu}$

$$\begin{aligned} f_n(z) &= \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^n} \\ f_{3/2}(z) &= \begin{cases} z - \frac{z^2}{2^{3/2}} + \cdots & z \quad \textbf{small} \\ \frac{4}{3\sqrt{\pi}} \Big[(\log z)^{3/2} + \frac{\pi^2}{8} (\log z)^{-1/2} + \cdots & z \quad \textbf{large} \end{cases} \end{aligned}$$

5.4.2 High temperature and low densities

$$
\begin{aligned}
\frac{\lambda^3}{v} &\ll 1 \qquad v = \frac{V}{N} \\
\frac{\lambda^3}{v} &= e^{\beta\mu} - \frac{(e^{\beta\mu})^2}{2^{3/2}} + \cdots \\
e^{\beta\mu} &= \frac{\lambda^3}{v} + \frac{1}{2^{3/2}}\left(\frac{\lambda^3}{v}\right)^2 + \cdots \\
\because \quad e^{\beta\mu} &= \frac{\lambda^3}{v} + \frac{1}{2^{3/2}}(e^{\beta\mu})^2 + \cdots \\
&= \frac{\lambda^3}{v} + \frac{1}{2^{3/2}}\left(\frac{\lambda^3}{v} + \cdots\right)^2 + \cdots
\end{aligned}
$$

In the limit $\lambda \to 0$ (i.e. $T \to \infty$)

$$e^{\beta\mu} \simeq \frac{\lambda^3}{v}$$

hence $< n_{\vec{p}} >$ reduces to Maxwell-Boltzmann form

$$< n_{\vec{p}} > \doteq \frac{\lambda^3}{v} e^{-\beta\epsilon_{\vec{p}}}$$

The equation of states

$$
\begin{aligned}
\frac{PV}{NkT} &= \frac{v}{\lambda^3}\left(e^{\beta\mu} - \frac{(e^{\beta\mu})^2}{2^{5/2}} + \cdots\right) \\
&= 1 + \frac{1}{2^{5/2}}\frac{\lambda^3}{v} + \cdots
\end{aligned}
$$

Here the corrections are due to quantum effects, not molecular interactions.

5.4.3 Low temperatures and high densities

$$\frac{\lambda^3}{v} \gg 1$$

At $T = 0$

$$\frac{\lambda^3}{v} = \frac{4}{3\sqrt{\pi}} (\beta\mu(0))^{3/2}$$

The chemical potential at $T = 0$, is called the Fermi energy

$$\mu(0) \equiv \epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{v} \right)^{2/3}$$

Then

$$\begin{aligned} < n_{\vec{p}} > &= \frac{1}{e^{\beta(\epsilon_{\vec{p}} - \epsilon_F)} + 1} \\ &= \begin{cases} 1 & \epsilon_{\vec{p}} < \epsilon_F \\ 0 & \epsilon_{\vec{p}} > \epsilon_F \end{cases} \end{aligned}$$

The physical meaning of ϵ_F is clear: at $T = 0$ energy levels are filled with particles up to ϵ_F. In momentum space the particles fill a sphere of radius p_F, with $\epsilon_F = \frac{p_F^2}{2m}$. The surface of the sphere is called the Fermi surface, and p_F is called the Fermi momentum.

From

$$\frac{\lambda^3}{v} = \frac{4}{3\sqrt{\pi}} \left[(\beta\mu)^{3/2} + \frac{\pi^2}{8} (\beta\mu)^{-1/2} + \cdots \right]$$

one solves

$$\mu = \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right]$$

The average occupation number

$$< n_{\vec{p}} >= \frac{1}{e^{\beta(\epsilon_{\vec{p}}-\mu)+1}}$$

The internal energy

$$U = \sum_{\vec{p}} \epsilon_{\vec{p}} < n_{\vec{p}} >= \frac{V}{h^3} \frac{4\pi}{2m} \int_0^{\infty} \mathrm{d}p \, p^4 < n_{\vec{p}} >$$
$$= \frac{3}{5} N \epsilon_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right]$$

We define the Fermi temperature T_F by

$$kT_F \equiv \epsilon_F$$

5.5 The theory of white dwarf stars

5.5.1 Introduction

An empirical rule for stars: brightness $I \sim k{\cdot}wavelength\ \lambda$, and k is roughly universal, as shown in Fig. 5.1.

Note: there are exceptional, e.g., red giant stars are bright, white dwarf stars are faint

The white dwarf star is
– interesting because it is approximately a Fermi gas at very low temperature.
– lack brightness because the hydrogen supply has been used up.

Some data for a white dwarf star

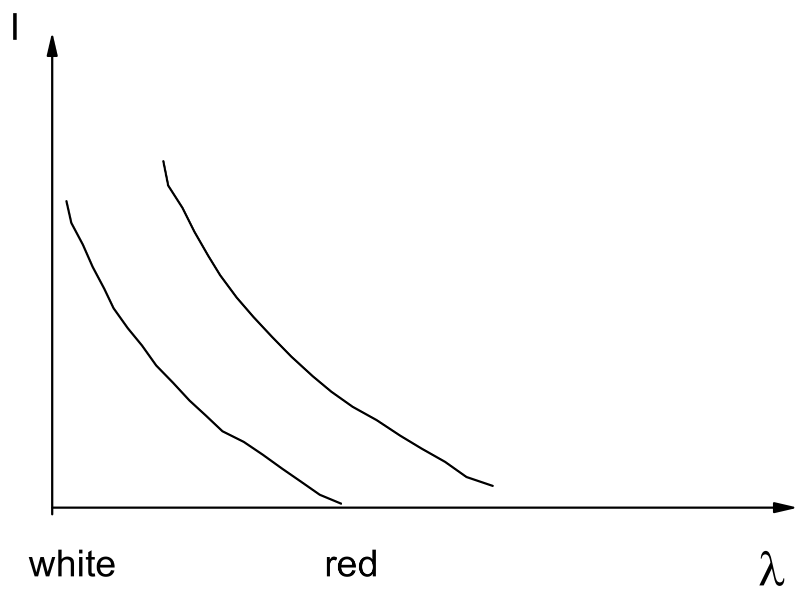

Figure 5.1:

Content: mostly helium

$$\begin{aligned} \textbf{density} &\simeq 10^7 g/cm^3 \simeq 10^7 \rho_\odot \\ \textbf{mass} &\simeq 10^{33} g \simeq M_\odot \\ \textbf{central temperature} &= 10^7 K = T_\odot \end{aligned}$$

Hence, high pressure leads to ionization, and the star can be regarded as a gas composed of helium nuclei and electrons.

We regard the electrons to be an ideal Fermi gas, $T_F \simeq 10^{11} K \gg T_\odot$**. Therefore the gas behaves as at zero temperature.**

Since the density is very high, ϵ_F **is very large. Electrons move very fast, and must be treated by relativistic dynamics.**

In contrast to electrons, the nuclei move slowly, and the corresponding pressure can be neglected. But the nuclei provide gravity to bind the star.

Three important effects

- **Pauli principle**
- **relativistic dynamics**
- **gravitational law**

5.5.2 Relativistic dynamics

a single electron is specified by $< \vec{p}, s >$

$$\epsilon_{\vec{p}s} = \sqrt{(pc)^2 + (m_e c)^2} \qquad p = |\vec{p}|$$

m_e**: mass of the electron**

Assume a Fermi momentum p_F

$$\frac{V}{h^3}\left(\frac{4}{3}\pi p_F^3\right) = \frac{N}{2}$$

$$\uparrow$$

2 is from spin degrees of freedom

$$\therefore \quad p_F = \hbar\left(\frac{3\pi^2}{v}\right)^{1/3}$$

The ground state energy

$$\begin{aligned}
E_0 &= 2\sum_{|\vec{p}|<p_F}\sqrt{(pc)^2+(m_e c)^2} \\
&= \frac{2V}{h^3}\int_0^{p_F} \mathrm{d}p\, 4\pi p^2\sqrt{(pc)^2+(m_e c)^2} \\
\therefore \quad \frac{E_0}{N} &= \frac{m_e^4 c^5}{\pi^2\hbar^3} v f(x_F) \\
f(x_F) &= \int_0^{x_F} \mathrm{d}x\, x^2\sqrt{1+x^2} \\
&= \begin{cases} \frac{1}{3}x_F^3(1+\frac{3}{10}x_F+\cdots & x_F \ll 1 \\ \frac{1}{4}x_F^4(1+\frac{1}{x_F^2}+\cdots & x_F \gg 1 \end{cases} \\
x_F &\equiv \frac{p_F}{m_e c}
\end{aligned}$$

M **and** R **denote the total mass and radius of the star respectively**

$$\begin{aligned}
M &\simeq 2m_p N \\
v &= \frac{8\pi}{3}\frac{m_p R^3}{M} \\
x_F &= \frac{\bar{M}^{1/3}}{\bar{R}}
\end{aligned}$$

$$\bar{M} = \frac{9\pi}{8}\frac{M}{m_p} \qquad \bar{R} = \frac{R}{(\hbar/m_e c)}$$

The pressure exerted by the Fermi gas

$$\begin{aligned} P_0 &= -\frac{\partial E_0}{\partial V} \\ &= \frac{m_e^4 c^5}{\pi^2 \hbar^3}\left[-f(x_F) - \frac{\partial f(x_F)}{\partial x_F}\frac{\partial x_F}{\partial v}v\right] \\ &= \left[\frac{1}{3}x_F^3\sqrt{1+x_F^2} - f(x_F)\right] \end{aligned}$$

5.5.3 Gravity

If there is no gravity, we need a wall to keep the star, and the density of the star will be uniform. The work one needs to do to compress the star from $r = \infty$ to $r = R$ is

$$-\int_{\infty}^{R} P_0(r) 4\pi r^2 \mathrm{d}r$$

Suppose now the gravity is "switched on", then the density becomes non-uniform. On dimensional grounds, one assumes it reduces the energy by an amount

$$-\frac{\alpha\gamma M^2}{R} \qquad \gamma : \quad \text{gracitational const.}$$

α is a constant of the order of unit. Hence

$$\int_{\infty}^{R} P_0 4\pi r^2 \mathrm{d}r = -\frac{\alpha\gamma M^2}{R}$$

$$\begin{aligned} \Rightarrow \qquad P_0(R) &= \frac{\alpha}{4\pi}\frac{\gamma M^2}{R^4} \\ &= \frac{\alpha}{4\pi}\gamma\left(\frac{8m_p}{9\pi}\right)^2\left(\frac{m_e c}{\hbar}\right)^4\frac{\bar{M}^2}{\bar{R}^4} \end{aligned}$$

For $x_F \gg 1$,

$$\begin{aligned} P_0(R) &= K\left(\frac{\bar{M}^{4/3}}{\bar{R}^4} - \frac{\bar{M}^{2/3}}{\bar{R}^2}\right) \\ K &= \frac{m_e c^2}{12\pi^2}\left(\frac{m_e c}{\hbar}\right)^3 \end{aligned}$$

From the above two equations for P_0

$$\begin{aligned} \therefore \quad \bar{R} &= \bar{M}^{1/3}\sqrt{1-(\bar{M}/\bar{M}_0)^{2/3}} \\ \bar{M}_0 &= \left(\frac{27\pi}{64\alpha}\right)^{3/2}\left(\frac{\hbar c}{\gamma m_p^2}\right)^{3/2} \end{aligned}$$

If $\alpha = 1$

$$M_0 = \frac{8}{9\pi} m_p \bar{M}_0 \simeq M_\odot$$

- **No white dwarf star can have a mass larger than M_0. Otherwise it collapses due to over strong gravity and induces, e.g. a supernova.**

- **More accurate calculations show that**

$$M_0 \simeq 1.4 M_\odot$$